Module - 4
LAPLACE TRANSFORMATION & APPLICATIONS

Objectives:
1. To study the concepts of step, ramp and impulse signals.
2. To study Laplace transforms technique.
3. To study Laplace transforms technique to circuits.
4. To find transfer function of a system response using Laplace transforms.

5.1 Introduction

In this chapter, we will introduce Laplace transform. This is an extremely important technique. For a given set of initial conditions, it will give the total response of the circuit comprising of both natural and forced responses in one operation. The idea of Laplace transform is analogous to any familiar transform. For example, Logarithms are used to change a multiplication or division problem into a simpler addition or subtraction problem and Antilogs are used to carry out the inverse process. This example points out the essential feature of a transform: They are designed to create a new domain to make mathematical manipulations easier. After evaluating the unknown in the new domain, we use inverse transform to get the evaluated unknown in the original domain. The Laplace transform enables the circuit analyst to convert the set of integrodifferential equations describing a circuit to the complex frequency domain, where thay become a set of linear algebraic equations. Then using algebraic manipulations, one may solve for the variables of interest. Finally, one uses the inverse transform to get the variable of interest in time domain. Also, in this chapter, we express the impedance in s domain or complex frequency domain. Hence, we may analyze a circuit using one of the reduction techniques such as Thevenin theorem or source transformation discussed in earlier chapters.

5.2 Definition of Laplace transorm

A transform is a change in the mathematical description of a physical variable to facilitate computation. Keeping this definition in mind, Laplace transform of a function $f(t)$ is defined as

$$\mathcal{L}\{f(t)\} = F(s) = \int_0^\infty f(t)e^{-st}dt \tag{5.1}$$

Here the complex frequency is $s = \sigma + j\omega$. Since the argument of the exponent e in equation (5.1) must be dimensionless, it follows that s has the dimensions of frequency and units of inverse seconds (sec^{-1}).

The notation implies that once the integral has been evaluated, $f(t)$, a time domain function is transformed to $F(s)$, a frequency domain function.

If the lower limit of integration in equation (5.1) is $-\infty$, then it is called the bilateral Laplace transform. However for circuit applications, the lower limit is taken as zero and accordingly the transform is unilateral in nature.

The lower limit of integration is sometimes chosen to be 0^- to permit $f(t)$ to include $\delta(t)$ or its derivatives. Thus we should note immediately that the integration from 0^- to 0^+ is zero except when an impulse function or its derivatives are present at the origin.

Region of convergence

The Laplace transform of a signal $f(t)$ as seen from equation (5.1) is an integral operation. It exists if $f(t)e^{-\sigma t}$ is absolutely integrable. That is $\int_0^\infty f(t)e^{-\sigma t}dt < \infty$. Cleary, only typical choices of σ will make the integral converge. The range of σ that ensures the existence of $X(s)$ defines the region of convergence (ROC) of the Laplace transform. As an example, let us take $x(t) = e^{3t}, t \geq 0$. Then

$$\begin{aligned} X(s) &= \int_0^\infty x(t)e^{-(\sigma+j\omega)t}dt \\ &= \int_0^\infty e^{(-\sigma+3)t}e^{-j\omega t}dt \end{aligned}$$

The above integral converges if and only if $-\sigma + 3 < 0$ or $\sigma > 3$. Thus, $\sigma > 3$ defines the ROC of $X(s)$. Since, we shall deal only with causal signals$(t \geq 0)$ we avoid explicit mention of ROC.

Due to the convergence factor, $e^{-\sigma t}$, a number of important functions have Laplace transforms, even though Fourier transforms for these functions do not exist. But this does not mean that every mathematical function has Laplace transform. The reader should be aware that, for example, a function of the form e^{t^2} does not have Laplace transform.

The inverse Laplace transform is defined by the relationship:

$$\mathcal{L}^{-1}\{F(s)\} = f(t) = \frac{1}{2\pi j}\int_{\sigma-j\infty}^{\sigma+j\infty} F(s)e^{st}ds \tag{5.2}$$

where σ is real. The evaluation of integral in equation (5.2) is based on complex variable theory, and hence we will avoid its use by developing a set of Laplace transform pairs.

5.3 Three important singularity functions

The three important singularity functions employed in circuit analysis are:

(i) unit step function, $u(t)$

(ii) delta function, $\delta(t)$

(iii) ramp function, $r(t)$.

They are called singularity functions because they are either not finite or they do not possess finite derivatives everywhere.

The mathematical definition of unit step function is

$$u(t) = \begin{cases} 0, & t < 0 \\ 1, & t > 0 \end{cases} \tag{5.3}$$

The step function is not defined at $t = 0$. Thus, the unit step function $u(t)$ is 0 for negative values of t, and 1 for positive values of t. Often it is advantageous to define the unit step function as follows:

$$u(t) = \begin{cases} 1, & t \geq 0^+ \\ 0, & t \leq 0^- \end{cases}$$

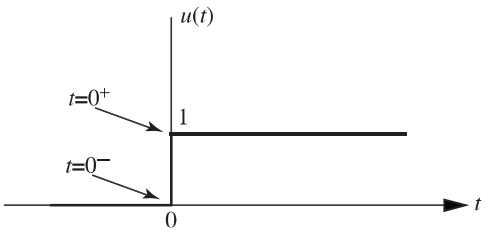

Figure 5.1 The unit step function

A discontinuity may occur at time other than $t = 0$; for example, in sequential switching, the unit step function that occurs at $t = a$ is expressed as $u(t - a)$.

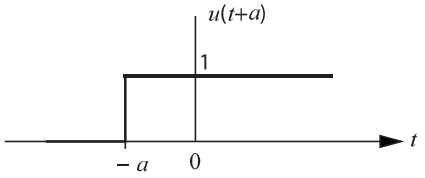

Figure 5.2 The step function occuring at $t = a$

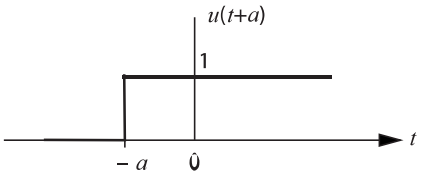

Figure 5.3 The step function occuring at $t = -a$

Thus, $$u(t - a) = \begin{cases} 0, & t - a < 0 \text{ or } t < a \\ 1, & t - a > 0 \text{ or } t > a \end{cases}$$

Similarly, the unit step function that occurs at $t = -a$ is expressed as $u(t + a)$.

Thus, $$u(t + a) = \begin{cases} 0, & t + a < 0 \text{ or } t < -a \\ 1, & t + a > 0 \text{ or } t > -a \end{cases}$$

We use step function to represent an abrupt change in voltage or current, like the changes that occur in the circuits of control engineering and digital systems. For example, the voltage

$$v(t) = \begin{cases} 0, & t < a \\ K, & t > a \end{cases}$$

may be expressed in terms of the unit step function as

$$v(t) = Ku(t-a) \tag{5.4}$$

The derivative of the unit step function $u(t)$ is the unit impulse function $\delta(t)$.

That is,
$$\delta(t) = \frac{d}{dt}u(t) = \begin{cases} 0, & t < 0 \\ \text{undefined}, & t = 0 \\ 0, & t > 0 \end{cases} \tag{5.5}$$

The unit impulse function also known as dirac delta fucntion is shown in Fig. 5.4.

The unit impulse may be visualized as very short duration pulse of unit area. This may be expressed mathematically as:

$$\int_{0^-}^{0^+} \delta(t)dt = 1 \tag{5.6}$$

where $t = 0^-$ denotes the time just before $t = 0$ and $t = 0^+$ denotes the time just after $t = 0$. Since the area under the unit impulse is unity, it is a practice to write '1' beside the arrow that is used to symbolize the unit impulse function as shown in Fig. 5.4. When the impulse has a strength other than unity, the area of the impulse function is equal to its strength. For example, an impulse function $5\delta(t)$ has an area of 5 units. Figure 5.5 shows impulse functions, $2\delta(t+2)$, $5\delta(t)$ and $-2\delta(t-3)$.

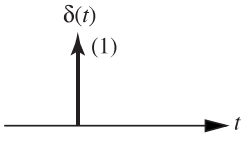

Figure 5.4 The circuit impulse function

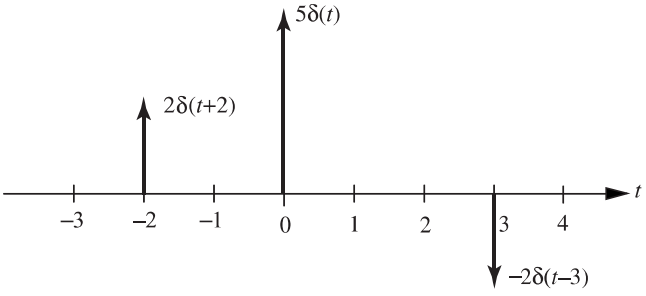

Figure 5.5 Three impulse functions

An important property of the unit impulse function is what is often called the sifting property; which is exhibited by the following integral:

$$\int_{t_1}^{t_2} f(t)\delta(t-t_0)dt = \begin{cases} f(t_0), & t_1 < t_0 < t_2 \\ 0, & t_1 > t_0 > t_2 \end{cases}$$

for a fintie t_0 and any $f(t)$ continuous at t_0.

Integrating the unit step function results in the unit ramp function $r(t)$.

$$r(t) = \int_{-\infty}^{t} u(\tau)d\tau = tu(t) \tag{5.7}$$

$$\text{or} \qquad r(t) = \begin{cases} 0, & t \leq 0 \\ t, & t \geq 0 \end{cases}$$

Figure 5.6 shows the ramp function.

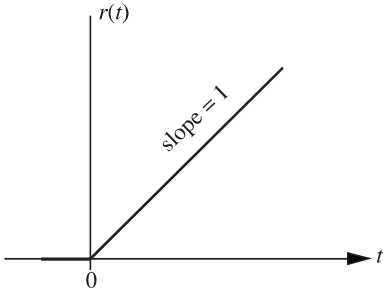

Figure 5.6 The unit ramp function

In general, a ramp is a function that changes at a constant rate.

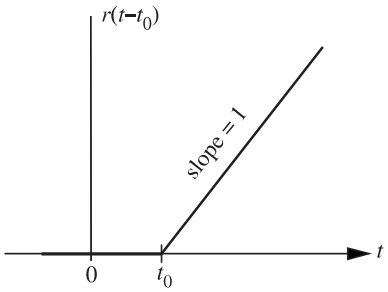

Figure 5.7 The unit ramp function delayed by t_0

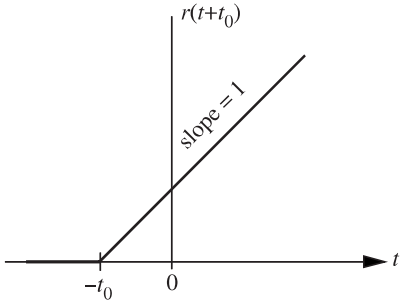

Figure 5.8 The unit ramp function advanced by t_0

A delayed ramp function is shown in Fig. 5.7. Mathematically, it is described as follows:

$$r(t-t_0) = \begin{cases} 0, & t \leq t_0 \\ t-t_0, & t \geq t_0 \end{cases}$$

An advanced ramp function is shown in Fig. 5.8. Mathematically, it is described as follows:

$$r(t+t_0) = \begin{cases} 0, & t \le -t_0 \\ t+t_0, & t \ge -t_0 \end{cases}$$

It is very important to note that the three sigularity functions are related by differentiation as

$$\delta(t) = \frac{du(t)}{dt}, \qquad u(t) = \frac{dr(t)}{dt}$$

or by integration as

$$u(t) = \int_{-\infty}^{t} \delta(t)dt, \quad r(t) = \int_{-\infty}^{t} u(\tau)d\tau$$

5.4 Functional transforms

A functional transform is simply the Laplace transform of a specified function of t. Here we make an assumption that $f(t)$ is zero for $t < 0$.

5.4.1 Decaying exponential function

$f(t) = e^{-at}u(t)$, where $a > 0$ and $u(t)$ is the unit step function.

$$\begin{aligned} \mathscr{L}\left\{e^{-at}u(t)\right\} = F(s) &= \int_0^{\infty} f(t)dt \\ &= \int_0^{\infty} e^{-at}e^{-st}dt \\ &= \frac{-e^{-(s+a)t}}{(s+a)}\Bigg|_{t=0}^{\infty} \\ &= \frac{1}{s+a} \end{aligned}$$

5.4.2 Unit step function

$$f(t) = u(t)$$

$$\mathscr{L}\{u(t)\} = F(s) = \int_0^{\infty} e^{-st}dt = \frac{1}{s}$$

5.4.3 Impulse function

$$f(t) = \delta(t)$$

$$\mathcal{L}\{\delta(t)\} = F(s) = \int_{0-}^{\infty} \delta(t) e^{-st} dt = e^{-st}\big|_{t=0} = 1$$

Please note that we have used the sifting property of an impulse function.

5.4.4 Sinusoidal function

$$f(t) = \sin \omega t, \qquad t \geq 0$$

Since

$$\sin \omega t = \frac{1}{2j}\left[e^{j\omega t} - e^{-j\omega t}\right]$$

and

$$\mathcal{L}\{e^{-at}\} = \frac{1}{s+a}$$

we have

$$\mathcal{L}\{\sin \omega t\} = F(s) = \frac{1}{2j}\int_{0}^{\infty} \left(e^{j\omega t} - e^{-j\omega t}\right) e^{-st} dt$$

$$= \frac{1}{2j}\left[\frac{1}{s - j\omega} - \frac{1}{s + j\omega}\right]$$

$$= \frac{\omega}{s^2 + \omega^2}$$

Table 5.1 gives a list of important Laplace transform pairs. It includes the functions of most interest in an introductory course on circuit applications.

Table 5.1 Important transform pairs

$f(t)(t \geq 0)$	$F(s)$
$\delta(t)$	1
$u(t)$	$\frac{1}{s}$
t	$\frac{1}{s^2}$
e^{-at}	$\frac{1}{s+a}$
$\sin \omega t$	$\frac{\omega}{s^2+\omega^2}$
$\cos \omega t$	$\frac{s}{s^2+\omega^2}$

$f(t)(t \geq 0)$	$F(s)$
t^n	$\dfrac{n!}{s^{n+1}}$
te^{-at}	$\dfrac{1}{(s+a)^2}$
$e^{-at}\sin\omega t$	$\dfrac{\omega}{(s+a)^2+\omega^2}$
$e^{-at}\cos\omega t$	$\dfrac{s+a}{(s+a)^2+\omega^2}$

All functions in the above table are represented without multiplied by $u(t)$, since we have explicity declared that $t \geq 0$.

5.5 Operational transforms (properties of Laplace transform)

Operational transforms indicate how mathematical operations performed on either $f(t)$ or $F(s)$ are converted into the opposite domain. Following operations are of primary interest.

***Note*:** The symbol $\triangleq$ means **"by the definition"**.

5.5.1 Linearity

If $$\mathcal{L}\{f_1(t)\} = F_1(s) \text{ and } \mathcal{L}\{f_2(t)\} = F_2(s)$$

then $$\mathcal{L}\{a_1 f_1(t) + a_2 f_2(t)\} = a_1 F_1(s) + a_2 F_2(s)$$

***Proof*:**

$$\begin{aligned}\mathcal{L}\{a_1 f_1(t) + a_2 f_2(t)\} &\triangleq \int_0^\infty [a_1 f_1(t) + a_2 f_2(t)]e^{-st}dt \\ &= a_1 \int_0^\infty f_1(t)e^{-st}dt + a_2 \int_0^\infty f_2(t)e^{-st}dt \\ &= a_1 F_1(s) + a_2 F_2(s)\end{aligned}$$

EXAMPLE 5.1

Find the Laplace transform of $f(t) = (A + Be^{-bt}u(t))$.

SOLUTION

We have the transform pair

$$\mathcal{L}\{u(t)\} = \frac{1}{s}$$

and $$\mathcal{L}\{e^{-bt}u(t)\} = \frac{1}{s+b}$$

Thus, using linearity property,

$$\begin{aligned}\mathcal{L}\{f(t)\} = F(s) &= \mathcal{L}\{Au(t)\} + \mathcal{L}\{Be^{-bt}u(t)\} \\ &= \frac{A}{s} + \frac{B}{s+b} \\ &= \frac{\boldsymbol{(A+B)s + Ab}}{\boldsymbol{s(s+b)}}\end{aligned}$$

5.5.2 Time shifting

If $\mathcal{L}\{x(t)\} = X(s)$, then for any real number t_0,

$$\mathcal{L}\{x(t-t_0)u(t-t_0)\} = e^{-t_0 s}X(s)$$

***Proof*:**

$$\mathcal{L}\{x(t-t_0)u(t-t_0)\} \triangleq \int_0^\infty x(t-t_0)u(t-t_0)e^{-st}dt$$

Since,
$$u(t-t_0) = \begin{cases} 1, & t-t_0 > 0 \text{ or } t > t_0 \\ 0, & t-t_0 < 0 \text{ or } t < t_0 \end{cases}$$

we get,
$$\mathcal{L}\{x(t-t_0)u(t-t_0)\} = \int_{t_0}^\infty x(t-t_0)e^{-st}dt$$

Using the transformation of variable,

$$t = \tau + t_0$$

we get,
$$\begin{aligned}\mathcal{L}\{x(t-t_0)u(t-t_0)\} &= \int_0^\infty x(\tau)e^{-s(\tau+t_0)}d\tau \\ &= e^{-t_0 s}\int_0^\infty x(\tau)e^{-s\tau}d\tau \\ &= e^{-t_0 s}X(s)\end{aligned}$$

EXAMPLE 5.2

Find the Laplace transform of $x(t)$, shown in Fig. 5.9.

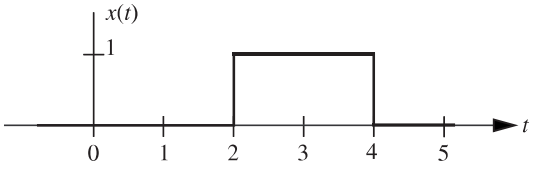

Figure 5.9

SOLUTION

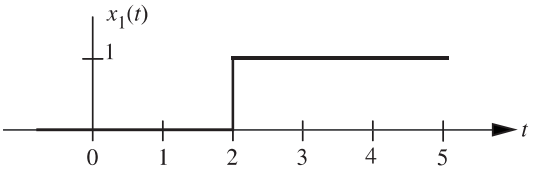

Figure 5.10(a)

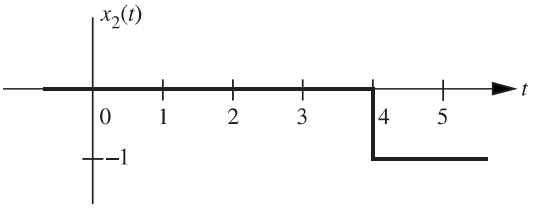

Figure 5.10(b)

Using Figs. 5.10(a) and 5.10(b), we can write

$$x(t) = x_1(t) + x_2(t) = u(t-2) - u(t-4)$$

We know that, $\mathcal{L}\{u(t)\} = \dfrac{1}{s}$ and using time shifting property, we have

$$\mathcal{L}\{x(t)\} = X(s) = \frac{1}{s}e^{-2s} - \frac{1}{s}e^{-4s}$$

$$\Rightarrow \qquad \boldsymbol{X(s) = \frac{1}{s}(e^{-2s} - e^{-4s})}$$

5.5.3 Shifting in s domain (Frequency-domain shifting)

If $\mathcal{L}\{x(t)\} = X(s)$, then

$$\mathcal{L}\{e^{s_0 t}x(t)\} = X(s - s_0)$$

***Proof*:**

$$\begin{aligned}\mathcal{L}\{e^{s_0 t}x(t)\} &\triangleq \int_0^\infty e^{s_0 t}x(t)e^{-st}dt \\ &= \int_0^\infty x(t)e^{-(s-s_0)t}dt \\ &= X(s - s_0)\end{aligned}$$

EXAMPLE 5.3

Find the Laplace transform of $x(t) = Ae^{-at}\cos(\omega_0 t + \theta)u(t)$.

SOLUTION

Given
$$\begin{aligned} x(t) &= Ae^{-at}\cos(\omega_0 t + \theta)u(t) \\ &= Ae^{-at}[\cos\omega_0 t\cos\theta - \sin\omega_0 t\sin\theta]u(t) \\ &= A\cos\theta e^{-at}\cos\omega_0 t u(t) - A\sin\theta e^{-at}\sin\omega_0 t u(t) \end{aligned}$$

We know the transform pairs,

$$\mathcal{L}\{\cos\omega_0 t u(t)\} = \frac{s}{s^2+\omega_0^2}$$

and
$$\mathcal{L}\{\sin\omega_0 t u(t)\} = \frac{\omega_0}{s^2+\omega_0^2}$$

Applying frequency shifting property, we get

$$\begin{aligned} \mathcal{L}\left\{e^{-at}\cos\omega_0 t u(t)\right\} &= \left.\frac{s}{s^2+\omega_0^2}\right|_{s\to s+a} \\ &= \frac{s+a}{(s+a)^2+\omega_0^2} \end{aligned}$$

and
$$\begin{aligned} \mathcal{L}\left\{e^{-at}\sin\omega_0 t u(t)\right\} &= \left.\frac{\omega_0}{s^2+\omega_0^2}\right|_{s\to s+a} \\ &= \frac{\omega_0}{(s+a)^2+\omega_0^2} \end{aligned}$$

Finally, applying linearity property, we get

$$\begin{aligned} \mathcal{L}\{Ae^{-at}\cos(\omega_0 t+\theta)u(t)\} &= A\cos\theta\,\mathcal{L}\{e^{-at}\cos\omega_0 t u(t)\} - A\sin\theta\,\mathcal{L}\{e^{-at}\sin\omega_0 t u(t)\} \\ &= \frac{A\cos\theta(s+a)}{(s+a)^2+\omega_0^2} - A\sin\theta\frac{\omega_0}{(s+a)^2+\omega_0^2} \\ &= \boldsymbol{\frac{A[(s+a)\cos\theta - \omega_0\sin\theta]}{(s+a)^2+\omega_0^2}} \end{aligned}$$

5.5.4 Time scaling

If $\mathcal{L}\{x(t)\} = X(s)$, then

$$\mathcal{L}\{x(at)\} = \frac{1}{a}X\left(\frac{s}{a}\right)$$

***Proof*:**

$$\mathcal{L}\{x(at)\} \triangleq \int_0^\infty x(at)e^{-st}dt$$

put $at = \tau$

$\Rightarrow$ $adt = d\tau$

Hence

$$\mathcal{L}\{x(at)\} = \int_0^\infty x(\tau)e^{-s\frac{\tau}{a}}\frac{1}{a}d\tau$$

$$= \frac{1}{a}\int_0^\infty x(\tau)e^{-\frac{s}{a}\tau}d\tau = \frac{1}{a}X\left(\frac{s}{a}\right)$$

EXAMPLE 5.4

Find the Laplace transform of $x(t) = \sin(2\omega_0 t)u(t)$.

SOLUTION

We know the transform pair,

$$\mathcal{L}\{\sin\omega_0 t u(t)\} = \frac{\omega_0}{s^2+\omega_0^2}$$

Applying scaling property,

$$\mathcal{L}\{\sin 2\omega_0 t u(t)\} = \frac{1}{2}\left[\frac{\omega_0}{\left(\frac{s}{2}\right)^2+\omega_0^2}\right]$$

$$= \boldsymbol{\frac{2\omega_0}{s^2+4\omega_0^2}}$$

5.5.5 Time differentiation

If $\mathcal{L}\{x(t)\} = X(s)$, then

$$\mathcal{L}\left\{\frac{dx(t)}{dt}\right\} = sX(s) - x(0)$$

***Proof*:**

Let $y(t) = \dfrac{dx(t)}{dt}$

Then

$$\mathcal{L}\{y(t)\} = Y(s) \triangleq \int_0^\infty y(t)e^{-st}dt$$

$$= \int_0^\infty \frac{dx(t)}{dt}e^{-st}dt$$

Integrating by parts yields

$$\begin{aligned} Y(s) &= e^{-st}x(t)\big|_0^\infty - \int_0^\infty x(t)(-se^{-st})dt \\ &= \lim_{t\to\infty}[e^{-st}x(t)] - x(0) + s\int_0^\infty x(t)e^{-st}dt \\ &= 0 - x(0) + sX(s) \end{aligned}$$

Hence, $$\mathcal{L}\left\{\frac{dx(t)}{dt}\right\} = Y(s) = sX(s) - x(0)$$

Therefore, differentiation in time domain is equivalent to multiplication by s in the s domain.

Whenever $x(t)$ is discontinuous at $t = 0$ (like a step function), then $x(0)$ should be read as $x(0^-)$.

The differentiation property can be extended to yield

$$\mathcal{L}\left\{\frac{d^n x(t)}{dt^n}\right\} = s^n X(s) - s^{n-1}x(0)\cdots - x^{n-1}(0)$$

When $x(t)$ is discontinuous at the origin, the argument 0 on the right side of the above equation should be read as 0^-. Accordingly for a discontinuous function $x(t)$ at the origin, we get

$$\mathcal{L}\left\{\frac{d^n x(t)}{dt^n}\right\} = s^n X(s) - s^{n-1}x(0^-)\cdots - x^{n-1}(0^-)$$

EXAMPLE 5.5

Find the Laplace transform of $x(t) = \sin^2 \omega_0 t u(t)$.

SOLUTION

We find that, $x(0) = 0$

$$\begin{aligned} \frac{dx(t)}{dt} &= 2\omega_0 \sin\omega_0 t \cos\omega_0 t u(t) \\ &= \omega_0 \sin 2\omega_0 t u(t) \end{aligned} \tag{5.8}$$

We know that, $$\mathcal{L}\{\sin\omega_0 t u(t)\} = \frac{\omega_0}{s^2 + \omega_0^2}$$

Applying time scaling property,

$$\mathcal{L}\{\sin 2\omega_0 t u(t)\} = \frac{1}{2}\left[\frac{\omega_0}{\left(\frac{s}{2}\right)^2 + \omega_0^2}\right]$$
$$= \frac{2\omega_0}{s^2 + 2(\omega_0)^2}$$

Taking Laplace transform on both the sides of equation (5.8), we get

$$\mathcal{L}\left\{\frac{dx(t)}{dt}\right\} = \omega_0\, \mathcal{L}\{\sin \omega_0 t u(t)\}$$
$$\Rightarrow \quad sX(s) - x(0) = \frac{2\omega_0^2}{s^2 + (2\omega_0)^2}$$
$$\Rightarrow \quad \boldsymbol{X(s) = \frac{2\omega_0^2}{s[s^2 + (2\omega_0)^2]}}$$

EXAMPLE 5.6

Solve the second order linear differential equation

$$y''(t) + 5y'(t) + 6y(t) = x(t)$$

with the initial conditions, $y(0) = 2$, $y'(0) = 1$ and $x(t) = e^{-t}u(t)$.

SOLUTION

Taking Laplace transform on both the sides of the given differential equation, we get

$$\left|s^2Y(s) - sy(0) - y'(0)\right| + 5\left|sY(s) - y(0)\right| + 6Y(s) = X(s)$$

where $$X(s) = \mathcal{L}\{e^{-t}u(t)\} = \frac{1}{s+1}$$

Substituting the initial conditions, we get

$$(s^2 + 5s + 6)Y(s) = \frac{1}{s+1} + 2s + 11$$
$$\Rightarrow \quad Y(s) = \frac{2s^2 + 13s + 12}{(s+1)(s+2)(s+3)}$$

Using partial fraction expansion, we get

$$Y(s) = \frac{1}{2}\left[\frac{1}{s+1}\right] + 6\left[\frac{1}{s+2}\right] - \frac{9}{2}\left[\frac{1}{s+3}\right]$$

Taking inverse Laplace transform, we get

$$\boldsymbol{y(t) = \frac{1}{2}e^{-t} + 6e^{-2t} - \frac{9}{2}e^{-3t}, \qquad t \geq 0}$$

5.5.6 Integration in time domain

For a causal signal $x(t)$,

If $$y(t) = \int_0^t x(\tau)d\tau,$$

then $$\mathcal{L}\{y(t)\} = Y(s) = \frac{X(s)}{s}$$

***Proof*:**

$$\mathcal{L}\{x(t)\} = X(s) \triangleq \int_0^\infty x(t)e^{-st}dt$$

Dividing both sides by s yields

$$\frac{X(s)}{s} = \int_0^\infty x(t)\frac{e^{-st}}{s}dt$$

Integrating the right-hand side by parts, we get

$$\frac{X(s)}{s} = \frac{e^{-ts}}{s}y(t)\Big|_{t=0}^{\infty} - \int_0^\infty y(t)\frac{e^{-ts}}{s}(-s)dt$$

$$\frac{X(s)}{s} = y(t)\ \frac{e^{-st}}{s}\Big|_{t=0}^{\infty} + \int_0^\infty y(t)e^{-st}dt$$

The first term on the right-hand side evaluates to zero at both limits, because

$$e^{-\infty} = 0 \ \text{ and } \ y(0) = \int_0^0 x(\tau)d\tau = 0$$

Hence, $$Y(s) = \frac{X(s)}{s}$$

Thus, integration in time domain is equivalent to division by s in the s domain.

EXAMPLE 5.7

Consider the RC circuit shown in Fig. 5.11. The input is the rectangular pulse shown in Fig. 5.12. Find $i(t)$ by assuming circuit is initially relaxed.

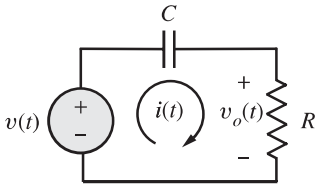

Figure 5.11

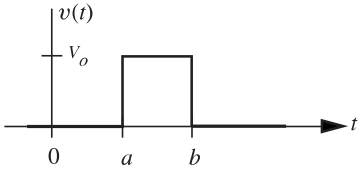

Figure 5.12

SOLUTION

Applying *KVL* to the circuit represented by Fig. 5.11, we get

$$Ri(t) + \frac{1}{C}\int_0^t i(\tau)d\tau = v(t)$$

$$\Rightarrow \qquad Ri(t) + \frac{1}{C}\int_0^t i(\tau)d\tau = V_o[u(t-a) - u(t-b)]$$

Taking Laplace transforms on both the sides, we get

$$R\mathbf{I}(s) + \frac{1}{Cs}\mathbf{I}(s) = \frac{V_o}{s}(e^{-as} - e^{-bs})$$

$$\Rightarrow \qquad \mathbf{I}(s) = \frac{\dfrac{V_o}{R}}{s + \dfrac{1}{RC}}(e^{-as} - e^{-bs})$$

We know the transform pair,

$$\mathscr{L}\{e^{-at}u(t)\} = \frac{1}{s+a}$$

and then using the time-shift property, we can find inverse of $\mathbf{I}(s)$.

That is, $$i(t) = \frac{V_o}{R}e^{-\frac{t}{RC}}u(t)\Big|_{t\to t-a} - \frac{v_o}{R}e^{-\frac{t}{RC}}u(t)\Big|_{t\to t-b}$$

$$\Rightarrow \quad \boldsymbol{i(t) = \frac{V_o}{R}\left[e^{-\frac{(t-a)}{RC}}u(t-a) - e^{-\frac{(t-b)}{RC}}u(t-b)\right]}$$

5.5.7 Differentiation in the s domain

For a signal $x(t)$, $t \geq 0$, we have

$$\mathscr{L}\{-tx(t)\} = \frac{dX(s)}{ds}$$

***Proof*:**
For a causal signal, $x(t)$, the Laplace transform is given by

$$\mathcal{L}\{x(t)\} = X(s) = \int_0^\infty x(t)e^{-st}dt$$

Differentiating both the sides with respect to s, we get

$$\frac{dX(s)}{ds} = \int_0^\infty x(t)(-te^{-st})dt$$

$$\Rightarrow \qquad \frac{dX(s)}{ds} = \int_0^\infty [-tx(t)]e^{-st}dt$$

Hence, $$\mathcal{L}\{-tx(t)\} = \frac{dX(s)}{ds} \quad \text{or} \quad \mathcal{L}\{tx(t)\} = \frac{-dX(s)}{ds}$$

In general, $$\mathcal{L}\{t^n x(t)\} = (-1)^n \frac{d^n X(s)}{ds^n}$$

EXAMPLE 5.8

Find the Laplace transform of $x_1(t) = te^{-3t}u(t)$.

SOLUTION

We know that,

$$\mathcal{L}\{e^{-at}u(t)\} = \frac{1}{s+a}$$

Hence $$\mathcal{L}\{e^{-3t}u(t)\} = \frac{1}{s+3}$$

Using the differentiation in s domain property,

$$\mathcal{L}\{x_1(t)\} = X_1(s) = \frac{-d}{ds}\left[\frac{1}{s+3}\right]$$

$$= \frac{\mathbf{1}}{\mathbf{(s+3)^2}}$$

5.5.8 Convolution

If $$\mathcal{L}\{x(t)\} = X(s)$$

and $$\mathcal{L}\{h(t)\} = H(s)$$

then $$\mathcal{L}\{x(t) * h(t)\} = X(s)H(s)$$

where $*$ indicates the convolution operator.

***Proof*:**

$$x(t) * h(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau$$

Since $x(t)$ and $h(t)$ are causal signals, the convolution in this case reduces to

$$x(t) * h(t) = \int_{0}^{\infty} x(\tau)h(t-\tau)d\tau$$

Hence, $$\mathcal{L}\{x(t) * h(t)\} = \int_{0}^{\infty} \left[\int_{0}^{\infty} x(\tau)h(t-\tau)d\tau\right] e^{-st}dt$$

Interchanging the order of integrals, we get

$$\mathcal{L}\{x(t) * h(t)\} = \int_{0}^{\infty} x(\tau) \left[\int_{0}^{\infty} h(t-\tau)e^{-st}dt\right] d\tau$$

Using the change of variable $\lambda = t - \tau$ in the inner integral, we get

$$\begin{aligned}\mathcal{L}\{x(t) * h(t)\} &= \int_{0}^{\infty} x(\tau)e^{-s\tau} \left[\int_{0}^{\infty} h(\lambda)e^{-s\lambda}d\lambda\right] d\tau \\ &= X(s)H(s)\end{aligned}$$

Please note that this theorem reduces the complexity of evaluating the convolution integral to a simple multiplication.

EXAMPLE 5.9

Find the convolution of $h(t) = e^{-t}$ and $f(t) = e^{-2t}$.

SOLUTION

$$\begin{aligned}h(t) * f(t) &= \mathcal{L}^{-1}\{H_1(s)F(s)\} \\ &= \mathcal{L}^{-1}\left\{\left(\frac{1}{s+1}\right)\left(\frac{1}{s+2}\right)\right\} \\ &= \mathcal{L}^{-1}\left\{\frac{1}{s+1} + \frac{-1}{s+2}\right\} \\ &= \boldsymbol{e^{-t} - e^{-2t}, \quad t \geq 0}\end{aligned}$$

EXAMPLE 5.10

Find the convolution of two indentical rectangular pulses. Each rectangular pulse has unit amplitude and duration equal to $2T$ seconds. Also, the pulse is centered at $t = T$.

SOLUTION

Let the pulse be as shown in Fig. 5.13.

From the Fig. 5.13, we can write

$$x(t) = u(t) - u(t - 2T)$$

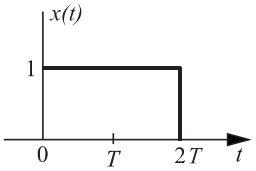

Figure 5.13

Taking Laplace transform, we get

$$X(s) = \frac{1}{s} - \frac{1}{s}e^{-2Ts}$$
$$= \frac{1}{s}(1 - e^{-2Ts})$$

Let $y(t) = x(t) * x(t)$

Then, $Y(s) = X^2(s)$

$$= \left[\frac{1 - e^{-2Ts}}{s}\right]^2$$

$$\Rightarrow \quad Y(s) = \frac{1}{s^2} - \frac{2}{s^2}e^{-2Ts} + \frac{1}{s^2}e^{-4Ts}$$

Taking inverse Laplace transform, we get

$$y(t) = tu(t) - 2(t - 2T)u(t - 2T) + (t - 4T)u(t - 4T)$$
$$= r(t) - 2r(t - 2T) + r(t - 4T)$$
$$\boldsymbol{y(t) = x(t) * x(t)}$$

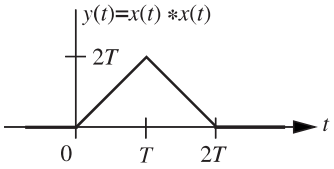

Figure 5.14

5.5.9 Initial-value theorem

The initial-value theorem allows us to find the initial value $x(0)$ directly from its Laplace transform $X(s)$.

If $x(t)$ is a causal signal,

then, $$x(0) = \lim_{s \to \infty} sX(s) \tag{5.9}$$

***Proof*:**

To prove this theorem, we use the time differentiation property.

$$\mathcal{L}\left\{\frac{dx(t)}{dt}\right\} = sX(s) - x(0) = \int_0^\infty \frac{dx}{dt}e^{-st}dt \tag{5.10}$$

* The problems with $*$ are better understood after the inverse Laplace transforms are studied.

If we let $s \to \infty$, then the integral on the right side of equation (5.10) vanishes due to damping factor, e^{-st}.

Thus, $$\lim_{s\to\infty}[sX(s) - x(0)] = 0$$

$$\Rightarrow \quad x(0) = \lim_{s\to\infty} sX(s)$$

EXAMPLE 5.11

Find the initial value of

$$F(s) = \frac{s+1}{(s+1)^2 + 3^2}$$

SOLUTION

$$f(0) = \lim_{s\to\infty} sX(s) = \lim_{s\to\infty} s\left[\frac{s+1}{(s+1)^2+3^2}\right]$$

$$= \lim_{s\to\infty} \frac{s^2+s}{(s+1)^2+3^2}$$

$$= \lim_{s\to\infty} \frac{s^2\left[1+\frac{1}{s}\right]}{s^2\left[1+\frac{2}{s}+\frac{10}{s^2}\right]} = 1$$

We know the transform pair:

$$\mathcal{L}\{e^{-bt}\cos at\} = \frac{s+b}{(s+b)^2+a^2}$$

Hence, inverse Laplace transform of $F(s)$ yields

$$\boldsymbol{f(t) = e^{-t}\cos 3t}$$

At $t = 0$, we get $f(0) = 1$.
This verifies the theorem.

5.5.10 Final-value theorem

The final-value theorem allows us to find the final value $x(\infty)$ directly from its Laplace transform $X(s)$.

If $x(t)$ is a causal signal,

then $$\lim_{t\to\infty} x(t) = \lim_{s\to 0} sX(s)$$

***Proof*:**

The Laplace transform of $\dfrac{dx(t)}{dt}$ is given by

$$sX(s) - x(0) = \int_0^\infty \frac{dx(t)}{dt} e^{-st} dt$$

Taking the limit $s \to 0$ on both the sides, we get

$$\begin{aligned}
\lim_{s\to 0}[sX(s) - x(0)] &= \lim_{s\to 0}\int_0^\infty \frac{dx(t)}{dt}e^{-st}dt \\
&= \int_0^\infty \frac{dx(t)}{dt}\left[\lim_{s\to 0} e^{-st}\right]dt \\
&= \int_0^\infty \frac{dx(t)}{dt}dt \\
&= x(t)|_0^\infty \\
&= x(\infty) - x(0)
\end{aligned}$$

Since, $\lim_{s\to 0}[sX(s) - x(0)] = \lim_{s\to 0}[sX(s)] - x(0)$

we get, $x(\infty) - x(0) = \lim_{s\to 0}[sX(s) - x(0)]$

Hence, $x(\infty) = \lim_{s\to 0}[sX(s)]$

This proves the final value theorem.

The final value theorem may be applied if, and only if, all the poles* of $X(s)$ have a real part that is negative.

The final value theorem is very useful since we can find $x(\infty)$ from $X(s)$. However, one must be careful in using final value theorem since the function $x(t)$ may not have a final value as $t \to \infty$. For example, consider $x(t) = \sin at$ having $X(s) = \dfrac{a}{s^2 + a^2}$. Now we know $\lim_{t\to\infty} \sin at$ does not exit. However, if we uncarefully use the final value theorem in this case, we would obtain:

$$\lim_{s\to 0} sX(s) = \lim_{s\to 0} s\frac{a}{s^2 + a^2} = 0$$

Note that the actual function $x(t)$ does not have a limiting value as $t \to \infty$. The final value theorem has failed because the poles of $X(s)$ lie on the $j\omega$ axis. Therefore, we conclude that for final value theorem to give a valid result, poles of $X(s)$ should not lie to right side of the s-plane or on the $j\omega$ axis.

EXAMPLE 5.12

Find the final value of

$$X(s) = \frac{10}{(s+1)^2 + 10^2}$$

* Consider a function, $X(s) = \dfrac{P(s)}{Q(s)}$. The roots of the denomoniator polymial, $Q(s)$ are called poles ($\times$) and the roots of the numerator polynomial, $P(s)$ are called zeros (O).

SOLUTION

$$\lim_{t\to\infty} x(t) = x(\infty)$$

$$= \lim_{s\to 0}[sX(s)] = \lim_{s\to 0} \frac{s10}{(s+1)^2 + 10^2} = \mathbf{0}$$

We know the Laplace transform pair

$$\mathcal{L}\left[e^{-at}\sin bt\right] = \frac{b}{(s+a)^2 + b^2}$$

Hence,
$$x(t) = \mathcal{L}^{-1}\{X(s)\}$$
$$= \mathcal{L}^{-1}\left\{\frac{10}{(s+1)^2 + 10^2}\right\} = e^{-t}\sin 10t$$

Thus,
$$x(\infty) = 0$$

This verifies the result obtained from final-value theorem.

5.5.11 Time periodicity

Let us consider a function $x(t)$ that is periodic as shown in Fig. 5.15. The function $x(t)$ can be represented as the sum of time-shifted functions as shown in Fig. 5.16.

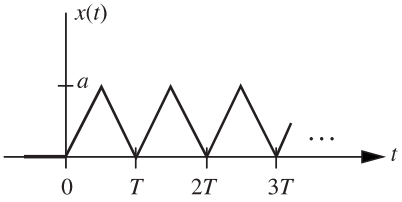

Figure 5.15 A periodic function

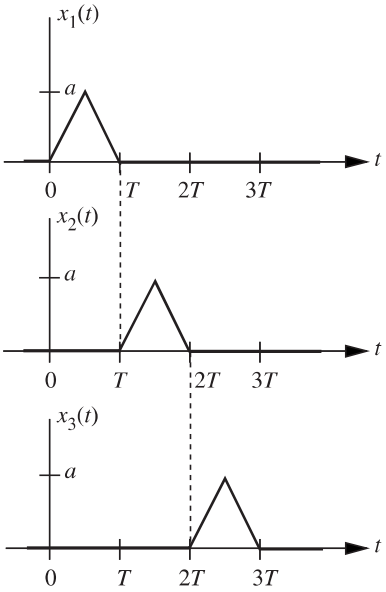

Figure 5.16 Decomposition of periodic function

Hence,
$$x(t) = x_1(t) + x_2(t) + x_3(t) + \cdots$$
$$= x_1(t) + x_1(t-T)u(t-T) + x_1(t-2T)u(t-2T) + \cdots \qquad (5.11)$$

where $x_1(t)$ is the waveform described over the first period of $x(t)$. That is, $x_1(t)$ is the same as the function $x(t)$ gated* over the interval $0 < t < T$.

*gating means the function $x(t)$ is multiplied by 1 over the interval $0 \le t \le T$ and elsewhere by 0.

Taking the Laplace transform on both sides of equation (5.11) with the time-shift property applied, we get

$$X(s) = X_1(s) + X_1(s)e^{-Ts} + X_1(s)e^{-2Ts} + \cdots$$

$$\Rightarrow \quad X(s) = X_1(s)(1 + e^{-Ts} + e^{-2Ts} + \cdots)$$

But $$1 + a + a^2 + \cdots = \frac{1}{1-a}, \qquad |a| < 1$$

Hence, we get
$$X(s) = X_1(s)\left[\frac{1}{1 - e^{-Ts}}\right] \tag{5.12}$$

In equation (5.12), $X_1(s)$ is the Laplace transform of $x(t)$ defined over first period only. Hence, we have shown that the Laplace transform of a periodic function is the Laplace transform evaluated over its first period divided by $1 - e^{-Ts}$.

EXAMPLE 5.13

Find the Laplace transform of the periodic signal $x(t)$ shown in Fig. 5.17.

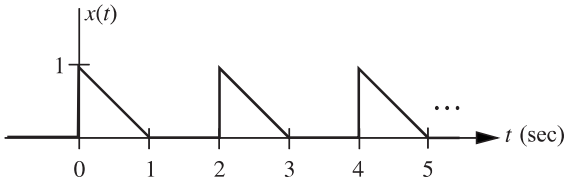

Figure 5.17

SOLUTION

From Fig. 5.17, we find that $T = 2$ Seconds.

The signal $x(t)$ considered over one period is donoted as $x_1(t)$ and shown in Fig. 5.18(a).

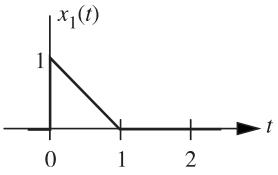

Figure 5.18(a)

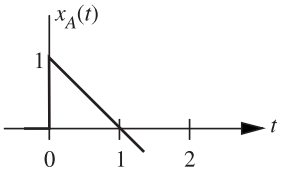

Figure 5.18(b)

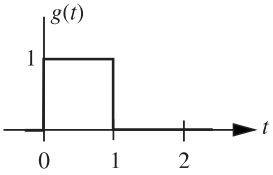

Figure 5.18(c)

The signal $x_1(t)$ may be viewed as the multiplication of $x_A(t)$ and $g(t)$.

$$\begin{aligned}
\text{That is,} \quad x_1(t) &= x_A(t)g(t) \\
&= [-t+1][u(t)-u(t-1)] \\
\Rightarrow \quad x_1(t) &= -tu(t)+tu(t-1)+u(t)-u(t-1) \\
&= -tu(t)+(t-1+1)u(t-1)+u(t)-u(t-1) \\
&= -tu(t)+(t-1)u(t-1)+u(t-1)+u(t)-u(t-1) \\
&= u(t)-tu(t)+(t-1)u(t-1) \\
&= u(t)-r(t)+r(t-1)
\end{aligned}$$

Taking Laplace Transform, we get

$$\begin{aligned}
X_1(s) &= \frac{1}{s}-\frac{1}{s^2}+\frac{1}{s^2}e^{-s} \\
&= \frac{s-1+e^{-s}}{s^2}
\end{aligned}$$

Hence,
$$\boldsymbol{X(s) = \frac{X_1(s)}{1-e^{-sT}} = \frac{(s-1+e^{-s})}{s^2(1-e^{-2s})}}$$

5.6 Inverse Laplace transform

The inverse Laplace transform of $X(s)$ is defined by an integral operation with respect to variable s as follows:

$$x(t) = \frac{1}{2\pi}\int_{\sigma-j\infty}^{\sigma+j\infty} X(s)e^{st}ds \tag{5.13}$$

Since s is complex, the solution requries a knowledge of complex variables. In otherwords, the evaluation of integral in equation (5.13) requires the use of contour integration in the complex plane, which is very difficult. Hence, we will avoid using equation (5.13) to compute inverse Laplace transform.

In many situations, the Laplace transform can be expressed in the form

$$X(s) = \frac{P(s)}{Q(s)} \tag{5.14}$$

where
$$\begin{aligned}
P(s) &= b_m s^m + b_{m-1}s^{m-1} + \cdots + b_1 s + b_0 \\
Q(s) &= a_n s^n + a_{n-1}s^{n-1} + \cdots + a_1 s + a_0, \qquad a_n \neq 0
\end{aligned}$$

The function $X(s)$ as defined by equation (5.14) is said to be rational function of s, since it is a ratio of two polynomials. The denominator $Q(s)$ can be factored into linear factors.

A partial fraction expansion allows a strictly proper rational function $\frac{P(s)}{Q(s)}$ to be expressed as a factor of terms whose numerators are constants and whose denominator corresponds to linear or a combination of linear and repeated factors. This in turn allows us to relate such terms to their corresponding inverse transform.

For performing partial fraction technique on $X(s)$, the function $X(s)$ has to meet the following conditions:

(i) $X(s)$ must be a proper fraction. That is, $m < n$. When $X(s)$ is improper, we can use long division to reduce it to proper fraction.

(ii) $Q(s)$ should be in the factored form.

EXAMPLE 5.14

Find the inverse Laplace transform of

$$X(s) = \frac{2s+4}{s^2+4s+3}$$

SOLUTION

$$X(s) = \frac{2s+4}{s^2+4s+3}$$

$$= \frac{2(s+2)}{(s+1)(s+3)} = \frac{K_1}{s+1} + \frac{K_2}{s+3}$$

where,

$$K_1 = (s+1)X(s)|_{s=-1}$$

$$= \left.\frac{2(s+2)}{(s+3)}\right|_{s=-1} = 1$$

$$K_2 = (s+3)X(s)|_{s=-3}$$

$$= \left.\frac{2(s+2)}{(s+1)}\right|_{s=-3} = 1$$

Hence, $$X(s) = \frac{1}{s+1} + \frac{1}{s+3}$$

We know that: $$\mathcal{L}\{e^{-\alpha t}u(t)\} = \frac{1}{s+\alpha}$$

Therefore, $$\boldsymbol{x(t) = [e^{-t} + e^{-3t}]u(t)}$$

EXAMPLE 5.15

Find the inverse Laplace transform of

$$X(s) = \frac{s^2+2s+5}{(s+3)(s+5)^2}$$

SOLUTION

Let $$X(s) = \frac{K_1}{s+3} + \frac{K_2}{s+5} + \frac{K_3}{(s+5)^2}$$

where
$$K_1 = (s+3)X(s)\big|_{s=-3}$$
$$= \frac{s^2+2s+5}{(s+5)^2}\bigg|_{s=-3} = 2$$
$$K_2 = \frac{1}{1!}\frac{d}{ds}[(s+5)^2X(s)]\bigg|_{s=-5}$$
$$= \frac{d}{ds}\left[\frac{s^2+2s+5}{s+3}\right]\bigg|_{s=-5}$$
$$= \frac{s^2+6s+1}{(s+3)^2}\bigg|_{s=-5} = -1$$
$$K_3 = (s+5\ )^2X(s)\big|_{s=-5}$$
$$= \frac{s^2+2s+5}{(s+3)}\bigg|_{s=-5} = -10$$

Then $$X(s) = \frac{2}{s+3} - \frac{1}{s+5} - \frac{10}{(s+5)^2}$$

Taking inverse Laplace transform, we get

$$x(t) = 2e^{-3t} - e^{-5t} - 10te^{-5t}, \qquad t \geq 0$$

or $$\boldsymbol{x(t) = (2e^{-3t} - e^{-5t} - 10te^{-5t})u(t)}$$

Reinforcement problems

R.P 5.1

Find the Laplace transform of: (a) $\cosh(at)$ (b) $\sinh(at)$

SOLUTION

(a) $\cosh(at) = \frac{1}{2}[e^{at} + e^{-at}]$

We know the Laplace transform pair:

$$\mathcal{L}\{e^{-at}\} = \frac{1}{s+a}$$

and $$\mathcal{L}\{e^{at}\} = \frac{1}{s-a}$$

Applying linearity property, we get,

$$\begin{aligned}\mathcal{L}\{\cosh(at)\} &= \frac{1}{2}\mathcal{L}\{e^{at}\} + \frac{1}{2}\mathcal{L}\{e^{-at}\} \\ &= \frac{1}{2}\left[\frac{1}{s-a} + \frac{1}{s+a}\right] \\ &= \frac{s}{s^2 - a^2}\end{aligned}$$

(b) $\sinh at = \frac{1}{2}[e^{at} - e^{-at}]$

Applying linearity property,

$$\begin{aligned}\mathcal{L}\{\sinh(at)\} &= \frac{1}{2}\left[\frac{1}{s-a} - \frac{1}{s+a}\right] \\ &= \boldsymbol{\frac{a}{s^2 - a^2}}\end{aligned}$$

R.P 5.2

Find the Laplace transform of $f(t) = \cos(\omega t + \theta)$.

SOLUTION

Given

$$\begin{aligned}f(t) &= \cos(\omega t + \theta) \\ &= \cos\theta\cos\omega t - \sin\theta\sin\omega t\end{aligned}$$

Applying linearity property, we get,

$$\begin{aligned}\mathcal{L}\{f(t)\} &= F(s) \\ &= \cos\theta\,\mathcal{L}\{\cos\omega t\} - \sin\theta\,\mathcal{L}\{\sin\omega t\} \\ &= \cos\theta\frac{s}{s^2+\omega^2} - \sin\theta\frac{\omega}{s^2+\omega^2} \\ &= \boldsymbol{\frac{s\cos\theta - \omega\sin\theta}{s^2+\omega^2}}\end{aligned}$$

R.P 5.3

Find the Laplace transform of each of the following functions:

(a) $x(t) = t^2\cos(2t + 30^\circ)u(t)$

(b) $x(t) = 2tu(t) - 4\frac{d}{dt}\delta(t)$

(c) $x(t) = 5u\left(\frac{t}{3}\right)$

(d) $x(t) = 5e^{-\frac{t}{2}}u(t)$

SOLUTION

(a) Let us first find the Laplace transform of $\cos(2t + 30^\circ)u(t)$

$$\begin{aligned}
\mathscr{L}\{\cos(2t + 30^\circ)u(t)\} &= \mathscr{L}\{\cos 30^\circ \cos 2tu(t) - \sin 30^\circ \sin 2tu(t)\} \\
&= \cos 30^\circ \mathscr{L}\{\cos 2tu(t)\} - \sin 30^\circ \mathscr{L}\{\sin 2tu(t)\} \\
&= \cos 30^\circ \left[\frac{s}{s^2+4}\right] - \sin 30^\circ \left[\frac{2}{s^2+4}\right] \\
&= \frac{s\cos 30^\circ - 2\sin 30^\circ}{s^2+4}
\end{aligned}$$

The Laplace transform of $x(t)$ is now found by using differentiation in s domain property.

$$\begin{aligned}
\mathscr{L}\left\{t^2\cos(2t + 30^\circ)\right\} &= \frac{d^2}{ds^2}\left[\mathscr{L}\{\cos(2t + 30^\circ)u(t)\}\right] \\
&= \frac{d^2}{ds^2}\left[\frac{s\cos 30^\circ - 2\sin 30^\circ}{s^2+4}\right] \\
&= \frac{d^2}{ds^2}\left[\frac{\frac{\sqrt{3}}{2}s - 1}{s^2+4}\right] \\
&= \frac{d}{ds}\frac{d}{ds}\left[\frac{\frac{\sqrt{3}}{2}s - 1}{s^2+4}\right] \\
&= \frac{d}{ds}\frac{d}{ds}\left[\left(\frac{\sqrt{3}}{2}s - 1\right)\left(s^2+4\right)^{-1}\right] \\
&= \frac{d}{ds}\left[\left(\frac{\sqrt{3}}{2}\left(s^2+4\right)^{-1}\right) - 2s\left(\frac{\sqrt{3}}{2}s - 1\right)\left(s^2+4\right)^{-2}\right] \\
&= \frac{\frac{\sqrt{3}}{2}(-2s)}{\left(s^2+4\right)^2} - 2\frac{\frac{\sqrt{3}}{2}s - 1}{\left(s^2+4\right)^2} - \frac{2s\frac{\sqrt{3}}{2}}{\left(s^2+4\right)^2} + \frac{8s^2\left(\frac{\sqrt{3}}{2}s - 1\right)}{\left(s^2+4\right)^3} \\
&= \boldsymbol{\frac{8 - 12\sqrt{3}s - 6s^2 + \sqrt{3}s^2}{\left(s^2+4\right)^3}}
\end{aligned}$$

(b) $x(t) = 2tu(t) - 4\dfrac{d}{dt}\delta(t)$

$$\mathscr{L}\{x(t)\} = X(s) = 2\mathscr{L}\{tu(t)\} - 4\mathscr{L}\left\{\frac{d}{dt}\delta(t)\right\}$$

We know that whenever a function $f(t)$ is discontinuous at the origin, we have $\mathcal{L}\left\{\frac{d}{dt}f(t)\right\} = sF(s) - f(0^-)$. Applying this relation to the second term on the right side of the above equation, we get

$$\begin{aligned} X(s) &= 2\frac{1}{s^2} - 4[s \times 1 - \delta(0^-)] \\ &= \frac{2}{s^2} - 4[s - 0] \\ &= \boldsymbol{\frac{2}{s^2} - 4s} \end{aligned}$$

(c) $x(t) = 5u\left(\frac{t}{3}\right)$

Using scaling property,

$$\mathcal{L}\{f(at)\} = \frac{1}{a}F\left(\frac{s}{a}\right)$$

we get,

$$\begin{aligned} \mathcal{L}\{x(t)\} = X(s) &= 5 \times \frac{1}{1/3}\mathcal{L}\{u(t)\}_{s\to\left(\frac{s}{\frac{1}{3}}\right)} \\ &= 5 \times \frac{1}{1/3} \times \left[\frac{1}{s}\right]_{s\to\left(\frac{s}{\frac{1}{3}}\right)} \\ &= \boldsymbol{\frac{5}{s}} \end{aligned}$$

(d) $x(t) = 5e^{-\frac{t}{2}}u(t)$

We know the Laplace transform pair:

$$\mathcal{L}\{e^{-at}u(t)\} = \frac{1}{s+a}$$

Hence,

$$\begin{aligned} \mathcal{L}\{x(t)\} = X(s) &= 5\mathcal{L}\left\{e^{-\frac{1}{2}t}u(t)\right\} \\ &= 5\frac{1}{s+\frac{1}{2}} = \boldsymbol{\frac{10}{2s+1}} \end{aligned}$$

R.P 5.4

Find the Laplace transform of the following functions:

(a) $x(t) = t\cos at$

(b) $x(t) = \frac{1}{2a^2}\sin at \sinh(at)$

(c) $x(t) = \frac{\sin^2 \omega t}{t}$

SOLUTION

(a) $x(t) = t\cos at$

We know that $\mathcal{L}\{tf(t)\} = -\dfrac{d}{ds}F(s)$

Let $f(t) = \cos at$

$\Rightarrow$ $F(s) = \dfrac{s}{s^2 + a^2}$

Hence
$$\mathcal{L}\{t\cos at\} = \mathcal{L}\{tf(t)\} = -\frac{d}{ds}\left[\frac{s}{s^2 + a^2}\right]$$
$$= \boldsymbol{\frac{s^2 - a^2}{(s^2 + a^2)^2}}$$

(b)
$$x(t) = \frac{1}{2a^2}\sin at \sinh at$$
$$= \frac{1}{2a^2}\left[\frac{1}{2}e^{at}\sin at - \frac{1}{2}e^{-at}\sin at\right]$$
$$= \frac{1}{4a^2}\left[e^{at}\sin at - e^{-at}\sin at\right]$$

We know the shifting in s domain property:

$$\mathcal{L}\{e^{s_0 t}f(t)\} = F(s)|_{s\to(s-s_0)}$$

Applying this property along with linearity property, we get

$$\mathcal{L}\{x(t)\} = X(s)$$
$$= \frac{1}{4a^2}\left[\mathcal{L}\{e^{at}\sin at\} - \mathcal{L}\{e^{-at}\sin at\}\right]$$
$$= \frac{1}{4a^2}\left[\frac{a}{s^2 + a^2}\bigg|_{s\to s-a} - \frac{a}{s^2 + a^2}\bigg|_{s\to s+a}\right]$$
$$= \frac{1}{4a^2}\left[\frac{a}{(s-a)^2 + a^2} - \frac{a}{(s+a)^2 + a^2}\right]$$
$$= \boldsymbol{\frac{s}{[(s-a)^2 + a^2]\,[(s+a)^2 + a^2]}}$$

(c) $x(t) = \dfrac{1}{t}\sin^2 \omega t$

We know that $\mathcal{L}\{f(t)\} = F(s) = \displaystyle\int_0^\infty f(t)e^{-st}dt$

Hence,
$$\int_s^\infty F(s)ds = \int_0^\infty f(t)\int_s^\infty e^{-st}dsdt$$
$$= \int_0^\infty f(t)\left[\frac{e^{-st}}{-t}\right]_s^\infty dt$$
$$= \int_0^\infty \frac{f(t)}{t}e^{-st}dt$$
$$= \mathscr{L}\left[\frac{f(t)}{t}\right]$$

In the present case,
$$f(t) = \sin^2\omega t$$
$$= \left[\frac{1}{j2}e^{j\omega t} - \frac{1}{j2}e^{-j\omega t}\right]^2$$
$$= \frac{e^{j2\omega t} - 2 + e^{-j2\omega t}}{-4}$$

Hence,
$$F(s) = -\frac{1}{4}\left[\frac{1}{s-j2\omega}\right] + \frac{1}{2}\left(\frac{1}{s}\right) - \frac{1}{4}\left[\frac{1}{s+j2\omega}\right]$$

Hence,

$$X(s) = \mathscr{L}\left\{\frac{1}{t}\sin^2\omega t\right\}$$
$$= \mathscr{L}\left\{\frac{1}{t}f(t)\right\}$$
$$= \int_s^\infty F(s)ds = \lim_{x\to\infty}\int_s^x f(x)dx$$
$$= \lim_{x\to\infty}\left[\frac{\ln(x-j2\omega) - \ln(s-j2\omega) - 2\ln x + 2\ln s + \ln(x+j2\omega) - \ln(s+j2\omega)}{-4}\right]$$
$$= -\frac{1}{4}\ln\left(\frac{x^2+4\omega^2}{x^2}\right)_{x\to\infty} + \frac{1}{4}\ln\left[\frac{s^2+4\omega^2}{s^2}\right]$$
$$= \boldsymbol{\frac{1}{4}\ln\left[\frac{s^2+4\omega^2}{s^2}\right]}$$

R.P 5.5

Consider the pulse shown in Fig. R.P. 5.5, where $f(t) = e^{2t}$ for $0 < t < T$. Find $F(s)$ for the pulse.

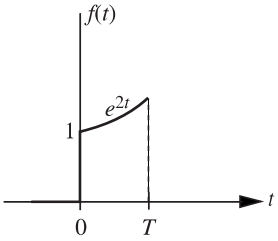

Figure R.P. 5.5

SOLUTION

The discrete pulse $f(t)$ could be imagined as the product of signal $x(t)$ and $g(t)$ as shown in Fig. R.P. 5.5(a) and (b) respectively.

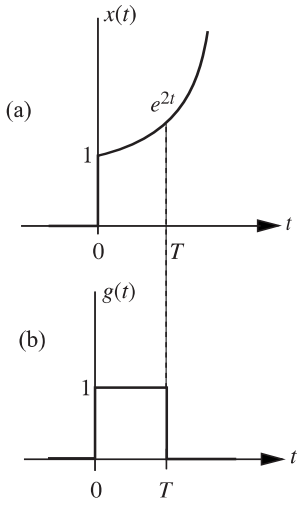

Figure R.P. 5.5(a)

That is,
$$\begin{aligned} f(t) &= x(t)g(t) \\ &= e^{2t}[u(t) - u(t-T)] \\ &= e^{2t}u(t) - e^{2t}u(t-T) \\ &= e^{2t}u(t) - e^{2(t-T+T)}u(t-T) \\ &= e^{2t}u(t) - e^{2T}e^{2(t-T)}u(t-T) \end{aligned}$$

Hence,
$$\begin{aligned} \mathcal{L}\{f(t)\} = F(s) &= \frac{1}{s-2} - \frac{e^{2T}e^{-Ts}}{s-2} \\ &= \frac{1}{s-2} - \frac{e^{-T(s-2)}}{s-2} \\ &= \boldsymbol{\frac{1 - e^{-T(s-2)}}{(s-2)}} \end{aligned}$$

Alternate method:

$$\begin{aligned} \mathcal{L}\{f(t)\} = F(s) &\triangleq \int_0^\infty f(t)e^{-st}dt \\ &= \int_0^T e^{2t}e^{-st}dt \\ &= \boldsymbol{\frac{1 - e^{-T(s-2)}}{(s-2)}} \end{aligned}$$

R.P 5.6

Find the Laplace transform of $f(t)$ shown in Fig. R.P. 5.6.

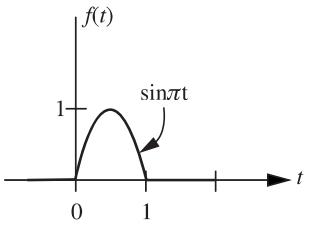

Figure R.P. 5.6

SOLUTION

$f(t)$ is a discrete pulse and can be expressed mathematically as:

$$\begin{aligned} f(t) &= x(t)g(t) \\ &= \sin \pi t[u(t) - u(t-1)] \\ &= \sin \pi t u(t) - \sin \pi t u(t-1) \\ &= \sin \pi t u(t) - \sin[\pi(t-1+1)]u(t-1) \\ \Rightarrow \quad f(t) &= \sin \pi t u(t) - \sin[\pi(t-1)] \cos \pi(t-1) \\ &\quad - \cos[\pi(t-1)] \sin \pi u(t-1) \\ &= \sin \pi t u(t) + \sin[\pi(t-1)]u(t-1) \end{aligned}$$

Hence, $$F(s) = \mathcal{L}\{f(t)\} = \frac{\pi}{s^2+\pi^2} + \frac{e^{-1s}\pi}{s^2+\pi^2}$$

$$= \frac{\boldsymbol{\pi}}{\boldsymbol{s^2+\pi^2}}[\mathbf{1} + \boldsymbol{e^{-s}}]$$

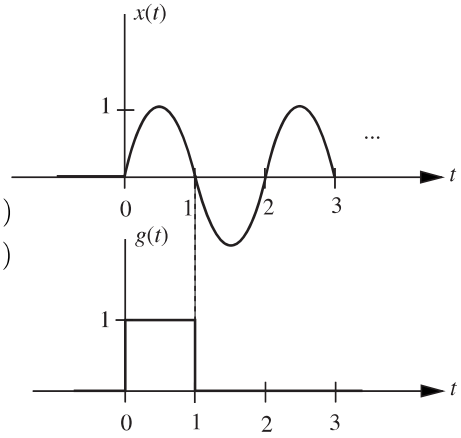

Figure R.P. 5.6(a)

R.P 5.7

Determine the Laplace transform of $f(t)$ shown in Fig. R.P. 5.7.

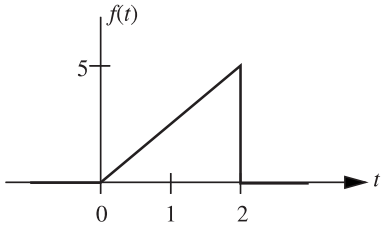

Figure R.P. 5.7

SOLUTION

We can write,

$$
\begin{aligned}
f(t) &= x(t)g(t) \\
&= \left[\frac{5}{2}t\right][u(t) - u(t-2)] \\
&= \frac{5}{2}tu(t) - \frac{5}{2}tu(t-2) \\
&= \frac{5}{2}tu(t) - \frac{5}{2}(t-2+2)u(t-2) \\
&= \frac{5}{2}tu(t) - \frac{5}{2}(t-2)u(t-2) - 5u(t-2)
\end{aligned}
$$

Hence, $\mathcal{L}\{f(t)\} = F(s) = \frac{5}{2}\left(\frac{1}{s^2}\right) - \frac{5}{2}\left(\frac{1}{s^2}\right)e^{-2s} - \frac{5}{s}e^{-2s}$

$$= \boldsymbol{\frac{5}{2s^2}[1 - e^{-2s} - 2se^{-2s}]}$$

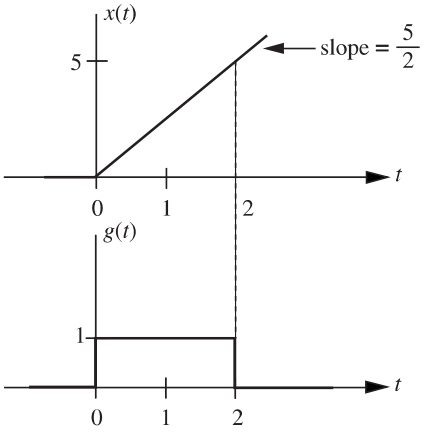

Figure R.P. 5.7(a)

R.P 5.8

Find the Laplace transform of $f(t)$ shown in Fig. R.P. 5.8.

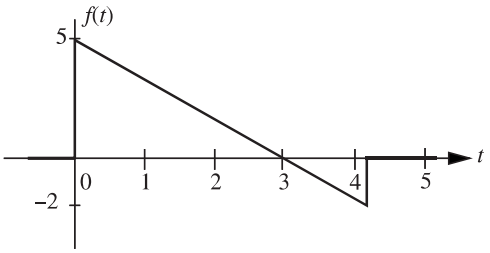

Figure R.P. 5.8

SOLUTION

The equation of a straight line is $y = mx + c$, where m = slope of the line and c = intercept on y-axis.

Hence, $f(t) = \frac{-5}{3}t + 5$

When $f(t) = -2$, let us find t.

That is, $-2 = \dfrac{-5}{3}t + 5$

$\Rightarrow \qquad t = 4.2$ Seconds

Mathematically,

$$f(t) = x(t)g(t)$$

$$= \left[\frac{-5}{3}t + 5\right][u(t) - u(t-4.2)]$$

$$= \frac{-5}{3}tu(t) + \frac{5}{3}tu(t-4.2) + 5u(t) - 5u(t-4.2)$$

$$= \frac{-5}{3}tu(t) + \frac{5}{3}(t - 4.2 + 4.2)u(t-4.2) + 5u(t) - 5u(t-4.2)$$

$$= \frac{-5}{3}tu(t) + \frac{5}{3}(t-4.2)u(t-4.2) + 7u(t-4.2) + 5u(t) - 5u(t-4.2)$$

$$= \frac{-5}{3}tu(t) + \frac{5}{3}(t-4.2)u(t-4.2) + 2u(t-4.2) + 5u(t)$$

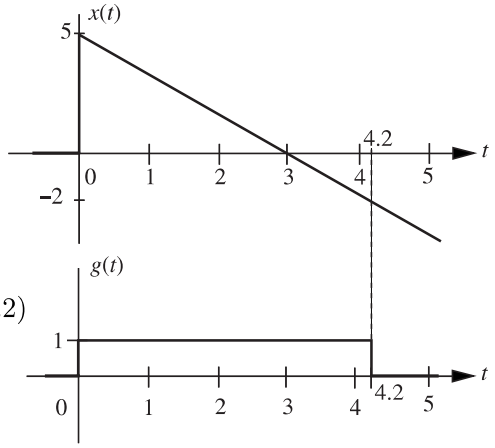

Figure R.P. 5.8(a)

Hence,

$$F(s) = \mathcal{L}\{f(t)\}$$

$$= \frac{-5}{3s^2} + \frac{5}{3s^2}e^{-4.2s} + \frac{2}{s}e^{-4.2s} + \frac{5}{s}$$

$$= \boldsymbol{\frac{-5 + 5e^{-4.2s} + 6se^{-4.2s} + 15s}{3s^2}}$$

R.P 5.9

If $f(0^-) = -3$ and $15u(t) - 4\delta(t) = 8f(t) + 6f'(t)$, find $f(t)$ (hint: by taking the Laplace transform of the differential equation, solving for $F(s)$ and by inverting, find $f(t)$).

SOLUTION

Given, $\qquad 15u(t) - 4\delta(t) = 8f(t) + 6f'(t)$

Taking Laplace transform on both the sides, we get

$$\frac{15}{s} - 4 = 8F(s) + 6[sF(s) - f(0^-)]$$

$$\Rightarrow \qquad \frac{15}{s} - 4 = 8F(s) + 6sF(s) + 18$$

Therefore, $\qquad F(s)(6s+8) = -18 + \dfrac{15 - 4s}{s}$

$$\Rightarrow \qquad F(s) = \frac{-18}{(6s+8)} + \frac{15 - 4s}{s(6s+8)}$$

$$= \frac{-22s + 15}{6s\left(s + \dfrac{4}{3}\right)} = \frac{K_1}{s} + \frac{K_2}{s + \dfrac{4}{3}}$$

The constants K_1 and K_2 are found using the theory of partial fractions.

$$K_1 = \left.\frac{-22s + 15}{6\left(s + \frac{4}{3}\right)}\right|_{s=0} = 1.875$$

$$K_2 = \left.\frac{-22s + 15}{6s}\right|_{s=\frac{-4}{3}} = -5.542$$

Hence, $$F(s) = \frac{1.875}{s} - \frac{5.542}{s + \frac{4}{3}}$$

Taking the inverse, we get $$\boldsymbol{f(t) = \left[1.875 - 5.542e^{\frac{-4}{3}t}\right] u(t)}$$

R.P 5.10

Find the inverse Laplace transform of the following functions:

(a) $F(s) = \dfrac{s+1}{s^2 + 4s + 13}$

(b) $F(s) = \dfrac{3e^{-s}}{s^2 + 2s + 17}$

SOLUTION

(a) $$F(s) = \frac{s+1}{(s+2)^2 + 9}$$

$$= \frac{(s+2) - 1}{(s+2)^2 + 9}$$

$$= \frac{s+2}{(s+2)^2 + 3^2} - \frac{1}{(s+2)^2 + 3^2}$$

$$= \frac{s+2}{(s+2)^2 + 3^2} - \frac{1}{3}\frac{3}{(s+2)^2 + 3^2}$$

The determination of the Laplace inverse makes use of the following two Laplace transform pairs:

$$\mathcal{L}\{e^{-bt}\sin at\} = \frac{a}{(s+b)^2 + a^2}$$

$$\mathcal{L}\{e^{-bt}\cos at\} = \frac{s+b}{(s+b)^2 + a^2}$$

Hence, $$f(t) = \mathcal{L}^{-1}\{F(s)\}$$

$$= \boldsymbol{e^{-2t}\cos 3t - \frac{1}{3}e^{-2t}\sin 3t}$$

(b) $$F(s) = \frac{3e^{-s}}{s^2 + 2s + 17}$$

Let $F(s) = e^{-s}X(s)$

where $X(s) = \dfrac{3}{s^2+2s+17} = \dfrac{3}{(s+1)^2+4^2}$

$= \dfrac{3}{4}\left[\dfrac{4}{(s+1)^2+4^2}\right]$

$\Rightarrow \quad x(t) = \dfrac{3}{4}e^{-t}\sin 4t$

Since $F(s) = e^{-s}X(s)$

we get, $f(t) = x(t-1)$

Therefore, $f(t) = \dfrac{3}{4}e^{-(t-1)}\sin[4(t-1)], \qquad t > 1$

$$\boldsymbol{f(t) = \frac{3}{4}e^{-(t-1)}\sin[4(t-1)]u(t-1)}$$

Laplace transform method for solving a set of differential equations:

1. Identify the circuit variables such as inductor currents and capacitor voltages.
2. Obtain the differential equations describing the circuit and keep a watch on the initial conditions of the circuit variables.
3. Obtain the Laplace transform of the various differential equations.
4. Using Cramer's rule or a similar technique, solve for one or more of the unknown variables, obtaining the solution in s domain.
5. Find the inverse transform of the unknown variables and thus obtain the solution in the time domain.

R.P 5.11

Referring to the RL circuit of Fig. R.P. 5.11, (a) write a differential equation for the inductor current $i_L(t)$. (b) Find $I_L(s)$, the Laplace transform of $i_L(t)$. (c) Solve for $i_L(t)$.

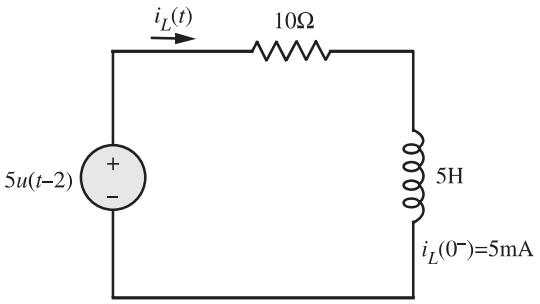

Figure R.P. 5.11

SOLUTION

(a) *Applying KVL clockwise*, we get

$$10i_L(t) + 5\frac{di_L}{dt} - 5u(t-2) = 0$$

(b) Taking Laplace transform of the above equation, we get

$$10I_L(s) + 5[sI_L(s) - i_L(0^-)] = \frac{5}{s}e^{-2s}$$

$$\Rightarrow \qquad I_L(s) = \frac{\frac{5}{s}e^{-2s} + 5i_L(0^-)}{5s+10}$$

$$= \frac{e^{-2s} + 5 \times 10^{-3}s}{s(s+2)}$$

$$= e^{-2s}\left[\frac{K_1}{s} + \frac{K_2}{s+2}\right] + \frac{5 \times 10^{-3}s}{s(s+2)}$$

where $$K_1 = \left.\frac{1}{s+2}\right|_{s=0} = \frac{1}{2}$$

$$K_2 = \left.\frac{1}{s}\right|_{s=-2} = -\frac{1}{2}$$

Hence, $$\boldsymbol{I_L(s) = \frac{1}{2}e^{-2s}\left[\frac{1}{s} - \frac{1}{s+2}\right] + \frac{5 \times 10^{-3}}{(s+2)}}$$

(c) Taking Inverse Laplace transform, we get

$$i_L(t) = \frac{1}{2}\left[u(t) - e^{-2t}u(t)\right]_{t \to t-2} + 5 \times 10^{-3}e^{-2t}u(t)$$

$$= \boldsymbol{\frac{1}{2}\left[u(t-2) - e^{-2t}u(t-2)\right] + 5 \times 10^{-3}e^{-2t}u(t)}$$

R.P 5.12

Obtain a single integrodifferential equation in terms of i_C for the circuit of Fig. R.P. 5.12. Take the Laplace transform, solve for $I_C(s)$, and then find $i_C(t)$ by making use of inverse transform.

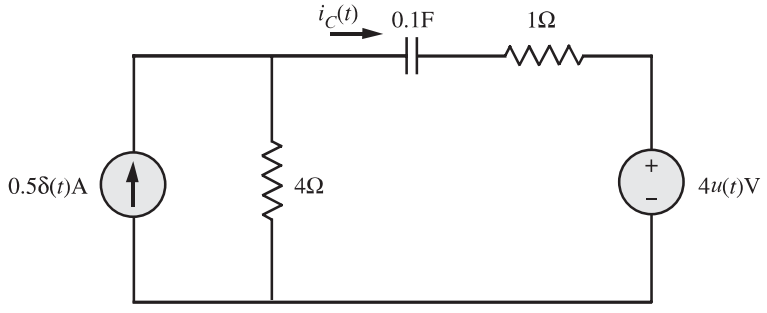

Figure R.P. 5.12

SOLUTION

Applying KVL clockwise to the right-mesh, we get

$$4u(t) + i_C + 10\int_0^{\infty} i_C dt + 4[i_C - 0.5\delta(t)] = 0$$

Taking Laplace transform, we get

$$4\frac{1}{s} + I_C(s) + \frac{10I_C(s)}{s} + 4I_C(s) - 2 = 0$$

$$\Rightarrow \quad I_C(s) = \frac{2s-4}{5s+10}$$

$$= \mathbf{0.4 - \frac{1.6}{s+2}}$$

Taking inverse Laplace transform, we get

$$\boldsymbol{i_C(t) = 0.4\delta(t) - 1.6e^{-2t}u(t)\ \text{Amps.}}$$

R.P 5.13

Refer the circuit shown in Fig. R.P. 5.13. Find $i(0)$ and $i(\infty)$ using initial and final value theorems.

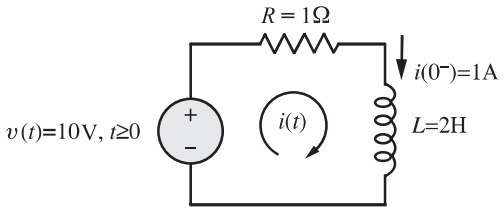

Figure R.P. 5.13

SOLUTION

Applying KVL we get

$$i + 2\frac{di}{dt} = 10$$

Taking Laplace transform, on both the sides, we get

$$I(s) + 2[sI(s) - i(0^-)] = \frac{10}{s}$$

$$\Rightarrow \quad I(s) + 2[sI(s) - 1] = \frac{10}{s}$$

$$\Rightarrow \quad I(s)[1 + 2s] = \frac{10}{s} + 2$$

$$\Rightarrow \qquad I(s) = \frac{10}{s(1+2s)} + \frac{2}{1+2s}$$
$$= \frac{10+2s}{s(1+2s)}$$
$$= \frac{5+s}{s\left(s+\frac{1}{2}\right)}$$

According to initial value theorem,

$$i(0) = \lim_{s\to\infty} sI(s)$$
$$= \lim_{s\to\infty} s\frac{(s+5)}{s\left(s+\frac{1}{2}\right)}$$
$$= \lim_{s\to\infty} \frac{1+\frac{5}{s}}{1+\frac{1}{2s}} = \mathbf{1}$$

We know from fundamentals for an inductor, $i(0^+) = i(0^-) = i(0)$. Hence, $i(0)$ found using initial value theorem verifies the initial value of $i(t)$ given in the problem.

From final value theorem,

$$i(\infty) = \lim_{s\to 0} sI(s)$$
$$= \lim_{s\to 0} \frac{s(s+5)}{s\left(s+\frac{1}{2}\right)} = \frac{5}{1/2} = \mathbf{10\ A}$$

R.P 5.14

Find $i(t)$ and $v_C(t)$ for the circuit shown in Fig. R.P. 5.14 when $v_C(0)$ = 10 V and $i(0)$ = 0 A. The input source is $v_i = 15u(t)$ V. Choose R so that the roots of the characteristic equation are real.

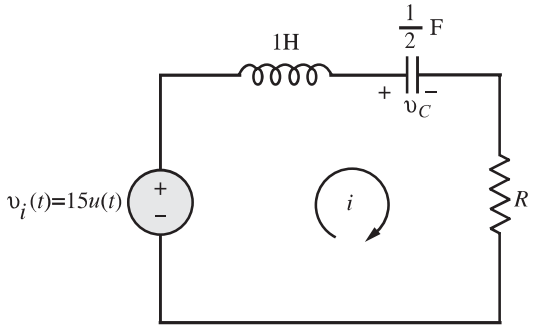

Figure R.P. 5.14

SOLUTION

Applying KVL clockwise, we get

$$L\frac{di}{dt} + v_C + R_i = v_i(t) \tag{5.15}$$

The differential equation describing the variable v_C is

$$C\frac{dv_C}{dt} = i \tag{5.16}$$

The Laplace transform of equation (5.15) is

$$L[sI(s) - i(0) + V_C(s) + RI(s) = V_i(s)] \tag{5.17}$$

The Laplace transform of equation (5.16) is

$$C[sV_C(s) - v_C(0) = I(s)] \tag{5.18}$$

Noting that $i(0) = 0$, substituting for C and L and rearranging equation (5.17) and (5.18), we get,

$$[R + s]I(s) + V_C(s) = V_i(s) = \frac{15}{s} \tag{5.19}$$

$$-I(s) + \frac{1}{2}sV_C(s) = 5 \tag{5.20}$$

Putting equations (5.19) and (5.20) in matrix form, we get

$$\begin{bmatrix} R+s & 1 \\ -1 & \frac{1}{2}s \end{bmatrix} \begin{bmatrix} I(s) \\ V_C(s) \end{bmatrix} = \begin{bmatrix} \frac{15}{s} \\ 5 \end{bmatrix} \tag{5.21}$$

Solving for $I(s)$ using Cramer's rule, we get

$$I(s) = \frac{5}{s^2 + Rs + 2}$$

The inverse Laplace transform of $I(s)$ will depend on the value of R. The equation $s^2 + Rs + 2 = 0$ is defined as the characteristic equation. For the roots of this equation to be real, it is essential that $b^2 - 4ac \geq 0$*.

This means that, $R^2 - 4 \times 1 \times 2 \geq 0$

$\Rightarrow$ $R \geq 2\sqrt{2}$

*The condition $b^2 - 4ac \geq 0$ is with respect to algebraic equaion $ax^2 + bx + c = 0$.

Let us choose the value of R as 3

Then $$I(s) = \frac{5}{s^2+3s+2} = \frac{5}{(s+1)(s+2)}$$

$$\Rightarrow \quad I(s) = \frac{K_1}{s+1} + \frac{K_2}{s+2}$$

where $$K_1 = \left.\frac{5}{s+2}\right|_{s=-1} = 5$$

$$K_2 = \left.\frac{5}{s+1}\right|_{s=-2} = -5$$

Hence, $$I(s) = \frac{5}{s+1} - \frac{5}{s+2}$$

Taking inverse Laplace transform, we get

$$\boldsymbol{i(t) = 5e^{-t}u(t) - 5e^{-2t}u(t)}$$

Please note that $t = 0$ gives $i(0) = 0$ and $t = \infty$ gives $i(\infty) = 0$.
Solving the matrix equation (5.21) for $V_C(s)$, using Cramer's rule, we get

$$V_C(s) = \frac{10s^2 + 10Rs + 30}{s(s^2 + Rs + 2)}$$

Substituting the value of R, we get

$$V_C(s) = \frac{10(s^2 + 3s + 3)}{s(s+1)(s+2)}$$

Using partial fraction expansion, we can write,

$$V_C(s) = \frac{K_1}{s} + \frac{K_2}{s+1} + \frac{K_3}{s+2}$$

where, $K_1 = 15$, $K_2 = -10$, $K_3 = 5$

Hence, $$V_C(s) = \frac{15}{s} - \frac{-10}{s+1} + \frac{5}{s+2}$$

Taking inverse Laplace transform, we get

$$\boldsymbol{v_C(t) = 15u(t) - 10e^{-t}u(t) + 5e^{-2t}u(t)}$$

Verification:
Putting $t = 0$, we get

$$v_C(0) = 15 - 10 + 5 = 10 \text{ V}$$
$$v_C(\infty) = 15 - 0 + 0 = 15 \text{ V}$$

This checks the validity of results obtained.

R.P 5.15

For the circuit shown in Fig. R.P. 5.15, the steady state is reached with the 100 V source. At $t = 0$, switch K is opened. What is the current through the inductor at $t = \frac{1}{2}$ seconds ?

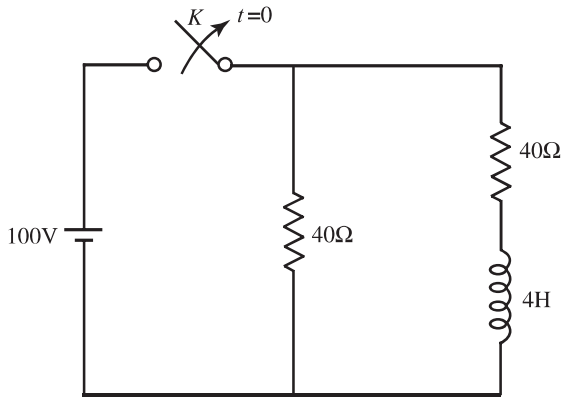

Figure R.P. 5.15

SOLUTION

At $t = 0^-$, the circuit is as shown in Fig. 5.15(a).

$$i_2(0^+) = i_2(0^-) = 2.5 \text{ A}$$

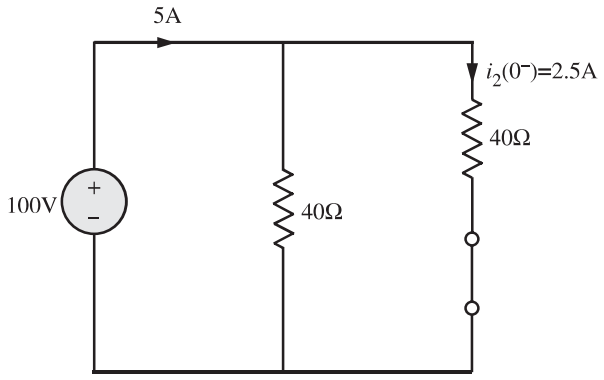

Figure R.P. 5.15(a)

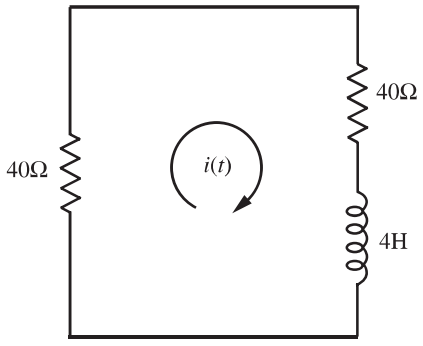

Figure R.P. 5.15(b)

For $t \geq 0^+$, the circuit diagram is as shown in Fig. 5.15(b). *Applying KVL clockwise* to the circuit, we get

$$80i(t) + 4\frac{di}{dt} = 0$$

Taking Laplace transform, we get

$$\begin{aligned} & 80I(s) + 4[sI(s) - i(0^-)] = 0 \\ \Rightarrow \quad & 80I(s) + 4sI(s) = 4 \times 2.5 \\ \Rightarrow \quad & [20 + s]I(s) = 2.5 \\ & I(s) = \frac{2.5}{s + 20} \end{aligned}$$

Taking inverse Laplace transform, we get,

$$i(t) = 2.5e^{-20t}$$

At $t = 0.5$ sec, we get

$$\boldsymbol{i(0.5) = 2.5e^{-10} = 1.135 \times 10^{-4}\ \mathrm{A}}$$

R.P 5.16

Refer the circuit shown in Fig. R.P. 5.16. Find:

(a) $v_o(t)$ for $t \geq 0$

(b) $i_o(t)$ for $t \geq 0$

(c) Does your solution for $i_o(t)$ make sense when $t = 0$? Explain.

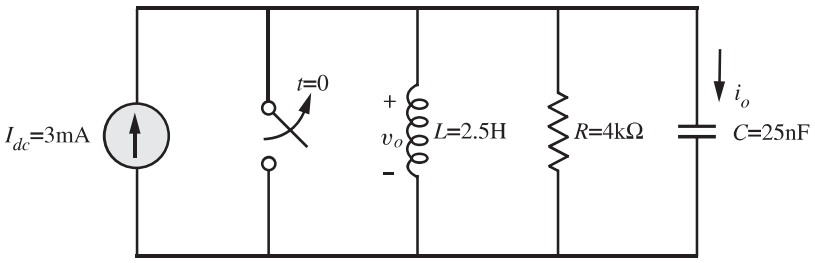

Figure R.P. 5.16

SOLUTION

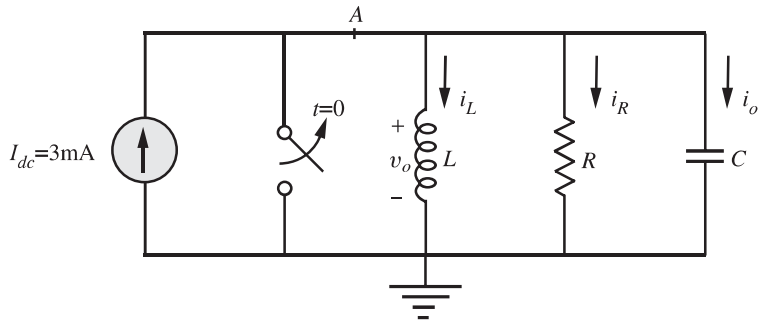

Figure R.P. 5.16(a)

KCL at node A: (for $t \geq 0$)

$$I_{dc} = \frac{1}{L}\int_0^t v_o dt + \frac{v_o}{R} + C\frac{dv_o}{dt}$$

Taking Laplace transform,

$$\frac{I_{dc}}{s} = \frac{V_o(s)}{sL} + \frac{V_o(s)}{R} + sCV_o(s)$$

Hence, $$V_o(s) = \frac{\frac{I_{dc}}{C}}{s^2 + \left(\frac{1}{RC}\right)s + \frac{1}{LC}}$$

Substituting the values of I_{dc}, R, L, and C, we find that

$$V_o(s) = \frac{120,000}{s^2 + 10,000s + 16 \times 10^6} = \frac{120,000}{(s+2000)(s+8000)}$$

Using partial fractions, we get

$$V_o(s) = \frac{K_1}{s+2000} + \frac{K_2}{s+8000}$$

where $K_1 = 20$, and $K_2 = -20$

Hence, $$V_o(s) = \frac{20}{s+2000} - \frac{20}{s+8000}$$

Taking inverse Laplace transform, we get

$$\boldsymbol{v_o(t) = 20e^{-2000t}u(t) - 20e^{-8000t}u(t)}$$

(b) $i_o(t) = C\dfrac{dv_o}{dt}$

Hence $$I_o(s) = C\left[sV_o(s) - v_o(0)\right]$$

For $t \leq 0^-$, since the switch was in closed state, the circuit was not activated by the source. This means that $v_o(0) = v_o(0^-) = v_o(0^+) = 0$ and $i_L(0^+) = i_L(0^-) = 0$.

Then,
$$\begin{aligned} I_o(s) &= CsV_o(s) \\ &= \frac{25 \times 10^{-9} \times s \times 120,000}{s^2 + 10,000s + 16 \times 10^6} \\ &= \frac{3 \times 10^{-3}s}{(s+2000)(s+8000)} \\ &= \frac{K_1}{s+2000} + \frac{K_2}{s+8000} \end{aligned}$$

We find that, $K_1 = -10^{-3}$, and $K_2 = 4 \times 10^{-3}$

Therefore, $$I_o(s) = \frac{-10^{-3}}{s + 2000} + \frac{4 \times 10^{-3}}{s + 8000}$$

Taking inverse Laplace transform, we get

$$\boldsymbol{i_o(t) = 4e^{-8000t}u(t) - e^{-2000t}u(t)\text{mA}}$$

(c) $i_o(0^+) = 4 - 1 = 3\text{mA}$

Yes. The initial inductor current is zero by hypothesis $\left(i_L(0^+) = I_L(0^-) = 0\right)$. Also, the initial resistor current is zero because $v_o(0^+) = v_o(0^-) = 0$. Thus at $t = 0^+$, the source current appears in the capacitor.

R.P 5.17

Refer the circuit shown in Fig. R.P. 5.17. The circuit parameters are $R = 10\text{k}\Omega$, $L = 800$ mH and $C = 100\text{nF}$, if $V_{dc} = 70\text{V}$, find:

(a) $v_o(t)$ for $t \geq 0$

(b) $i_o(t)$ for $t \geq 0$

(c) Use initial and final value theorems to check the inital and final values of current and voltage.

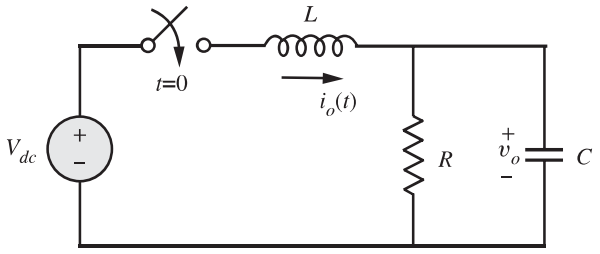

Figure R.P. 5.17

SOLUTION

At $t = 0^-$, switch is open and at $t = 0^+$, the switch is closed. Since at $t = 0^-$, the circuit is not energized by dc source, $i_o(0^-) = 0$ and $v_o(0^-) = 0$. Then by the hypothesis, that the current in an inductor and voltage across a capacitor cannot change instantaneously,

$$i_o(0^+) = i_o(0^-) = 0 \text{ and } \quad v_o(0^+) = v_o(0^-) = 0$$

The *KCL equation* when the switch is closed (for $t \geq 0^+$) is given by

$$C\frac{dv_o}{dt} + \frac{v_o}{R} + \frac{1}{L}\int_0^t (v_o - V_{dc})d\tau = 0$$

$$\Rightarrow \quad C\frac{dv_o}{dt} + \frac{v_o}{R} + \frac{1}{L}\int_0^t v_o d\tau = \frac{1}{L}\int_0^t V_{dc} d\tau$$

$$\Rightarrow \quad C\frac{dv_o}{dt} + \frac{v_o}{R} + \frac{1}{L}\int_0^t v_o d\tau = \frac{1}{L}V_{dc}t$$

Laplace transform of the above equation gives

$$C\left[sV_o(s) - v_o(0)\right] + \frac{V_o(s)}{R} + \frac{1}{L}\frac{V_o(s)}{s} = \frac{1}{L}\frac{V_{dc}}{s^2}$$

Since $v_o(0)$ is same as $v_o(0^-)$, we get

$$CsV_o(s) + \frac{V_o(s)}{R} + \frac{1}{L}\frac{V_o(s)}{s} = \frac{V_{dc}}{Ls^2}$$

$$\Rightarrow \quad V_o(s) = \frac{\dfrac{V_{dc}}{LC}}{s\left(s^2 + \dfrac{1}{RC}s + \dfrac{1}{LC}\right)}$$

Substituting the values of V_{dc}, R, L, and C, we get

$$V_o(s) = \frac{875 \times 10^6}{s\left[s^2 + 1000s + 1250 \times 10^4\right]}$$

$$= \frac{875 \times 10^6}{s(s - s_1)(s - s_2)}$$

where $s_1, s_2 = -500 \pm \sqrt{25 \times 10^4 - 1250 \times 10^4}$

$$= -500 \pm j3500$$

Hence, $$V_o(s) = \frac{875 \times 10^6}{s(s + 500) - j3500)(s + 500 + j3500)}$$

Using partial fractions, we get

$$V_o(s) = \frac{K_1}{s} + \frac{K_2}{s + 500 - j3500} + \frac{K_2^*}{s + 500 + j3500}$$

We find that $$K_1 = \frac{875 \times 10^6}{125 \times 10^5} = 70$$

$$K_2 = \frac{875 \times 10^6}{(-500 + j3500)(j7000)}$$

$$= 5\sqrt{50}\,\underline{/171.87^\circ}$$

$$V_o(s) = \frac{70}{s} + \frac{5\sqrt{50}\,\underline{/171.87^\circ}}{s+500-j3500} + \frac{5\sqrt{50}\,\underline{/-171.87^\circ}}{s+500+j3500}$$

Taking inverse Laplace transform, we get

$$v_o(t) = \left[70 + 5\sqrt{50}\,\underline{/171.87^\circ}\; e^{-(500-j3500)t} + 5\sqrt{50}\,\underline{/-171.87^\circ}\; e^{-(500+j3500)t}\right] u(t)$$

The inverse of $V_o(s)$ can be expressed in a better form by following the technique described below:

Let us consider a transformed function

$$F(s) = \frac{C+jd}{s+a-j\omega} + \frac{C-jd}{s+a+j\omega}$$

$$= \frac{m\,\underline{/\theta}}{s+a-j\omega} + \frac{m\,\underline{/-\theta}}{s+a+j\omega}$$

where $m = \sqrt{c^2+d^2}$ and $\theta = \tan^{-1}\left[\frac{d}{c}\right]$

The inverse transform of $F(s)$ is given by

$$f(t) = 2me^{-at}\cos(\omega t+\theta)u(t)$$

(For the proof see R.P. 5.19)

In the present context,

$$m = 5\sqrt{50}, \theta = 171.87^\circ$$

$$\omega = 3500 \text{ and } a = 500$$

This means that, $\mathcal{L}^{-1}\left\{\frac{5\sqrt{50}\,\underline{/171.87^\circ}}{s+500-j3500} + \frac{5\sqrt{50}\,\underline{/-171.87^\circ}}{s+500+j3500}\right\}$

$$= 2\times 5\sqrt{50}e^{-500t}\cos\left(3500t+171.87^\circ\right)$$

$$= 10\sqrt{50}e^{-500t}\cos\left(3500t+171.87^\circ\right)$$

Hence, $\boldsymbol{v_o(t) = \left[70 + 10\sqrt{50}e^{-500t}\cos\left(3500t+171.87^\circ\right)\right]u(t)}$

(b) $i_o(t) = \dfrac{v_o}{R} + C\dfrac{dv_o}{dt}$

Taking Laplace transforms on both the sides, we get

$$I_o(s) = \frac{V_o(s)}{R} + C\left[sV_o(s) - v_o(0^-)\right]$$

$$\Rightarrow \quad I_o(s) = \frac{V_o(s)}{R} + CsV_o(s)$$

$$\Rightarrow \quad I_o(s) = CV_o(s)\left[s + \frac{1}{RC}\right]$$

$$= \left(\frac{V_{dc}}{L}\right)\left[\frac{s + \dfrac{1}{RC}}{s\left(s^2 + \dfrac{1}{RC}s + \dfrac{1}{LC}\right)}\right]$$

Substituting the values of V_{dc}, R, L, and C, we get

$$I_o(s) = \frac{87.5(s+1000)}{s(s+500-j3500)(s+500+j3500)}$$
$$= \frac{K_1}{s} + \frac{K_2}{s+500-j3500} + \frac{K_2^*}{s+500+j3500}$$

We find that,

$$K_1 = \frac{87.5 \times 1000}{1250 \times 10^4} = 7\text{mA}$$
$$K_2 = \frac{87.5(500+j3500)}{(-500+j3500)(j7000)}$$
$$= 12.5\,\underline{/-106.26^\circ}\ \text{mA}$$
$$I_o(s) = \frac{7}{s} + \frac{12.5\,\underline{/-106.26^\circ}}{s+500-j3500} + \frac{12.5\,\underline{/106.26^\circ}}{s+500+j3500}$$

The inverse Laplace transform yields,

$$i_o(t) = \left[7 + 12.5\,\underline{/-106.26^\circ}\ e^{-(500-j3500)t} + 12.5\,\underline{/106.26^\circ}\ e^{-(500+j3500)t}\right]u(t)$$
$$= \boldsymbol{\left[7 + 25e^{-500t}\cos(3500t - 106.26^\circ)\right]u(t)\text{mA}}$$

(c) $V_o(s) = \dfrac{\dfrac{V_{dc}}{LC}}{s\left(s^2 + \left(\dfrac{1}{RC}\right)s + \dfrac{1}{LC}\right)}$

From Final Value theorem: $v_o(\infty) = \lim\limits_{t\to\infty} v_o(t) = \lim\limits_{s\to 0} sV_o(s) = \dfrac{V_{dc} \times LC}{LC}$ **= 70V**

The same result may be obtained by putting $t = \infty$ in the expression for $v_o(t)$.

From initial value theorem : $v_o(0) = \lim_{s\to\infty} sV_o(s)$

$$= \lim_{s\to\infty} \frac{\cancel{s}\dfrac{V_{dc}}{LC}}{\cancel{s}\left(s^2 + \dfrac{1}{RC}s + \dfrac{1}{LC}\right)}$$

$$= 0$$

This verifies our beginning analysis that $v_o(0^+) = v_o(0^-) = 0$. The same result may be obtained by putting $t = 0$ in the expression for $v_o(t)$.

We know that, $$I_o(s) = \frac{V_{dc}}{L} \frac{\left(s + \dfrac{1}{RC}\right)}{s\left(s^2 + \dfrac{1}{RC}s + \dfrac{1}{LC}\right)}$$

From final value theorem : $I_o(\infty) = \lim_{s\to 0} sI_o(s)$

$$= \lim_{s\to 0} \cancel{s}\frac{V_{dc}}{L} \frac{\left(s + \dfrac{1}{RC}\right)}{\cancel{s}\left(s^2 + \dfrac{1}{RC}s + \dfrac{1}{LC}\right)}$$

$$= \frac{V_{dc}}{L} \frac{\dfrac{1}{RC}}{\dfrac{1}{LC}}$$

$$= \frac{V_{dc}}{R} = \frac{70}{10 \times 10^3} = \textbf{7 mA}$$

The same result may be obtained by putting $t = \infty$ in the expression for $i_o(t)$.

From initial value theorem : $i_o(0) = \lim_{s\to\infty} sI_o(s)$

$$= \lim_{s\to\infty} s\left[\frac{V_{dc}}{L} \frac{\left(s + \dfrac{1}{RC}\right)}{s\left(s^2 + \dfrac{1}{RC}s + \dfrac{1}{LC}\right)}\right]$$

$$= \mathbf{0}$$

This agrees with our initial analysis that the initial current through the inductor is zero. The same result can be obtained by putting $t = 0$ in the expression for $i_o(t)$.

R.P 5.18

Apply the initial and final value theorems to each of the functions given below:

(a) $F(s) = \dfrac{s^2 + 5s + 10}{s + 6}$ (b) $F(s) = \dfrac{s^2 + 5s + 10}{5(s^2 + 6s + 8)}$

SOLUTION

Since in $F(s)$ referred in (a) and (b) are improper [1] fractions, the corresponding time domain counterparts, $f(t)$ contain impulses.

Thus, neither the initial value theorem nor the final value theorems may be applied to these transformed functions.

R.P 5.19

Find the inverse Laplace transform of $F(s) = \dfrac{c + jd}{s + a - j\omega} + \dfrac{c - jd}{s + a + j\omega}$

SOLUTION

Expressing $c + jd$ and $c - jd$ in the exponenetial from, we get,

$$F(s) = \frac{me^{j\theta}}{s + a - j\omega} + \frac{me^{-j\theta}}{s + a + j\omega}$$

where $m = \sqrt{c^2 + d^2}$ and $\theta = \tan^{-1}\left[\dfrac{d}{c}\right]$

Hence,

$$\begin{aligned} f(t) &= \mathcal{L}^{-1}\{F(s)\} \\ &= me^{j\theta}e^{-(a-j\omega)t}u(t) + me^{-j\theta}e^{-(a+j\omega)t}u(t) \\ &= me^{-at}e^{j(\theta+\omega t)}u(t) + me^{-at}e^{-j(\theta+\omega t)}u(t) \\ &= 2me^{-at}\left[\frac{e^{j(\theta+\omega t)} + e^{-j(\theta+\omega t)}}{2}\right]u(t) \\ &= \boldsymbol{2me^{-at}\cos(\theta + \omega t)u(t)} \end{aligned}$$

R.P 5.20

Find the initial and final values of $f(t)$ when $F(s) = \dfrac{60}{s^2 - 2s + 1}$

SOLUTION

Initial value theorem

$$\begin{aligned} f(0) &= \lim_{s\to\infty} sF(s) \\ &= \lim_{s\to\infty} s\frac{60}{s^2 - 2s + 1} = \mathbf{0} \end{aligned}$$

[1] If the degree of the numerator polynomial is greater than or equal to the degree of the denominator polynomial, the fraction is said to be improper.

Final value theorem:

The poles of $F(s)$ are given by finding the roots of the denominator polynomial. That is,

$$s^2 - 2s + 1 = 0 \quad \Rightarrow \quad (s-1)^2 = 0 \quad \Rightarrow \quad s = 1, 1$$

Since both the poles of $F(s)$ lie to the right of the s plane, final value theorem cannot be used to find $f(\infty)$.

R.P 5.21

Find $i(t)$ for the circuit of Fig.R.P. 5.21, when $i_1(t) = 7e^{-6t}$ A for $t \geq 0$ and $i(0) = 0$. Also find $i(\infty)$.

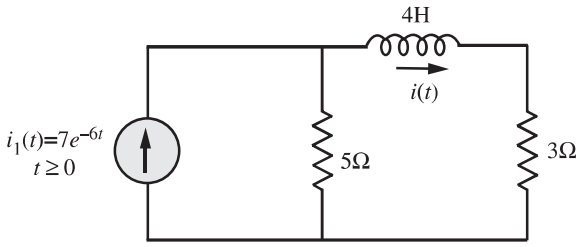

Figure R.P. 5.21

SOLUTION

Refer Fig. R.P. 5.21(a).

KCL at node v_1: $\dfrac{v_1}{5} + i = 7e^{-6t}$

Also, $\quad v_1 = 3i + 4\dfrac{di}{dt}$

Hence, $\quad \dfrac{1}{5}\left[3i + 4\dfrac{di}{dt}\right] + i = 7e^{-6t}$

$$\Rightarrow \qquad \frac{4}{5}\frac{di}{dt} + \frac{8}{5}i = 7e^{-6t}$$

$$\Rightarrow \qquad \frac{di}{dt} + 2i = \frac{35}{4}e^{-6t}$$

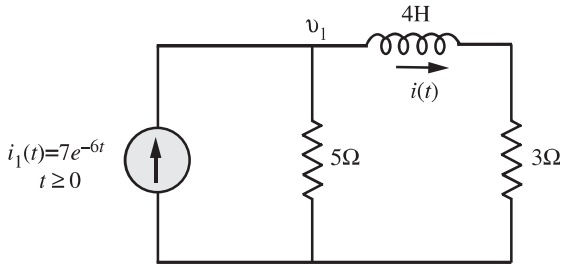

Figure R.P. 5.21(a)

Taking Laplace transform of the differential equation, we get

$$[sI(s) - i(0)] + 2I(s) = \frac{35}{4}\frac{1}{s+6}$$

$$\Rightarrow \qquad I(s) = \frac{35}{4}\frac{1}{(s+2)(s+6)}$$

Using partial fraction expansion, we get

$$I(s) = \frac{K_1}{s+2} + \frac{K_2}{s+6}$$

and find that $K_1 = \frac{35}{16}$ and $K_2 = \frac{-35}{16}$

Hence, $$I(s) = \frac{35}{16}\left[\frac{1}{s+2}\right] - \frac{35}{16}\left[\frac{1}{s+6}\right]$$

$$\Rightarrow \quad \boldsymbol{i(t) = \frac{35}{16}\left[e^{-2t} - e^{-6t}\right] u(t)}$$

5.7 Circuit element models and initial conditions

In the analysis of a circuit, the Laplace transform can be carried one step further by transforming the circuit itself rather than the differential equation. Earlier we have seen how to represent a circuit in time domain by differential equations and then use Laplace transform to transform the differential equations into algebraic equations. In this section, we will see how we can represent a circuit in s domain using the Laplace transform and then analyze it using algebraic equations.

5.7.1 Resistor

The voltage-current relationship for a resistor R is given by Ohm's law:

$$v(t) = i(t)R \tag{5.22}$$

Taking Laplace transform on both the sides, we get

$$V(s) = I(s)R \tag{5.23}$$

Fig. 5.19 (a) shows the representation of a resistor in time domain and Fig. 5.19(b) in frequency domain using Laplace transform.

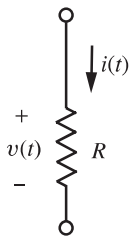

Figure 5.19(a) Resistor represented in time domain

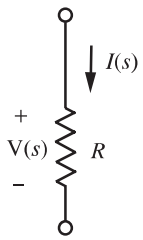

Figure 5.19(b) Resistor represented in the frequency domain using Laplace transform

The impedance of an element is defined as

$$Z(s) = \frac{V(s)}{I(s)}$$

provided all *initial conditions are zero*. Please note that the impedance is a concept defined only in frequency domain and not in time domain. In the case of a resistor, there is no initial condition to be set to zero. Comparision of equations (5.22) and (5.23) reveals that, resistor R has same representation in both time and frequency domains.

5.7.2 Capacitor

For a capacitor with capacitance C, the time-domain voltage-current relationship is

$$v(t) = \frac{1}{C}\int_0^t i(\tau)d\tau + v(0) \tag{5.24a}$$

The s domain characterization is obtained by taking the Laplace transform of the above equation. That is,

$$V(s) = \frac{1}{Cs}I(s) + \frac{v(0)}{s} \tag{5.24b}$$

To find the impedance of a capacitor, set the initial condition $v(0)$ to zero. Then from equation (5.24b), we get $Z(s) = \dfrac{V(s)}{I(s)} = \dfrac{1}{Cs}$ as the impedance of the capacitor. With the help of equation (5.24b), we can draw the frequency domain representation of a Capacitor and the same is shown in Fig. 5.20(b). This equivalent circuit is drawn so that the *KVL equation* represented by equation (5.24 b) is satisfied. Performing source transformation on the equivalent s domain circuit for a capacitor which is shown in Fig. 5.20(b), we get an alternate frequency domain representation as shown in Fig. 5.20(c).

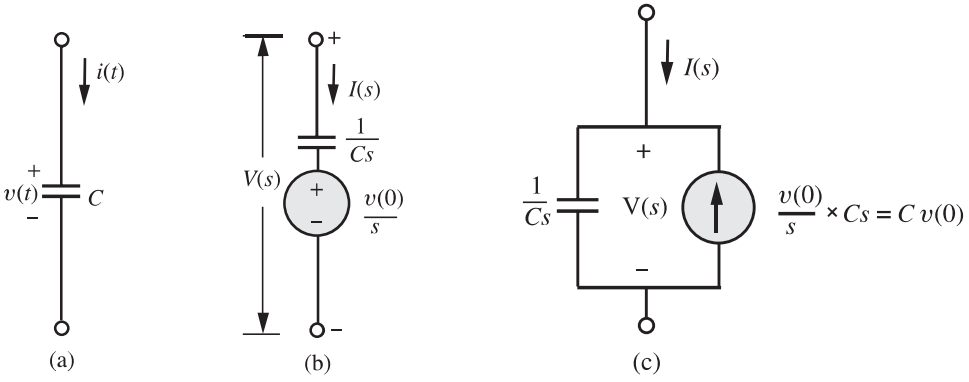

Figure 5.20(a) A capacitor represented in time domain
(b) A capacitor represented in the frequency domain
(c) Alternate frequency domain representation for a capacitor

5.7.3 Inductor

For an inductor with inductance L, the time domain voltage-current relation is

$$v(t) = L\frac{di(t)}{dt} \tag{5.25}$$

The Laplace transform of equation (5.25) yields,

$$V(s) = LsI(s) - Li(0) \tag{5.26}$$

To find impedance of an inductor, set the initial condition $i(0)$ to zero. Then from equation (5.26), we get

$$Z(s) = \frac{V(s)}{I(s)} = Ls \tag{5.27}$$

which represents the impedance of the inductor. Equation (5.26) is used to get the frequency domain representation of an inductor and the same is shown in Fig. 5.21(b). The series connection of elements corresponds to sum of the voltages in equation (5.26). Converting the voltage source in Fig.5.21(b) into an equivalent current source, we get an alternate representation for the inductor in frequency domain which is as shown in Fig. 5.21(c).

To find the frequency domain representation of a circuit, we replace the time domain representation of each element in the circuit by its frequency domain representation.

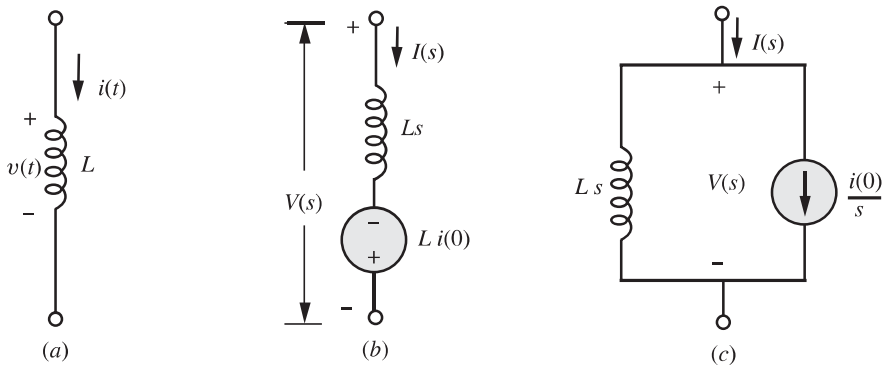

Figure 5.21(a) An inductor represented in time domain
(b) An inductor represented in the frequency domain
(c) An alternate frequency domain representation

To find the complete response of a circuit, we first get its frequency domain representation. Next, using KVL or KCL, we find the variables of interest in s doamin. Finally, we use the inverse Laplace transform to represent the variables of interest in time domain.

EXAMPLE 5.16

Determine the voltage $v_C(t)$ and the current $i_C(t)$ for $t \geq 0$ for the circuit shown in Fig. 5.22.

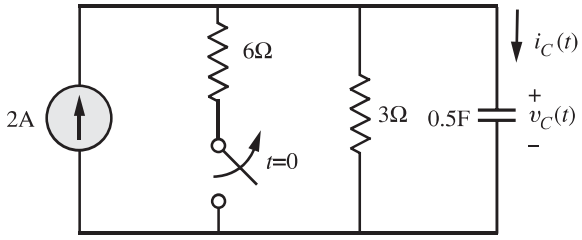

Figure 5.22

SOLUTION

We shall analyze this circuit using nodal technique. Hence we represent the capacitor in frequency domain by a parallel circuit since it is easier to account for current sources than voltage sources while handling nodal equations.

The symbol for switch indicates that at $t = 0^-$ it is closed and at $t = 0^+$, it is open. The circuit at $t = 0^-$ is shown in Fig. 5.23(a). Let us assume that at $t = 0^-$, the circuit is in steady state. Under steady state condition, capacitor acts as on open circuit as shown in Fig. 5.23(a).

$$i_1(0^-) = \frac{2 \times 6}{6+3} = \frac{12}{9} = \frac{4}{3}$$

$$v_C(0^-) = \frac{4}{3} \times 3 = 4\mathbf{V}$$

Hence, $\quad v_C(0) = v_C(0^+) = v_C(0^-) = 4\mathbf{V}$

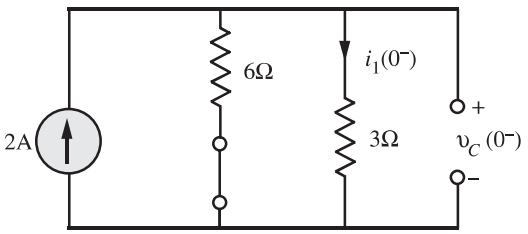

Figure 5.23(a)

Fig. 5.23(b) represents the frequency domain representation of the circuit shown in Fig. 5.22.

KCL at top node:

$$\frac{V_C(s)}{3} + \frac{s}{2}V_C(s) = 2 + \frac{2}{s}$$

$$\Rightarrow \qquad V_C(s) = \frac{6}{s} - \frac{2}{s + \frac{2}{3}}$$

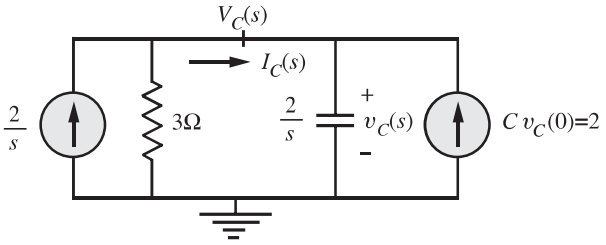

Figure 5.23(b)

Inverse Laplace transform yields

$$\boldsymbol{v_C(t) = \left[6 - 2e^{-\frac{2}{3}t}\right] u(t)\text{V}}$$

Also,
$$I_C(s) = \frac{V_C(s)}{\frac{2}{s}} - 2 = \frac{\frac{2}{3}}{s + \frac{2}{3}}$$

$$\Rightarrow \qquad \boldsymbol{i_C(t) = \frac{2}{3} e^{-\frac{2}{3}t} u(t)\text{A}}$$

EXAMPLE 5.17

Determine the current $i_L(t)$ for $t \geq 0$ for the circuit shown in Fig. 5.24.

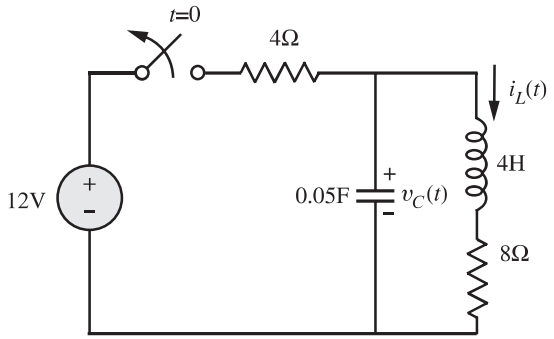

Figure 5.24

SOLUTION

At $t = 0^-$, switch is closed and at $t = 0^+$, it is open. Let us assume that at $t = 0^-$, the circuit is in steady state. In steady state, capacitor is open and inductor is short. The equivalent circuit at $t = 0^-$ is as shown in Fig. 5.25(a).

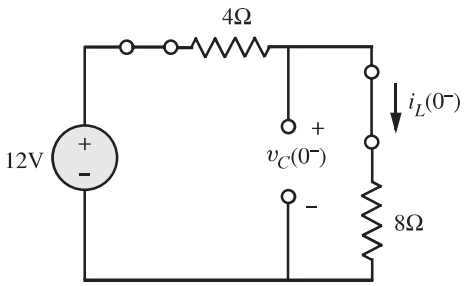

Figure 5.25(a)

$$i_L(0^-) = \frac{12}{8+4} = 1\text{A}$$

$$v_C(0^-) = 1 \times 8 = 8\text{V}$$

Therefore,
$$i_L(0) = i_L(0^+) = i_L(0^-) = 1\text{A}$$

$$v_C(0) = v_C(0^+) = v_C(0^-) = 8\text{V}$$

For $t \geq 0^+$, the circuit in frequency domain is as shown in Fig. 5.25(b). We will use *KVL* to find $i_L(t)$. Hence, we use series circuits to represent both the capacitor and inductor in the frequency domain. These series circuits contain voltage sources rather than current sources. It is easier to account for voltage sources than current sources when writing mesh equations. This justifies the selection of series representation for both the capacitor and inductor.

Applying *KVL* clockwise to the right mesh, we get

$$\frac{-8}{s}+\frac{20}{s}I_L(s)+4sI_L(s)-4+8I_L(s)=0$$

$$\Rightarrow \quad \frac{8}{s}+4=\left[\frac{20}{s}+8+4s\right]I_L(s)$$

$$\Rightarrow \quad I_L(s)=\frac{2+s}{s^2+2s+5}=\frac{(s+1)+1}{(s+1)^2+4}$$

$$\Rightarrow \quad I_L(s)=\frac{s+1}{(s+1)^2+2^2}+\frac{1}{2}\left[\frac{2}{(s+1)^2\times 2^2}\right]$$

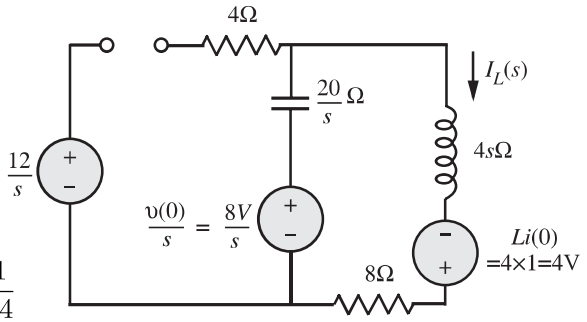

Figure 5.25(b)

We know the Laplace transform pairs:

$$\mathcal{L}\{e^{-at}\cos bt\}=\frac{s+a}{(s+a)^2+b^2}$$

and

$$\mathcal{L}\{e^{-at}\sin bt\}=\frac{b}{(s+a)^2+b^2}$$

Hence,

$$\boldsymbol{i_L(t)=\left[e^{-t}\cos 2t+\frac{1}{2}e^{-t}\sin 2t\right]u(t)\,\mathrm{A}}$$

EXAMPLE 5.18

Find $v_o(t)$ of the circuit shown in Fig. 5.26.

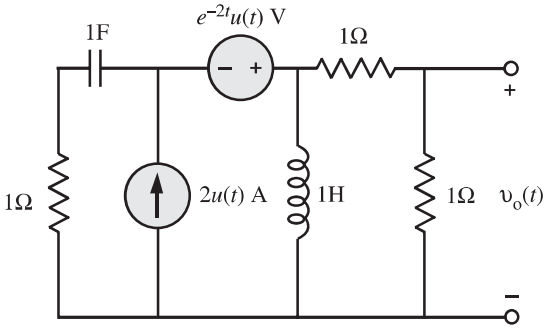

Figure 5.26

SOLUTION

The unit step function $u(t)$ is defined as follows:

$$u(t)=\begin{cases}1, & t\geq 0^+\\ 0, & t\leq 0^-\end{cases}$$

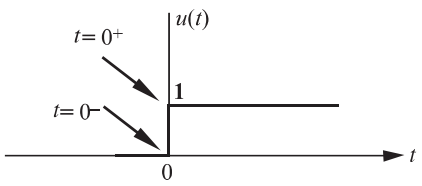

Figure 5.27(a)

Since the circuit has two independent sources with $u(t)$ associated with them, the circuit is not energized for $t \leq 0^-$. Hence the initial current through the inductor is zero. That is, $i_L(0^-) = 0$. Since current through an inductor cannot change instantaneously,

$$i_L(0) = i_L(0^+) = i_L(0^-) = 0$$

Also,
$$v_C(0) = v_C(0^+) = v_C(0^-) = 0$$

The equivalent circuit for $t \geq 0^+$ in frequency domain is as shown in Fig. 5.27(b).

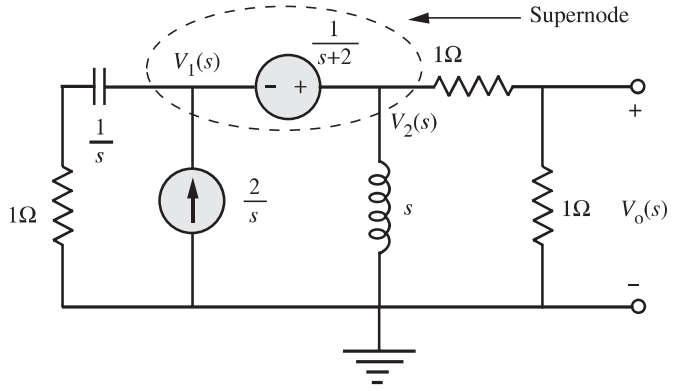

Figure 5.27(b)

KCL at supernode:

$$\frac{V_1(s)}{1+\frac{1}{s}} + \frac{V_2(s)}{s} + \frac{V_2(s)}{2} = \frac{2}{s}$$

$$\Rightarrow \quad V_1(s)\left[\frac{1}{1+\frac{1}{s}}\right] + V_2(s)\left[\frac{1}{s}+\frac{1}{2}\right] = \frac{2}{s}$$

$$\Rightarrow \quad V_1(s)\left[\frac{s}{s+1}\right] + V_2(s)\left[\frac{2+s}{2s}\right] = \frac{2}{s}$$

The constraint equation:

Applying KVL to the path comprising of current source → voltage source → inductor,

we get,
$$-V_1(s) - \frac{1}{s+2} + V_2(s) = 0$$

$$V_2(s) - V_1(s) = \frac{1}{s+2}$$

$$\Rightarrow \quad V_1(s) - V_2(s) = -\frac{1}{s+2}$$

Putting the above two equations in matrix form, we get

$$\begin{bmatrix} \dfrac{s}{s+1} & \dfrac{2+s}{2s} \\ 1 & -1 \end{bmatrix} \begin{bmatrix} V_1(s) \\ V_2(s) \end{bmatrix} = \begin{bmatrix} \dfrac{2}{s} \\ \dfrac{-1}{s+2} \end{bmatrix}$$

Solving for $V_2(s)$ and then applying the principle of voltage division, we get

$$V_o(s) = \frac{1}{2}V_2(s) = \frac{2(3s^2+6s+4)}{2(s+2)(3s^2+3s+2)}$$

$$\Rightarrow \qquad V_o(s) = \frac{\left(s^2+2s+\dfrac{4}{3}\right)}{(s+2)(s+0.5-j0.646)(s+0.5+j0.646)}$$

Using partial fractions, we can write

$$V_o(s) = \frac{K_1}{s+2} + \frac{K_2}{s+0.5-j0.646} + \frac{K_2^*}{s+0.5+j0.646}$$

We find that

$$K_1 = 0.5$$

$$K_2 = 0.316\,\underline{/-37.76}$$

Hence,

$$V_o(s) = \frac{0.5}{s+2} + \frac{0.316\,\underline{/-37.76}}{s+0.5-j0.646} + \frac{0.316\,\underline{/37.76}}{s+0.5+j0.646}$$

We know that, $\mathcal{L}^{-1}\left[\dfrac{1}{s+a}\right] = e^{-at}u(t)$

$$\mathcal{L}^{-1}\left[\frac{m\,\underline{/\theta}}{s+a-j\omega} + \frac{m\,\underline{/-\theta}}{s+a+j\omega}\right]$$

$$= 2me^{-at}\cos(\omega t+\theta)u(t)$$

Hence, $\boldsymbol{v_o(t) = 0.5e^{-2t}u(t) + 0.632e^{-0.5t}\cos\left[0.646t - 37.76°\right]u(t)}$

EXAMPLE 5.19

For the network shown in Fig. 5.28, find $v_o(t)$, $t > 0$, using mesh equations.

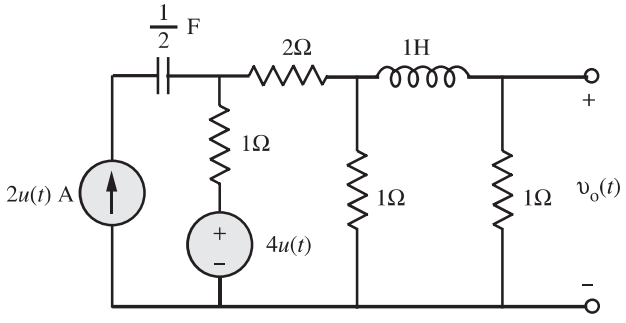

Figure 5.28

SOLUTION

The step function $u(t)$ is defined as follows.

$$u(t) = \begin{cases} 1, & t \geq 0^+ \\ 0, & t \leq 0^- \end{cases}$$

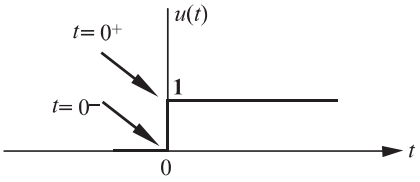

Figure 5.29(a)

Since the circuit is not energized for $t \leq 0^-$, there are no initial conditions in the circuit. For $t \geq 0^+$, the frequency domain equivalent circuit is shown in Fig. 5.29(b).

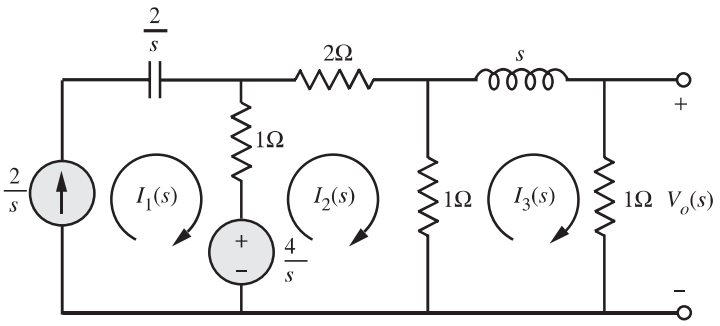

Figure 5.29(b)

By inspection, we find that $I_1(s) = \dfrac{2}{s}$

KVL clockwise for mesh 2:

$$\frac{-4}{s} + 1\,[I_2(s) - I_1(s)] + 2I_2(s) + 1\,[I_2(s) - I_3(s)] = 0$$

$$\Rightarrow \quad \frac{-4}{s} - I_1(s) + I_2(s)\,[1 + 2 + 1] - I_3(s) = 0$$

Substituting the value of $I_1(s)$, we get

$$\frac{-4}{s} + 4I_2(s) - I_3(s) = \frac{2}{s}$$

$$\Rightarrow \quad 4I_2(s) - I_3(s) = \frac{6}{s}$$

KVL clockwise for mesh 3:

$$1\,[I_3(s) - I_2(s)] + sI_3(s) + 1I_3(s) = 0$$

$$\Rightarrow \quad -I_2(s) + I_3(s)\,[s + 2] = 0$$

Putting the *KVL equations* for mesh 2 and mesh 3 in matrix form, we get

$$\begin{bmatrix} 4 & -1 \\ -1 & s+2 \end{bmatrix} \begin{bmatrix} I_2(s) \\ I_3(s) \end{bmatrix} = \begin{bmatrix} \dfrac{6}{s} \\ 0 \end{bmatrix}$$

Solving for $I_3(s)$, using Cramer's rule, we get

$$I_3(s) = \frac{1.5}{s\left(s + \frac{7}{4}\right)}$$

$$\Rightarrow \qquad V_o(s) = I_3(s) \times 1 = \frac{1.5}{s\left(s + \frac{7}{4}\right)}$$

Using partial fractions, we can write

$$V_o(s) = \frac{K_1}{s} + \frac{K_2}{s + \frac{7}{4}}$$

We find that, $\quad K_1 = \frac{6}{7}$, and $K_2 = \frac{-6}{7}$

Hence, $\quad V_o(s) = \frac{6}{7}\left[\frac{1}{s} - \frac{1}{s + \frac{7}{4}}\right]$

$$\Rightarrow \qquad \boldsymbol{v_o(t) = \frac{6}{7}\left[1 - e^{-\frac{7}{4}t}\right] u(t)}$$

EXAMPLE 5.20

Use mesh analysis to find $v_o(t)$, $t > 0$ in the network shown in Fig. 5.30.

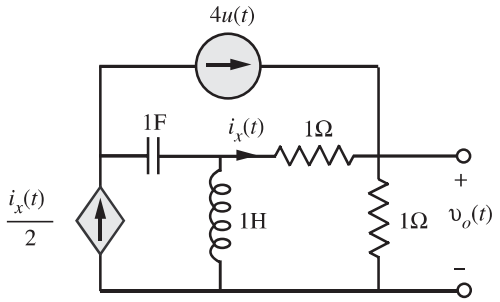

Figure 5.30

SOLUTION

The circuit is not energized for $t \leq 0^-$ because the independent current source is associated with $u(t)$. This means that there are no initial conditions in the circuit. The frequency domain circuit for $t \geq 0^+$ is shown in Fig. 5.31.

By inspection we find that:

$$I_1(s) = \frac{4}{s}, \quad I_2(s) = \frac{I_x(s)}{2}$$

$$I_x(s) = I_3(s) - \frac{4}{s} \quad \Rightarrow \quad 2I_2(s) = I_3(s) - \frac{4}{s} \quad \Rightarrow \quad I_2(s) = \frac{1}{2}\left[I_3(s) - \frac{4}{s}\right]$$

KVL for mesh 3 gives

$$s\left[I_3(s) - I_2(s)\right] + 1\left[I_3(s) - \frac{4}{s}\right] + 1 \times I_3(s) = 0$$

$$\Rightarrow \quad s\left[I_3(s) - \frac{1}{2}\left(I_3(s) - \frac{4}{s}\right)\right] + \left[I_3(s) - \frac{4}{s}\right] + I_3(s) = 0$$

$$\Rightarrow \quad I_3(s) = \frac{-4(s-2)}{s(s+4)}$$

and $$V_o(s) = 1[I_3(s)] = \frac{-4(s-2)}{s(s+4)}$$

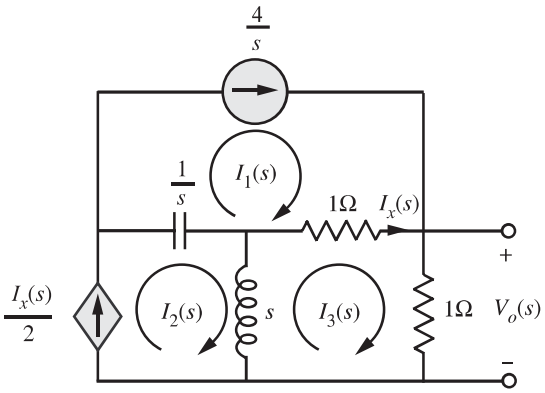

Figure 5.31

By partial fractions, we can write

$$V_o(s) = \frac{K_1}{s} + \frac{K_2}{s+4}$$

We find that $K_1 = 2,\ K_2 = -6$

Hence, $$V_o(s) = \frac{2}{s} - \frac{6}{s+4}$$

Taking inverse Laplace transform, we get

$$\boldsymbol{v_o(t) = 2u(t) - 6e^{-4t}u(t)}$$

EXAMPLE 5.21

Using the principle of superposition, find $v_o(t)$ for $t > 0$. Refer the circuit shown in Fig. 5.32.

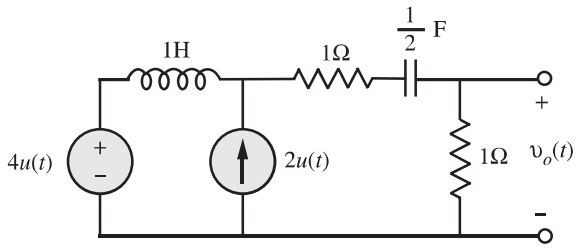

Figure 5.32

SOLUTION

Since both the independent sources are associated with $u(t)$, which is zero for $t \leq 0^-$, the circuit will not have any initial conditions. The frequency domain circuit for $t \geq 0^+$ is shown in Fig. 5.33(a).

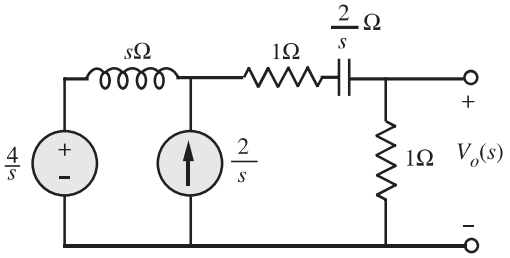

Figure 5.33(a)

As a first step, let us find the contribution to $V_o(s)$ due to voltage source alone. This needs the deactivation of the current source.

Referring to Fig. 5.33(b), we find that

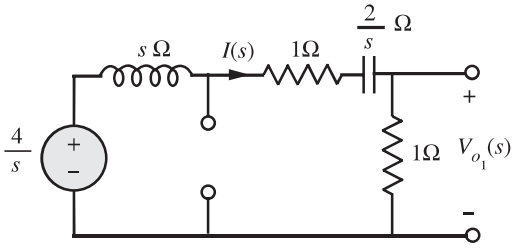

Figure 5.33(b)

$$I(s) = \frac{\frac{4}{s}}{s + 1 + \frac{2}{s} + 1}$$

$$\Rightarrow \qquad V_{o_1}(s) = I(s)[1] = \frac{4}{s^2 + 2s + 2}$$

Next let us find the contribution to the output due to current source alone.

Refer to Fig. 5.33(c). Using the principle of current division,

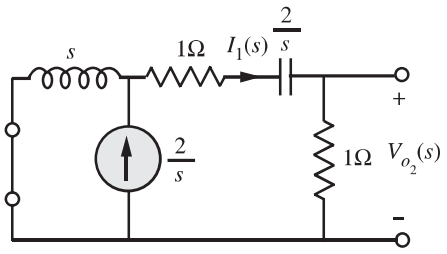

Figure 5.33(c)

$$I_1(s) = \frac{\frac{2}{s} \times s}{s + 1 + \frac{2}{s} + 1}$$

$$\Rightarrow \qquad V_{o_2}(s) = 1\,[I_1(s)] = \frac{2s}{s^2 + 2s + 2}$$

Finally adding the two contributions, we get

$$\begin{aligned} V_o(s) &= V_{o_1}(s) + V_{o_2}(s) \\ &= \frac{4}{s^2 + 2s + 2} + \frac{2s}{s^2 + 2s + 2} = \frac{2s + 4}{s^2 + 2s + 2} \\ &= \frac{K_1}{s + 1 - j1} + \frac{K_1^*}{s + 1 + j1} \end{aligned}$$

We find that, $\qquad K_1 = \sqrt{2}\,\underline{/-45^\circ}$

Hence, $\qquad V_o(s) = \dfrac{\sqrt{2}\,\underline{/-45^\circ}}{s + 1 - j1} + \dfrac{\sqrt{2}\,\underline{/+45^\circ}}{s + 1 + j1}$

We know that: $$\mathcal{L}^{-1}\left\{\frac{m\angle\theta}{s+a-jb}+\frac{m\angle-\theta}{s+a+jb}\right\}=2me^{-at}\cos(bt+\theta)u(t)$$

Hence, $$\boldsymbol{v_o(t)=2\sqrt{2}e^{-t}\cos(t+45^\circ)u(t)}$$

EXAMPLE 5.22

(a) Convert the circuit in Fig. 5.34 to an appropriate s domain representation.

(b) Find the Thevein equivalent seen by 1Ω resistor.

(c) Analyze the simplified circuit to find an expression for $i(t)$.

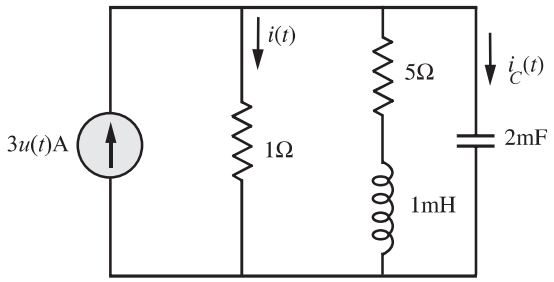

Figure 5.34

SOLUTION

(a) Since the independent current source has $u(t)$ in it, the circuit is not activated for $t \leq 0^-$. In otherwords, all the initial conditions are zero. Fig.5.35 (a) shows the s domain equivalent circuit for $t \geq 0^+$.

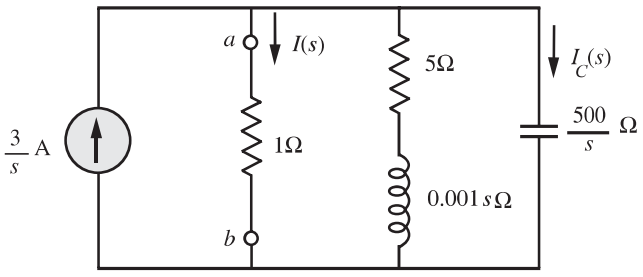

Figure 5.35(a)

(b) Sine we are interested in the current in 1Ω using the Thevenin theorem, remove the 1Ω resistor from the circuit shown in Fig. 5.35(a). The resulting circuit thus obtained is shown in Fig. 5.35(b).

$Z_t(s)$ is found by deactivating the independent current source.

$$Z_t(s) = (5 + 0.001s)||\frac{500}{s}$$
$$= \frac{2500 + 0.5s}{0.001s^2 + 5s + 500}\Omega$$

Referring to Fig. 5.35 (b),

$$V_t(s) = \frac{3}{s}[Z_t(s)]$$
$$= \frac{7.5 \times 10^6 + 1500s}{s(s^2 + 5000s + 5 \times 10^5)}\text{Volts}$$

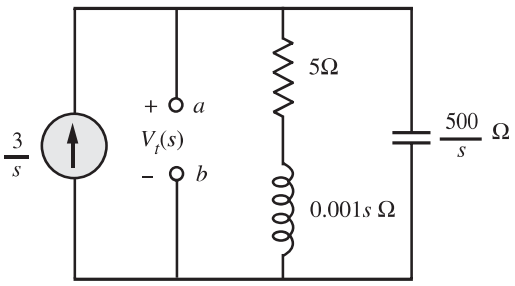

Figure 5.35(b)

The Thevenin equivalent circuit along with 1Ω resistor is shown in Fig. 5.35 (c).

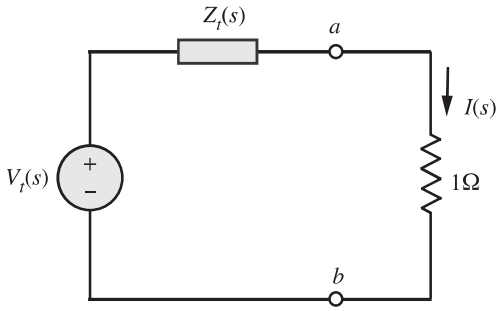

Figure 5.35(c)

$$I(s) = \frac{V_t(s)}{Z_t(s) + 1}$$
$$= \frac{7.5 \times 10^6 + 1500s}{s(s^2 + 5500s + 3 \times 10^6)}$$
$$= \frac{7.5 \times 10^6 + 1500s}{s(s + 4886)(s + 614)}$$

Using partial fractions, we get

$$I(s) = \frac{2.5}{s} + \frac{0.008}{s + 4886} - \frac{2.508}{s + 614}$$

Taking inverse Laplace transforms, we get

$$\boldsymbol{i(t) = \left(2.5 + 0.008e^{-4886t} - 2.508e^{-614t}\right) u(t)\text{A}}$$

Check:

$$i(0) = 2.5 + 0.008 - 2.508 = 0$$

and

$$i(\infty) = 2.5.$$

These could be verified by evaluating $i(t)$ at $t = 0$ and $t = \infty$ using the concepts explained in Chapter 4.

EXAMPLE 5.23

Refer the RLC circuit shown in Fig. 5.36. Find the complete response for $v(t)$ if $t \geq 0^+$. Take $v(0) = 2$V.

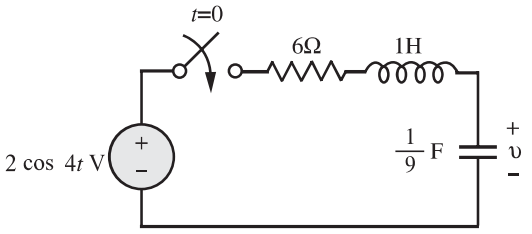

Figure 5.36

SOLUTION

Since we wish to analyze the circuit given in Fig. 5.36 using KVL, we shall represent L and C in frequency domain using series circuits to accomodate the initial conditions. Accordingly, we get the frequency domain circuit shown in Fig. 5.36 (a).

Applying KVL clockwise to the circuit shown in Fig. 5.36 (a), we get

$$\frac{-2s}{s^2+16} + \left(6 + s + \frac{9}{s}\right) I(s) + \frac{2}{s} = 0$$

$$\Rightarrow \quad I(s) = \frac{-32}{(s^2+6s+9)(s^2+16)}$$

Hence, $V(s) = I(s)\left[\frac{9}{s}\right] + \frac{2}{s}$

$$= \frac{2}{s} + \frac{-288}{s(s+3)^2(s^2+16)}$$

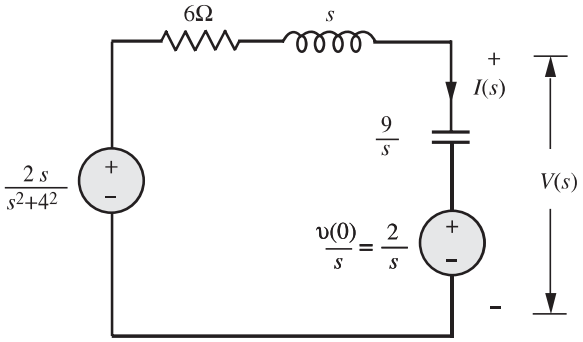

Figure 5.36(a)

Using partial fraction, we get

$$V(s) = \frac{2}{s} + \left[\frac{K_1}{s} + \frac{K_2}{s+3} + \frac{K_3}{(s+3)^2} + \frac{K_4}{s-j4} + \frac{K_4^*}{s+j4}\right]$$

Solving for K_1, K_2, K_3, and K_4, we get

$$K_1 = \left.\frac{-288}{(s+3)^2(s^2+16)}\right|_{s=0} = -2$$

$$K_2 = \frac{d}{ds}\left[\frac{-288}{s(s^2+16)}\right]_{s=-3} = 2.2$$

$$K_3 = \left.\frac{-288}{s(s^2+16)}\right|_{s=-3} = 3.84$$

$$K_4 = \left.\frac{-288}{s(s+3)^2(s+j4)}\right|_{s=j4} = 0.36\,\underline{/-106.2}$$

Therefore

$$V(s) = \frac{2}{s} - \frac{2}{s} + \frac{2.2}{s+3} + \frac{3.84}{(s+3)^2} + \frac{0.36\angle -106.2}{s-j4} + \frac{0.36\angle 106.2}{s+j4}$$

Taking inverse Laplace Transform we get,

$$\boldsymbol{v(t) = 2.2e^{-3t} + 3.84te^{-3t} + 0.72\cos(4t - 106.2^\circ)}$$

Verification:
Putting $t = 0$ in the above equation

$$\begin{aligned} v(0) &= 2.2 + 0 + 0.72\cos(-106.2^\circ) \\ &= 2.2 - 0.2 = 2\text{V} \end{aligned}$$

(The same quantity is given in the problem)

5.8 Waveform synthesis

The three important singularity functions explained in section 5.3 are very useful as building blocks in constructing other waveforms. In this section, we illustrate the concept of waveform synthesis with a number of exmaples, and also determine expressions for these waveforms.

EXAMPLE 5.24

Express the voltage pulse shown in Fig.5.37 in terms of unit step function and then find $V(s)$. Also find $\mathcal{L}\left\{\frac{dv(t)}{dt}\right\}$.

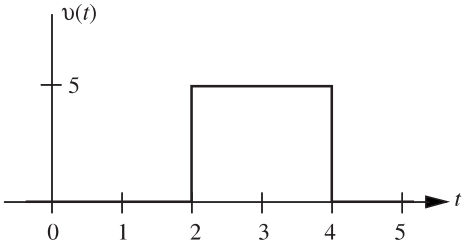

Figure 5.37

SOLUTION

The pulse shown in Fig. 5.37 is the gate function. This function may be regarded as a step function that switches on at $t = 2$ secs and switches off at $t = 4$ secs.

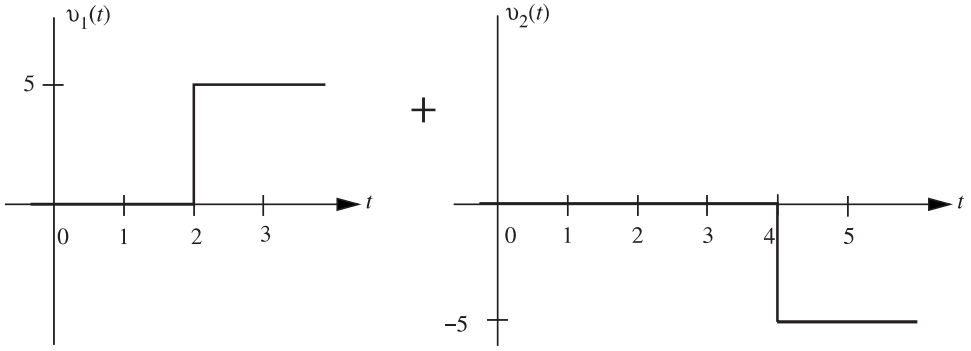

Figure 5.37(a)

Referring to Figs. 5.37 and 5.37 (a), we can write

$$v(t) = v_1(t) + v_2(t)$$

$$\Rightarrow \quad v(t) = 5u(t-2) - 5u(t-4)$$

$$\text{Hence,} \quad V(s) = \frac{5}{s}e^{-2s} - \frac{5}{s}e^{-4s}$$

$$= \frac{\mathbf{5}}{\boldsymbol{s}}\left[\boldsymbol{e^{-2s} - e^{-4s}}\right]$$

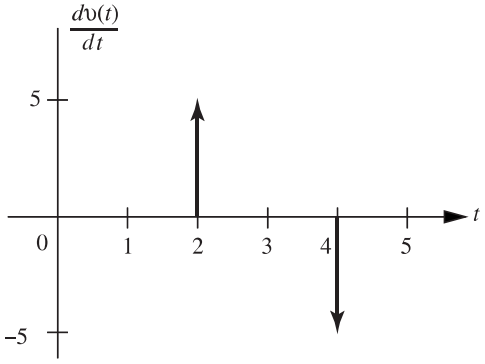

Figure 5.37(b)

Taking the derivative of $v(t)$, we get

$$\frac{dv(t)}{dt} = 5\left[\delta(t-2) - \delta(t-4)\right]$$

Fig. 5.37(b) shows the graph of $\dfrac{dv(t)}{dt}$.

We can obtain Fig. 5.37(b) directly from Fig. 5.36 by observing that at $t = 2$ seconds, there is a sudden rise of 5V leading to $5\delta(t-2)$. Similarly, at $t = 4$ seconds, a sudden fall of 5V leading to $-5\delta(t-4)$.

We know the Laplace trasnform pair

$$\mathcal{L}\{\delta(t-a)\} = e^{-as}\,\mathcal{L}\{\delta(t)\}$$

$$= e^{-as}$$

$$\text{Hence,} \quad \mathcal{L}\left\{\frac{\boldsymbol{dv(t)}}{\boldsymbol{dt}}\right\} = \mathbf{5}\left[\boldsymbol{e^{-2s} - e^{-4s}}\right]$$

EXAMPLE 5.25

Express the current pulse in Fig.5.38 in terms of the unit step.

Find: (i) $\mathcal{L}\{i(t)\}$ (ii) $\mathcal{L}\{\int i(t)dt\}$.

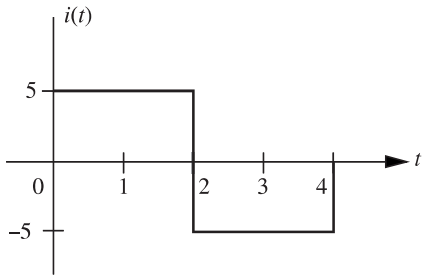

Figure 5.38

SOLUTION

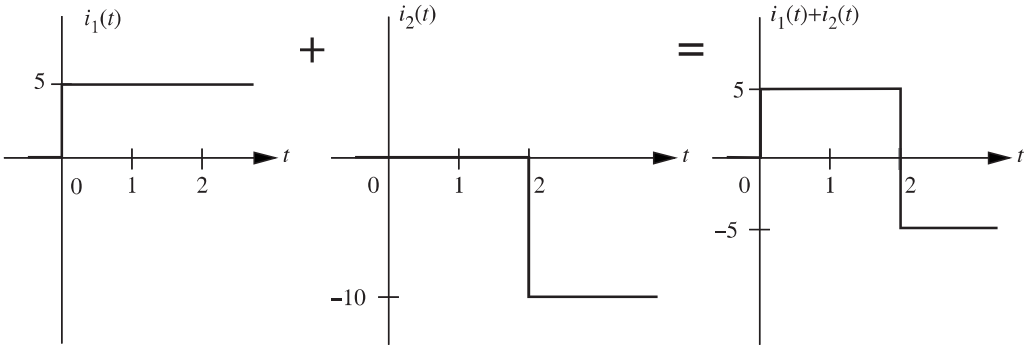

Figure 5.39(a)

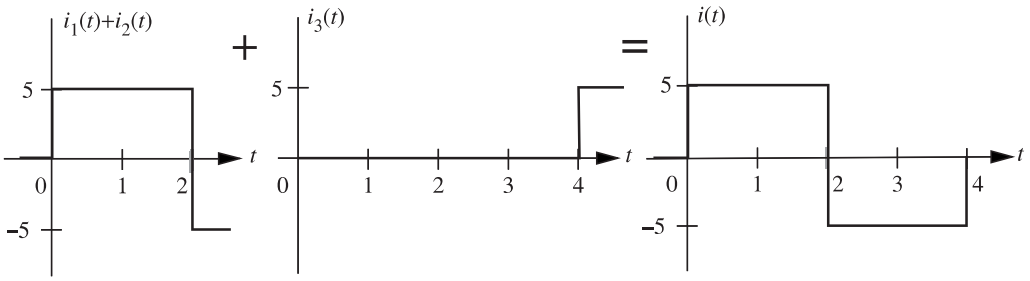

Figure 5.39(b)

Referring to Figs. 5.39 (a) and (b), using the principle of synthesis, we can write

$$\begin{aligned} i(t) &= i_1(t) + i_2(t) + i_3(t) \\ &= 5u(t) - 10u(t-2) + 5u(t-4) \end{aligned}$$

The Laplace transform of the above equation yields

$$\begin{aligned} I(s) &= \frac{5}{s} - \frac{10}{s}e^{-2s} + \frac{5}{s}e^{-4s} \\ &= \frac{5}{s}\left[1 - 2e^{-2s} + e^{-4s}\right] \\ &= \boldsymbol{\frac{5}{s}\left[1 - e^{-2s}\right]^2} \end{aligned}$$

Let $$f(t) = \int i(t)dt$$

then, $$\begin{aligned} f(t) &= \int [5u(t) - 10u(t-2) + 5u(t-4)]dt \\ &= 5r(t) - 10r(t-2) + 5r(t-4) \\ &= f_1(t) + f_2(t) + f_3(t) \end{aligned}$$

The function $f_1(t)$ is a ramp of slope = 5 as shown in Fig. 5.39 (c). To this, if we add a ramp of slope = -10, the effect of this addition is, we get a ramp of slope = $5 - 10 = -5$ for $t \geq 2$ secs till we encounter the next ramp. At $t = 4$ seconds, if we add a ramp with a slope of +5, the net slope beyond $t = 4$ seconds is $-5 + 5 = 0$. Thus figure $f(t)$ is drawn as shown in Fig. 5.39 (d).

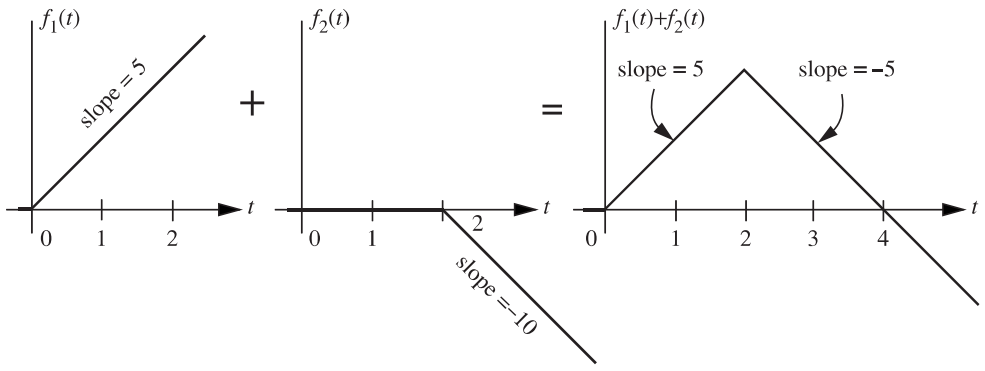

Figure 5.39(c)

$$\begin{aligned} \mathcal{L}\{f(t)\} &= F(s) \\ &= \mathcal{L}\{5r(t) - 10r(t-2) + 5r(t-4)\} \\ &= \frac{5}{s^2} - \frac{10}{s^2}e^{-2s} + \frac{5}{s^2}e^{-4s} \\ &= \boldsymbol{\frac{5}{s^2}\left[1 - 2e^{-2s} + e^{-4s}\right]} \end{aligned}$$

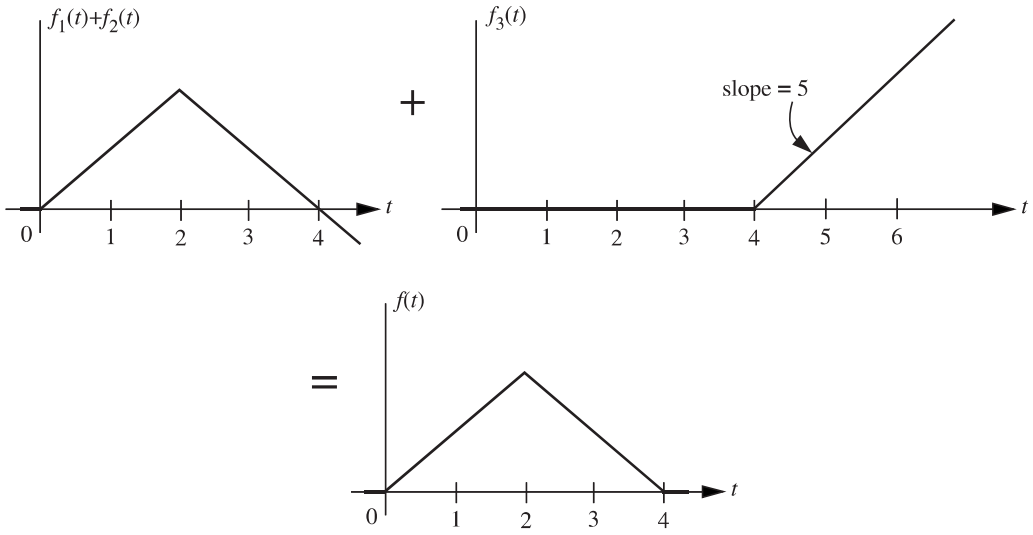

Figure 5.39(d)

EXAMPLE 5.26

Express the sawtooth function in terms of singularity functions. Then find $\mathcal{L}\{v(t)\}$.

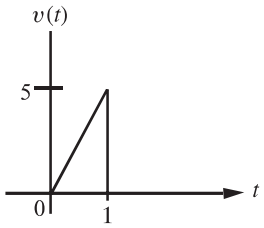

Figure 5.40

SOLUTION

There are three methods to solve this problem.

Method 1:

The function $v_1(t)$ is a ramp function of slope = +5. This slope +5 should continue till $t = 1$ second. Hence at $t = 1$ second, a ramp of slope $t = -5$ is added to $v_1(t)$. The graph of $v_1(t) + v_2(t)$ is shown in Fig. 5.41(a). Next, to $v_1(t) + v_2(t)$, a step of -5V is added at $t = 1$ second.

Hence,

$$\begin{aligned} v(t) &= v_1(t) + v_2(t) + v_3(t) \\ &= 5r(t) - 5r(t-1) - 5u(t-1) \\ V(s) &= \mathcal{L}\{f(t)\} = \frac{5}{s^2} - \frac{5}{s^2}e^{-s} - \frac{5}{s}e^{-s} \\ &= \frac{5}{s^2}\left[1 - e^{-s} - se^{-s}\right] \end{aligned}$$

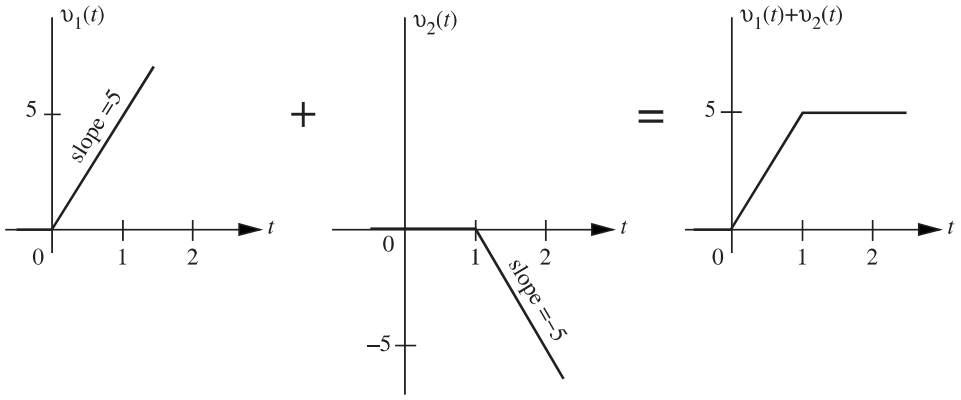

Figure 5.41(a)

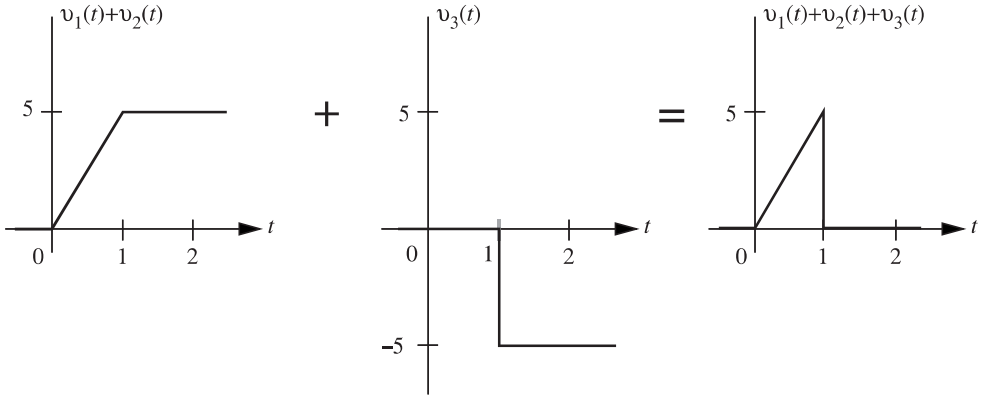

Figure 5.41(b)

Method 2:

This method involves graphical manipulation.

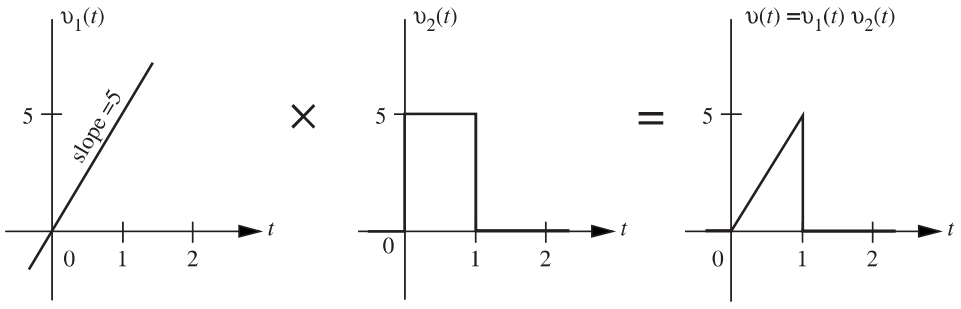

Figure 5.41(c)

The equation of a straight line passing through the origin is $y = mx$, where $m =$ slope of the line. This allows us to write $v_1(t) = 5t$. From Fig. 5.41(c), we can write

$$\begin{aligned} v(t) &= v_1(t)v_2(t) \\ &= 5t\,[u(t) - u(t-1)] \\ &= 5tu(t) - 5tu(t-1) \\ &= 5tu(t) - 5(t-1+1)u(t-1) \\ &= 5tu(t) - 5(t-1)u(t-1) - 5u(t-1) \\ &= 5r(t) - 5r(t-1) - 5u(t-1) \end{aligned}$$

Hence, $$\boldsymbol{V(s) = \frac{5}{s^2}\left[1 - e^{-s} - se^{-s}\right]}$$

Method 3:

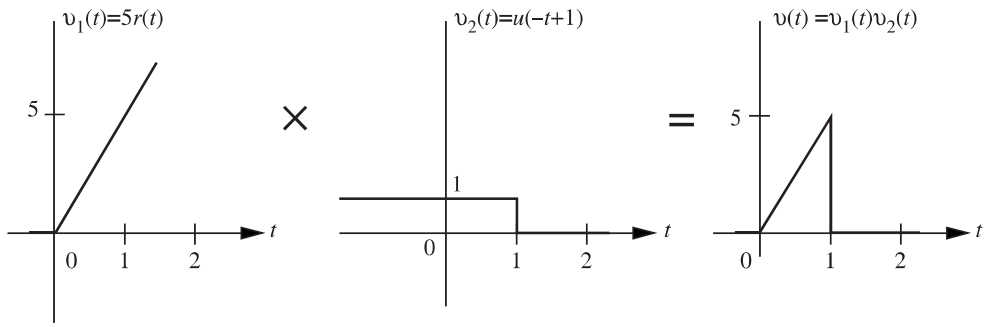

Figure 5.41(d)

This method also involves graphical manipulation. We observe from Fig. 5.41(d) that $v(t)$ is a multiplication of a ramp function and a unit step function.

Thus, $$\begin{aligned} v(t) &= v_1(t)v_2(t) \\ &= 5r(t)\,[u(-t+1)] \end{aligned}$$

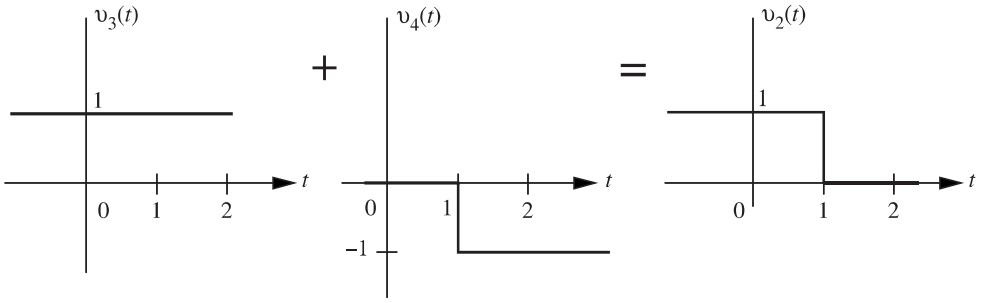

Figure 5.41(e)

From Fig. 5.41(e), we can write

$$v_2(t) = v_3(t) + v_4(t)$$

$$\Rightarrow \quad v_2(t) = 1 - u(t-1)$$

$$\Rightarrow \quad u(-t+1) = 1 - u(t-1)$$

Hence, $v(t) = 5r(t)\,[1 - u(t-1)]$

$$= 5r(t) - 5r(t)u(t-1)$$

We know that, $r(t) = tu(t)$

Hence, $v(t) = 5r(t) - 5tu(t)u(t-1)$

$$= 5r(t) - 5(t-1+1)u(t)u(t-1)$$

$$= 5r(t) - 5(t-1)u(t)u(t-1) - 5u(t)u(t-1)$$

Please note that, $u(t)u(t-1) = u(t-1)$ [Refer Fig. 5.41(f)]

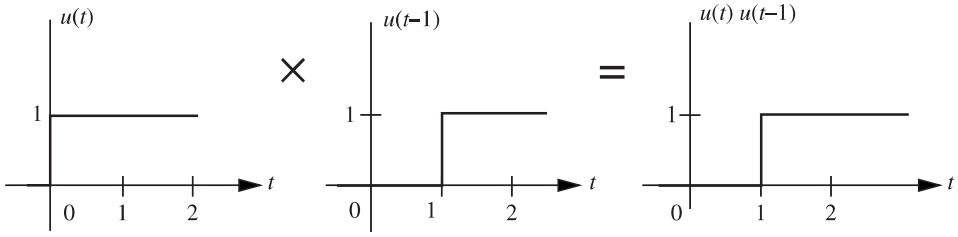

Figure 5.41(f)

Thus, $v(t) = 5r(t) - 5(t-1)u(t-1) - 5u(t-1)$

$$= 5r(t) - 5r(t-1) - 5u(t-1)$$

Hence, $\boldsymbol{V(s) = \frac{5}{s^2}\left[1 - e^{-s} - se^{-s}\right]}$

EXAMPLE 5.27

Given the signal

$$x(t) = \begin{cases} 3, & t < 0 \\ -2, & 0 < t < 1 \\ 2t - 4, & t > 1 \end{cases}$$

Express $x(t)$ in terms of singularity functions. Also find $\mathcal{L}\{x(t)\}$.

SOLUTION

The signal $x(t)$ may be viewed as follows:

(i) in the interval, $t < 0$, $x(t)$ may be regarded as $3u(-t)$

(ii) in the interval, $0 < t < 1$, $x(t)$ may be viewed as $-2[u(t) - u(t-1)]$ and

(iii) for $t > 1$, $x(t)$ may be viewed as $(2t-4)u(t-1)$

$$\text{Thus,} \qquad x(t) = 3u(-t) - 2\,[u(t) - u(t-1)] + (2t-4)u(t-1)$$

$$\Rightarrow \qquad x(t) = 3\,[1 - u(t)] - 2u(t) + 2u(t-1) + 2tu(t-1) - 4u(t-1)$$

$$= 3 - 5u(t) - 2u(t-1) + 2(t-1+1)u(t-1)$$

$$= 3 - 5u(t) - 2u(t-1) + 2(t-1)u(t-1) + 2u(t-1)$$

$$= \mathbf{3 - 5u(t) + 2r(t-1)}$$

$\mathcal{L}\{x(t)\}$ cannot be found because $x(t)$ contains a constant 3 for $-\infty < t < 0$ (a noncausal signal).

EXAMPLE 5.28

Express $f(t)$ in terms of singularity functions and then find $F(s)$.

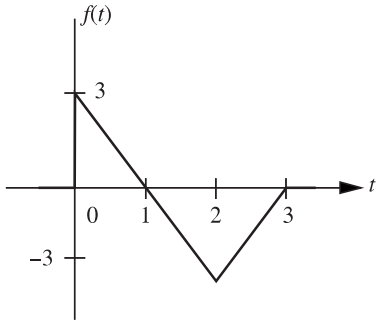

Figure 5.42

SOLUTION

To find $f(t)$ for $0 < t < 2$:
Equation of the straight line 1 is

$$\frac{y - y_1}{x - x_1} = \frac{y_2 - y_1}{x_2 - x_1}$$

Here, y is $f(t)$ and x is t.
Hence,

$$\frac{f(t) - 3}{t - 0} = \frac{-3 - 3}{2 - 0}$$

$$\Rightarrow \qquad 2f(t) - 6 = -6t$$

$$\Rightarrow \qquad f(t) = 3 - 3t$$

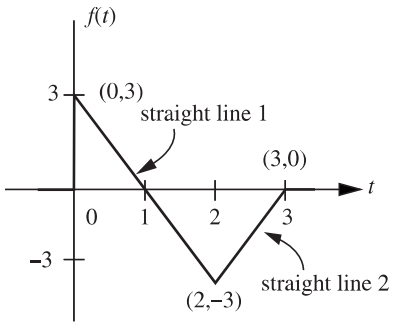

Figure 5.43

To find $f(t)$ for $2 < t < 3$:

Here,

$$\frac{f(t)+3}{t-2} = \frac{0+3}{3-2}$$

$$\Rightarrow \quad f(t)+3 = 3t-6$$

$$\Rightarrow \quad f(t) = 3t-9$$

Hence

$$f(t) = \begin{cases} 3-3t, & 0<t<2 \\ 3t-9, & 2<t<3 \\ 0, & \text{otherwise} \end{cases}$$

The above equation may also be written as :

$$\begin{aligned} f(t) &= [3-3t]\,[u(t)-u(t-2)] + [3t-9]\,[u(t-2)-u(t-3)] \\ &= 3u(t) - 3u(t-2) - 3tu(t) + 3tu(t-2) + 3tu(t-2) \\ &\quad -3tu(t-3) - 9u(t-2) + 9u(t-3) \\ \Rightarrow \quad f(t) &= 3u(t) - 12u(t-2) - 3tu(t) + 6tu(t-2) - 3tu(t-3) + 9u(t-3) \\ &= 3u(t) - 12u(t-2) - 3tu(t) + 6(t-2+2)\,u(t-2) \\ &\quad -3(t-3+3)\,u(t-3) + 9u(t-3) \\ &= 3u(t) - 12u(t-2) - 3tu(t) + 6(t-2)\,u(t-2) \\ &\quad +12u(t-2) - 3(t-3)\,u(t-3) - 9u(t-3) + 9u(t-3) \\ \boldsymbol{f(t)} &\boldsymbol{= 3u(t) - 3tu(t) + 6(t-2)\,u(t-2) - 3(t-3)\,u(t-3)} \end{aligned}$$

Hence,

$$\begin{aligned} F(s) &= \mathcal{L}\{f(t)\} \\ &= \frac{\mathbf{3}}{\boldsymbol{s}} - \frac{\mathbf{3}}{\boldsymbol{s^2}} + \frac{\mathbf{6}}{\boldsymbol{s^2}}\boldsymbol{e^{-2s}} - \frac{\mathbf{3}}{\boldsymbol{s^2}}\boldsymbol{e^{-3s}} \end{aligned}$$

EXAMPLE 5.29

Express the function $f(t)$ shown in Fig. 5.44 using singularity functions and then find $F(s)$.

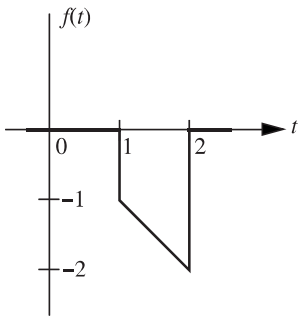

Figure 5.44

SOLUTION

Equation of the straight line shown in Fig. 5.45(a) is

$$\frac{f_1(t)+1}{t-1} = \frac{-2+1}{2-1}$$

$$\Rightarrow \quad f_1(t)+1 = -t+1$$

$$\Rightarrow \quad f_1(t) = -t$$

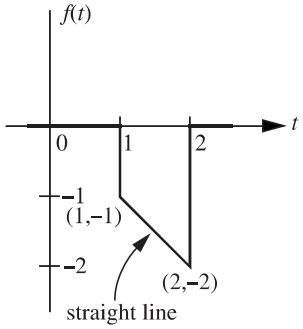

Figure 5.45(a)

The above equation is for the values t lying between 1 and 2. This could be expressed, by writing

$$f(t) = f_1(t)\, g(t)$$

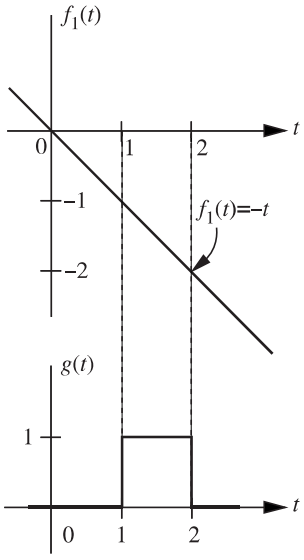

Figure 5.45(b)

$$\Rightarrow \quad f(t) = -t\,[u(t-1) - u(t-2)]$$
$$= -(t-1+1)\,u(t-1) + (t-2+2)\,u(t-2)$$
$$= -(t-1)\,u(t-1) - u(t-1) + (t-2)\,u(t-2) + 2u(t-2)$$
$$= -r(t-1) - u(t-1) + r(t-2) + 2u(t-2)$$

Hence,

$$F(s) = \mathcal{L}\{f(t)\}$$
$$= -\frac{1}{s^2}e^{-s} - \frac{1}{s}e^{-s} + \frac{1}{s^2}e^{-2s} + \frac{2}{s}e^{-2s}$$

EXAMPLE 5.30

Find the Laplace transform of the function $f(t)$ shown in Fig. 5.46.

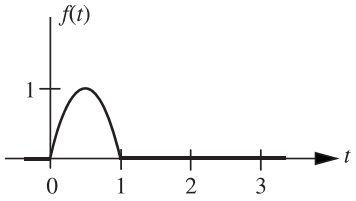

Figure 5.46

SOLUTION

Method 1:

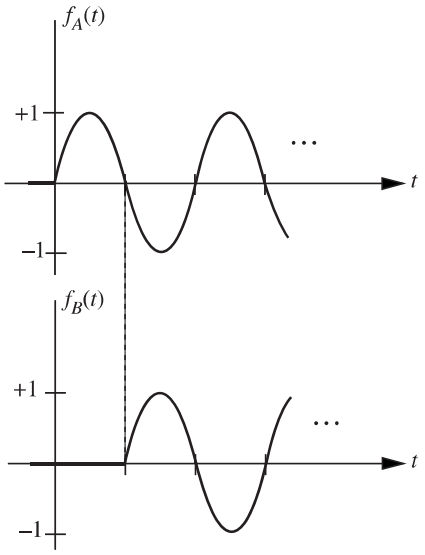

Figure 5.47(a)

We can write,
$$\begin{aligned} f(t) &= f_A(t) + f_B(t) \\ &= \sin \pi t u(t) + \sin \pi (t-1)\, u(t-1) \end{aligned}$$

Hence,
$$\begin{aligned} F(s) &= \mathcal{L}\{f(t)\} = \frac{\pi}{s^2+\pi^2} + \frac{\pi}{s^2+\pi^2} e^{-s} \\ &= \boldsymbol{\frac{\pi}{s^2+\pi^2}\left[1+e^{-s}\right]} \end{aligned}$$

Method 2 :

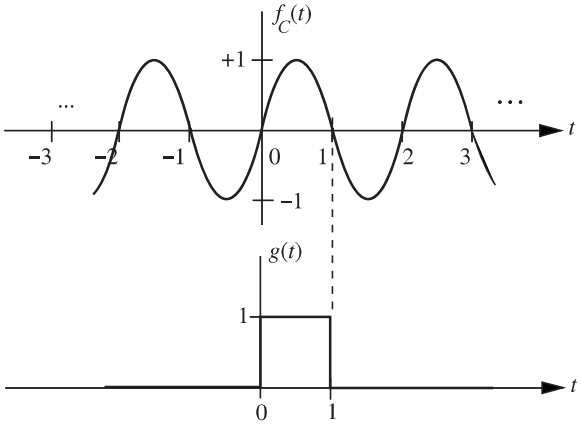

Figure 5.47(b)

Graphically, we can manipulate $f(t)$ as

$$\begin{aligned} f(t) &= f_{\mathrm{C}}(t)\, g(t) \\ &= \sin \pi t\, [u(t) - u(t-1)] \\ &= \sin \pi t u(t) - \sin \pi t u(t-1) \\ &= \sin \pi t u(t) - \sin \pi t\, (t-1+1)\, [u(t-1)] \\ &= \sin \pi t u(t) - \sin(\pi(t-1) + \pi)\, u(t-1) \\ &= \sin \pi t u(t) + \sin \pi (t-1)\, u(t-1) \end{aligned}$$

Hence,
$$F(s) = \mathcal{L}\{f(t)\} = \frac{\pi}{s^2+\pi^2} + \frac{s}{s^2+\pi^2} e^{-s}$$
$$= \boldsymbol{\frac{\pi}{s^2+\pi^2}\left[1 + e^{-s}\right]}$$

EXAMPLE 5.31

Find the Laplace transform of the signal $x(t)$ shown in Fig. 5.48.

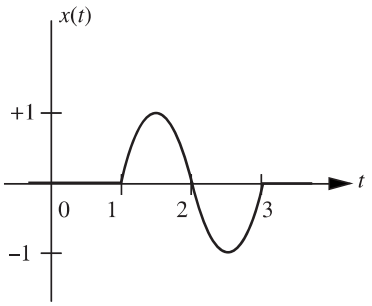

Figure 5.48

SOLUTION

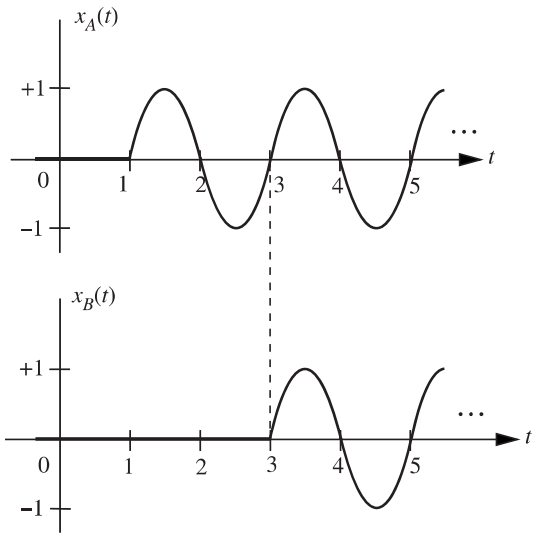

Figure 5.49

Mathematically, we can write $x(t)$ as

$$\begin{aligned} x(t) &= x_A(t) - x_B(t) \\ &= \sin \pi (t-1)\, u(t-1) - \sin \pi (t-3)\, u(t-3) \end{aligned}$$

Hence,
$$\mathcal{L}\{x(t)\} = X(s) = \frac{\pi}{s^2+\pi^2} e^{-s} - \frac{\pi}{s^2+\pi^2} e^{-3s}$$
$$= \boldsymbol{\frac{\pi}{s^2+\pi^2}\left[e^{-s} - e^{-3s}\right]}$$

EXAMPLE 5.32

Refer the waveform shown in Fig. 5.50. The equation for the waveform is $\sin t$ from 0 to π, $-\sin t$ from π to 2π. Show that the Lapalce transform of this waveform is $F(s) = \dfrac{1}{s^2+1} \coth\left(\dfrac{\pi s}{2}\right)$.

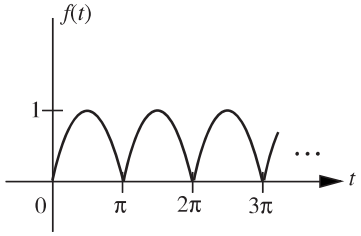

Figure 5.50

SOLUTION

$f(t)$ is a periodic waveform with a period $T = \pi$ seconds. Let $f_1(t)$ be the waveform $f(t)$ described over only one period. The Laplace transform of $f(t)$ and $f_1(t)$ are related as

$$F(s) = \frac{F_1(s)}{1 - e^{-sT}}$$

Let us now proceed to find $F_1(s)$. From Fig. 5.51 (b), we can write

$$\begin{aligned} f_1(t) &= f_A(t) + f_B(t) \\ &= \sin t\, u(t) + \sin(t - \pi)\, u(t - \pi) \\ \Rightarrow \quad F_1(s) &= \frac{1}{s^2 + 1} + \frac{1}{s^2 + 1} e^{-\pi s} \\ &= \frac{(1 + e^{-\pi s})}{s^2 + 1} \end{aligned}$$

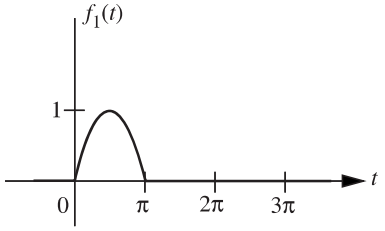

Figure 5.51(a)

Hence, $$F(s) = \frac{F_1(s)}{1 - e^{-sT}} = \frac{F_1(s)}{1 - e^{-\pi s}}$$

$$\Rightarrow \quad F(s) = \frac{(1 + e^{-\pi s})}{(s^2 + 1)(1 - e^{-\pi s})}$$

$$= \frac{1}{s^2 + 1} \frac{e^{-\pi s/2} \dfrac{\left[e^{\pi s/2} + e^{-\pi s/2}\right]}{2}}{e^{-\pi s/2} \dfrac{\left[e^{\pi s/2} - e^{-\pi s/2}\right]}{2}}$$

$$\Rightarrow \quad F(s) = \frac{1}{s^2 + 1} \frac{\cosh\left(\frac{\pi s}{2}\right)}{\sinh\left(\frac{\pi s}{2}\right)}$$

$$= \frac{\mathbf{1}}{\boldsymbol{s^2 + 1}} \mathbf{coth}\left(\frac{\boldsymbol{\pi s}}{\mathbf{2}}\right)$$

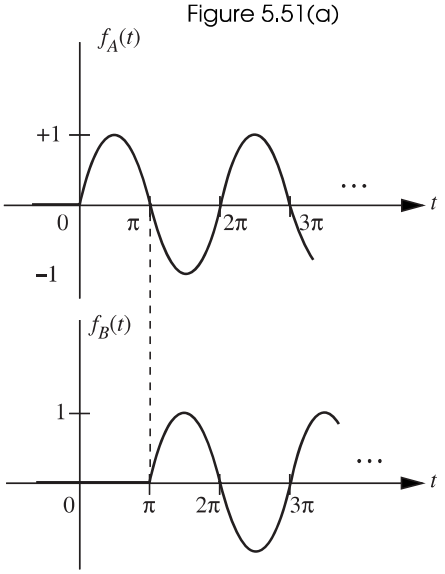

Figure 5.51(b)

EXAMPLE 5.33

Find the Laplace transform of the pulse shown in Fig. 5.52.

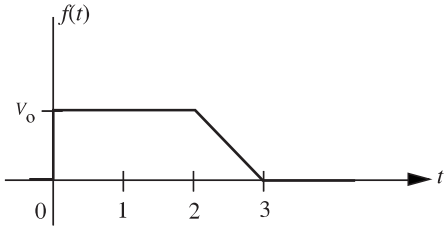

Figure 5.52

SOLUTION

We can describe Fig. 5.52 mathematically as

$$f(t) = \begin{cases} V_o, & 0 < t < 2 \\ -V_o t + 3V_o, & 2 < t < 3 \end{cases}$$

The expression for $f(t)$ for $2 < t < 3$ is obtianed as follows :
Equation of a straight line between two points is given by

$$\frac{y - y_1}{x - x_1} = \frac{y_2 - y_1}{x_2 - x_1}$$

In the present context, $y = f(t), x = t, (x_1, y_1) = (2, V_o)$ and $(x_2, y_2) = (3, 0)$

Hence, $$\frac{f(t) - V_o}{t - 2} = \frac{0 - V_o}{3 - 2}$$

$$\Rightarrow \qquad f(t) = -V_o t + 3V_o$$

The time domain expression for $f(t)$ between $t = 0$ and 3 could be written using graphical manipulation as

$$f(t) = V_o [u(t) - u(t-2)] + [-V_o t + 3V_o] [u(t-2) - u(t-3)]$$

The first term on the right-side of the above equation defines $f(t)$ for $0 < t < 2$ and the second term on the right-side defines $f(t)$ for $2 < t < 3$.

$$\begin{aligned} f(t) &= V_o u(t) - V_o u(t-2) - V_o t u(t-2) + V_o t u(t-3) + 3V_o u(t-2) - 3V_o u(t-3) \\ &= V_o u(t) - V_o u(t-2) - V_o(t - 2 + 2) u(t-2) \\ &\quad + V_o(t - 3 + 3) u(t-3) + 3V_o u(t-2) - 3V_o u(t-3) \\ &= V_o u(t) - V_o u(t-2) - V_o(t-2) u(t-2) - 2V_o u(t-2) + V_o(t-3) u(t-3) \\ &\quad + 3V_o u(t-3) + 3V_o u(t-2) - 3V_o u(t-3) \\ &= V_o u(t) - V_o(t-2) u(t-2) + V_o(t-3) u(t-3) \\ \Rightarrow \quad f(t) &= V_o u(t) - V_o r(t-2) + V_o r(t-3) \end{aligned}$$

Hence, $$\begin{aligned} F(s) &= \mathcal{L}\{f(t)\} \\ &= \boldsymbol{\frac{V_o}{s} - \frac{V_o}{s^2} e^{-2s} + \frac{V_o}{s^2} e^{-3s}} \end{aligned}$$

EXAMPLE 5.34

Consider a staircase waveform which extends to infinity and at $t = nt_0$ jumps to the value $n + 1$, being a superposition of unit step functions. Determine the Laplace transform of this waveform.

SOLUTION

We can write,

$$f(t) = u(t) + u(t - t_0) + u(t - 2t_0) + \cdots$$

$$F(s) = \mathcal{L}\{f(t)\} = \frac{1}{s} + \frac{1}{s}e^{-t_0 s} + \frac{1}{s}e^{-2t_0 s} + \cdots$$

$$= \frac{1}{s}\left[1 + e^{-t_0 s} + e^{-2t_0 s} + \cdots\right]$$

Let $e^{-t_0 s} = x$

then $F(s) = \frac{1}{s}\left[1 + x + x^2 + \cdots\right]$

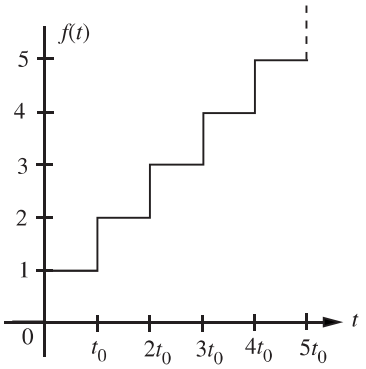

Figure 5.53

From Binomial theorem, we have

$$(1 - x)^{-1} = 1 + x + x^2 + \cdots$$

Hence, $$F(s) = \frac{1}{s(1 - x)}$$

$$= \frac{\mathbf{1}}{\boldsymbol{s\left(1 - e^{-t_0 s}\right)}}$$

EXAMPLE 5.35

(a) Find the Laplace transform of the staircase waveform shown in Fig. 5.54. (b) If this voltage were applied to an RL series circuit with $R = 1\Omega$ and $L = 1H$, find the current $i(t)$.

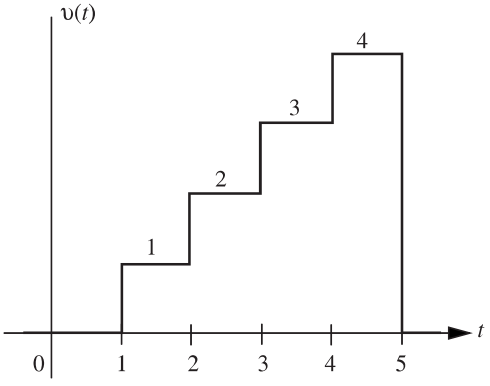

Figure 5.54

SOLUTION

(a) We can express mathematically, the voltage waveform shown in Fig. 5.54 as,

$$v(t) = \begin{cases} 1, & 1 < t < 2 \\ 2, & 2 < t < 3 \\ 3, & 3 < t < 4 \\ 4, & 4 < t < 5 \\ 0, & \text{elsewhere} \end{cases}$$

or $$\begin{aligned} v(t) &= [u(t-1) - u(t-2)] + 2[u(t-2) - u(t-3)] \\ &\quad +3[u(t-3) - u(t-4)] + 4[u(t-4) - u(t-5)] \\ &= u(t-1) + u(t-2) + u(t-3) + u(t-4) - 4u(t-5) \end{aligned}$$

Taking the Laplace transform, we get

$$\boldsymbol{V(s) = \frac{1}{s}\left[e^{-s} + e^{-2s} + e^{-3s} + e^{-4s} - 4e^{-5s}\right]}$$

(b) Assuming all initial conditions to be zero, the time domian circuit shown in Fig. 5.55 gets transformed to a circuit as shown in Fig. 5.56.

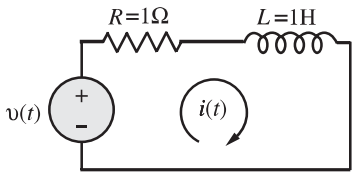

Figure 5.55 Time Domain Circuit

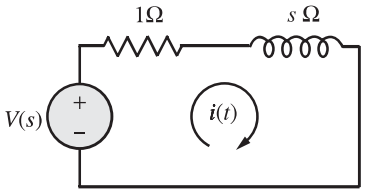

Figure 5.56 Frequency Domain Circuit

From Fig. 5.56, we can write

$$I(s) = \frac{V(s)}{s+1}$$

$$\Rightarrow\ I(s) = \frac{1}{s(s+1)}e^{-s} + \frac{1}{s(s+1)}e^{-2s} + \frac{1}{s(s+1)}e^{-3s} + \frac{1}{s(s+1)}e^{-4s} - \frac{4}{s(s+1)}e^{-5s}$$

$$\begin{aligned} \Rightarrow\ I(s) = \Bigg[&\left(\frac{1}{s} - \frac{1}{s+1}\right)e^{-s} + \left(\frac{1}{s} - \frac{1}{s+1}\right)e^{-2s} + \left(\frac{1}{s} - \frac{1}{s+1}\right)e^{-3s} \\ &+ \left(\frac{1}{s} - \frac{1}{s+1}\right)e^{-4s} - 4\left(\frac{1}{s} - \frac{1}{s+1}\right)e^{-5s}\Bigg] \end{aligned}$$

Taking the inverse Laplace transform, we get

$$\begin{aligned} i(t) &= \left[u(t) - e^{-t}u(t)\right]_{t\to t-1} + \left[u(t) - e^{-t}u(t)\right]_{t\to t-2} + \left[u(t) - e^{-t}u(t)\right]_{t\to t-3} \\ &\quad + \left[u(t) - e^{-t}u(t)\right]_{t\to t-4} - 4\left[u(t) - e^{-t}u(t)\right]_{t\to t-5} \end{aligned}$$

$$\begin{aligned} \boldsymbol{\Rightarrow i(t)} &= \boldsymbol{\left[1 - e^{-(t-1)}\right]u(t-1) + \left[1 - e^{-(t-2)}\right]u(t-2) + \left[1 - e^{-(t-3)}\right]u(t-3)} \\ &\quad \boldsymbol{+ \left[1 - e^{-(t-4)}\right]u(t-4) - 4\left[1 - e^{-(t-5)}\right]u(t-5)} \end{aligned}$$

EXAMPLE 5.36

A voltage pulse of 10 V magnitude and 5 μsec duration is applied to the RC network shown in Fig. 5.57. Find the current $i(t)$ if $R = 10\Omega$ and $C = 0.05\mu F$.

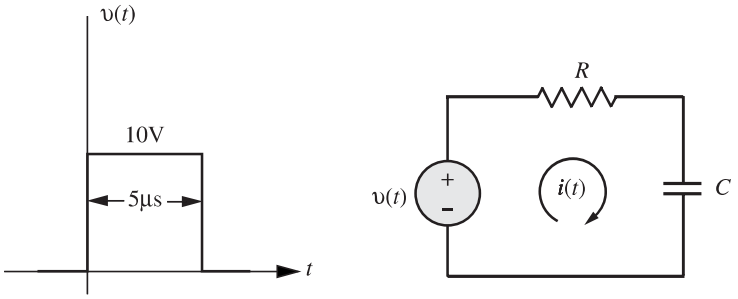

Figure 5.57

SOLUTION

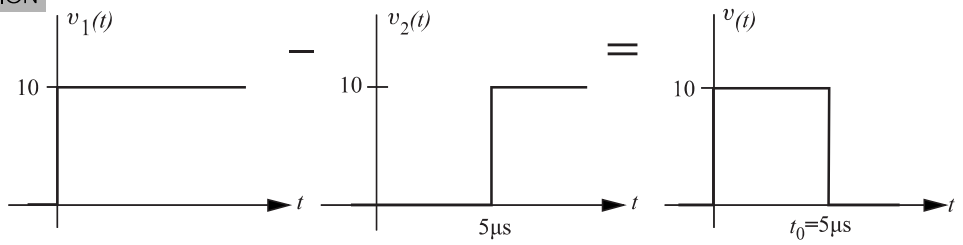

Figure 5.58(a)

Mathematically, we can express $v(t)$ as follows :

$$\begin{aligned} v(t) &= v_1(t) - v_2(t) \\ &= 10u(t) - 10u(t - t_0) \end{aligned}$$

Hence,
$$\begin{aligned} V(s) &= \mathcal{L}\{v(t)\} \\ &= \frac{10}{s}\left[1 - e^{-t_0 s}\right] \end{aligned}$$

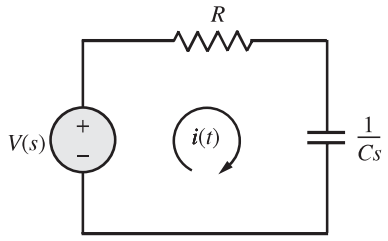

Figure 5.58(b)

Assuming all initial conditions to be zero, the Laplace transformed network is as shown in Fig. 5.58(b).

$$\begin{aligned} I(s) &= \frac{V(s)}{R + \dfrac{1}{Cs}} \\ &= \frac{10\left(1 - e^{-t_0 s}\right)}{s\left[R + \dfrac{1}{Cs}\right]} \end{aligned}$$

$$I(s) = \frac{10Cs}{s(RCs+1)}\left(1 - e^{-t_0 s}\right)$$
$$= \frac{10}{R}\frac{1}{s + \dfrac{1}{RC}}\left(1 - e^{-t_0 s}\right)$$
$$= \frac{10}{R}\left[\frac{1}{s + \dfrac{1}{RC}} - \frac{1}{s + \dfrac{1}{RC}}e^{-t_0 s}\right]$$

Taking inverse Laplace transform yields

$$i(t) = \frac{10}{R}e^{\frac{-t}{RC}}u(t) - \frac{10}{R}e^{\frac{-t}{RC}}u(t)\Big|_{t \to t - t_0}$$
$$= \frac{10}{R}e^{\frac{-t}{RC}}u(t) - \frac{10}{R}e^{\frac{-(t-t_0)}{RC}}u(t - t_0)$$
$$\boldsymbol{i(t) = e^{\frac{-t}{0.5\times10^{-6}}}u(t) - e^{\frac{-\left(t-5\times10^{-6}\right)}{0.5\times10^{-6}}}u\left(t - 5\times10^{-6}\right)}$$

EXAMPLE 5.37

Find the Laplace transform of the waveform shown in Fig. 5.59.

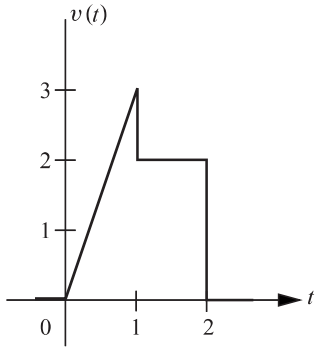

Figure 5.59

SOLUTION

$$v(t) = \begin{cases} 3t, & 0 < t < 1 \\ 2, & 1 < t < 2 \end{cases}$$

or
$$v(t) = 3t\left[u(t) - u(t-1)\right] + 2\left[u(t-1) - u(t-2)\right]$$
$$= 3tu(t) - 3tu(t-1) + 2u(t-1) - 2u(t-2)$$
$$= 3tu(t) - 3(t - 1 + 1)u(t-1) + 2u(t-1) - 2u(t-2)$$
$$= 3tu(t) - 3(t-1)u(t-1) - 3u(t-1) + 2u(t-1) - 2u(t-2)$$

$$v(t) = 3tu(t) - 3(t-1)u(t-1) - u(t-1) - 2u(t-2)$$
$$= 3r(t) - 3r(t-1) - u(t-1) - 2u(t-2)$$

Hence, $V(s) = \mathcal{L}\{v(t)\}$

$$= \frac{\mathbf{3}}{\boldsymbol{s^2}} - \frac{\mathbf{3}}{\boldsymbol{s^2}}\boldsymbol{e^{-s}} - \frac{\mathbf{1}}{\boldsymbol{s}}\boldsymbol{e^{-s}} - \frac{\mathbf{2}}{\boldsymbol{s}}\boldsymbol{e^{-2s}}$$

5.9 The System function

The system function or transfer function of a linear time-invariant system is defined as the ratio of Laplace transform of the output to Laplace transform of the input under the assumption that all initial conditions are zero.

Hence, for relaxed LTI system, the response $Y(s)$ to an input $X(s)$ is $H(s)\,X(s)$, where $H(s)$ is the system function. The system function $H(s)$ may be found in several ways:

1. For a system defined by a linear differential equation, by taking Laplace transform of the differential equation and then finding the ratio $\dfrac{Y(s)}{X(s)}$.
2. From the Laplace transform of impulse response $h(t)$.
3. From the s domain model of a physical system like an electrical system.

EXAMPLE 5.38

The output $y(t)$ of an LTI system is found to be $e^{-3t}u(t)$ when the input $x(t)$ is $0.5u(t)$.

(a) Find the impulse response $h(t)$ of the system.

(b) Find the output when the input is $x(t) = e^{-t}u(t)$.

SOLUTION

(a) Taking Laplace transforms of $x(t)$ and $y(t)$, we get

$$Y(s) = \frac{1}{s+3},\ X(s) = \frac{0.5}{s}$$

Hence $$H(s) = \frac{Y(s)}{X(s)} = \frac{2s}{s+3}$$

$$\Rightarrow \quad H(s) = \frac{2(s+3)-6}{(s+3)} = 2 - \frac{6}{s+3}$$

Taking inverse Laplace transform, we get

$$\boldsymbol{h(t) = 2\delta(t) - 6e^{-3t}u(t)}$$

(b) $$x(t) = e^{-t}u(t)$$

$$\Rightarrow \qquad X(s) = \frac{1}{s+1}$$

Thus,
$$\begin{aligned} Y(s) &= X(s)\,H(s) \\ &= \frac{2s}{(s+1)(s+3)} \\ &= \frac{K_1}{s+1} + \frac{K_2}{s+3} \end{aligned}$$

where
$$K_1 = \left.\frac{2s}{s+3}\right|_{s=-1} = -1$$

$$K_2 = \left.\frac{2s}{s+1}\right|_{s=-3} = 3$$

Therefore,
$$Y(s) = \frac{-1}{s+1} + \frac{3}{s+3}$$

Taking inverse Laplace transform of $Y(s)$, we get

$$y(t) = -e^{-t} + 3e^{-3t},\ t \geq 0$$

or
$$y(t) = \left(-e^{-t} + 3e^{-3t}\right), u(t)$$

EXAMPLE 5.39

Determine the output $v(t)$ for the circuit shown in Fig. 5.60.

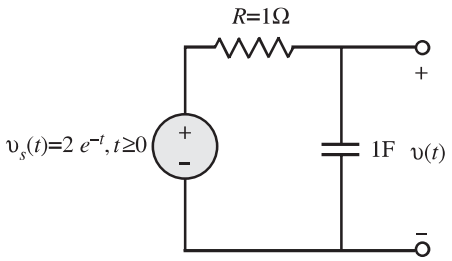

Figure 5.60

SOLUTION

The transformed network of Fig. 5.60 with the assumption that all initial conditions are zero is shown in Fig. 5.61(a).

$$V(s) = \frac{1}{s} I(s)$$

$$= \frac{1}{s}\left[\frac{V_s(s)}{1+\frac{1}{s}}\right]$$

$$\Rightarrow \qquad H(s) = \frac{V(s)}{V_s(s)} = \frac{1}{s+1}$$

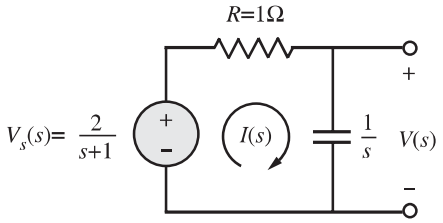

Figure 5.61(a)

The inverse Laplace transform of $H(s)$ is called the impulse response of the circuit and is denoted by $h(t)$.

$$h(t) = e^{-t}u(t)$$

I method :

From Convolution theorem, we have,

$$v(t) = h(t) * v_s(t)$$

$$= \int_0^\infty h(\tau)\, v_s(t-\tau)\, d\tau$$

$$= \int_0^\infty e^{-\tau}u(\tau) \times 2e^{-(t-\tau)}u(t-\tau)\, d\tau$$

$$= 2e^{-t}\int_0^\infty u(\tau)\, u(t-\tau)\, d\tau$$

Let us compute the product $u(\tau)\, u(t-\tau)$ for different values of τ

$$u(\tau) = \begin{cases} 1, & \tau < 0 \\ 0, & \tau > 0 \end{cases}$$

$$u(t-\tau) = \begin{cases} 1, & t-\tau > 0 \quad \text{or} \quad \tau < t \\ 0, & t-\tau < 0 \quad \text{or} \quad \tau > t \end{cases}$$

Hence, $$u(\tau)\, u(t-\tau) = \begin{cases} 1, & 0 < \tau < t, \ t > 0 \\ 0, & \text{otherwise} \end{cases}$$

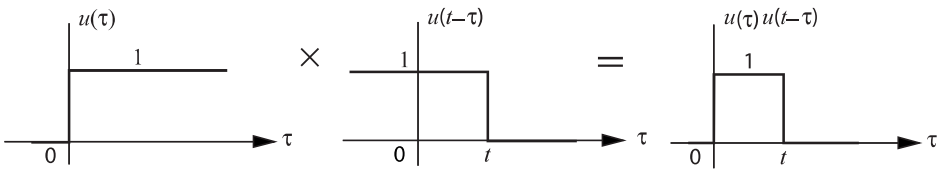

Figure 5.61(b)

Therefore, $$v(t) = 2e^{-t}\int_0^t d\tau = 2te^{-t},\ t \geq 0$$
$$= \mathbf{2te^{-t}u(t)}$$

II method :

In the frequency domain, convolution operation is transformed into a multiplicative operation.

That is, $$V(s) = H(s)\,V_s(s)$$
$$= \frac{1}{(s+1)} \times \frac{2}{(s+1)}$$
$$= \frac{2}{(s+1)^2}$$

Inverse Laplace transform yields,

$$\boldsymbol{v(t) = 2te^{-t}u(t)\text{Volts}}$$

Reinforcement problems

R.P 5.22

(a) Find $H(s) = \dfrac{V_o(s)}{V_i(s)}$ for the circuit shown in Fig. R.P. 5.22. (b) Determine $v_o(t)$ when the intital current in the inductor is zero.

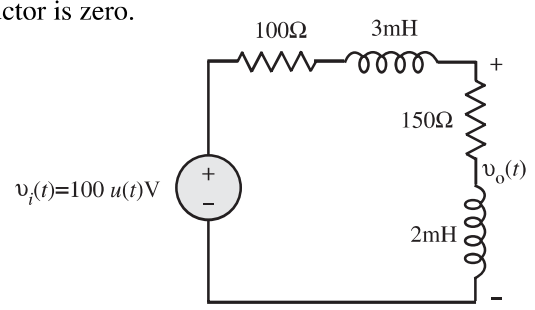

Figure R.P.5.22

SOLUTION

The Laplace transformed network with all initial conditions set to zero is shown in Fig. R.P. 5.22(a).

$$V_o(s) = I(s)\left[150 + 2\times 10^{-3}s\right]$$
$$= \frac{V_i(s)\left[150 + 2\times 10^{-3}s\right]}{100 + 3\times 10^{-3}s + 150 + 2\times 10^{-3}s}$$

$$\Rightarrow \qquad H(s) = \frac{V_o(s)}{V_i(s)} = \frac{\mathbf{1.5\times 10^5 + 2s}}{\mathbf{2.5\times 10^5 + 5s}}$$

(b) $V_o(s) = H(s)\,V_i(s)$

$$= \frac{1.5 \times 10^5 + 2s}{2.5 \times 10^5 + 5s} \times \frac{100}{s}$$

$$= \frac{40\left[s + 0.75 \times 10^5\right]}{s\left[s + 0.5 \times 10^5\right]}$$

$$= \frac{K_1}{s} + \frac{K_2}{s + 0.5 \times 10^5}$$

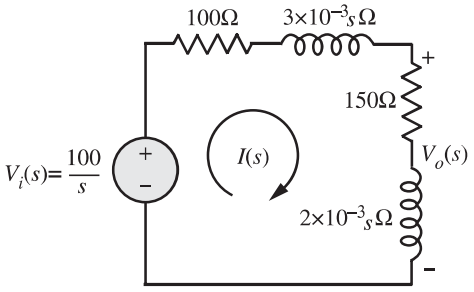

Figure R.P. 5.22(a)

where

$$K_1 = \left.\frac{40\left[s + 0.75 \times 10^5\right]}{\left[s + 0.5 \times 10^5\right]}\right|_{s=0} = 60$$

$$K_2 = \left.\frac{40\left[s + 0.75 \times 10^5\right]}{s}\right|_{s=-0.5\times10^5} = -20$$

Hence, $$V_o(s) = \frac{60}{s} - \frac{20}{s + 0.5 \times 10^5}$$

Taking inverse Laplace transform, we get

$$\boldsymbol{v_o(t) = \left[60 - 20e^{-0.5\times105t}\right] u(t)\ \text{Volts}}$$

R.P 5.23

Refer the circuit shown in Fig. R.P. 5.23. The switch closes at $t = 0$. Determine the voltage $v(t)$ after the switch closes.

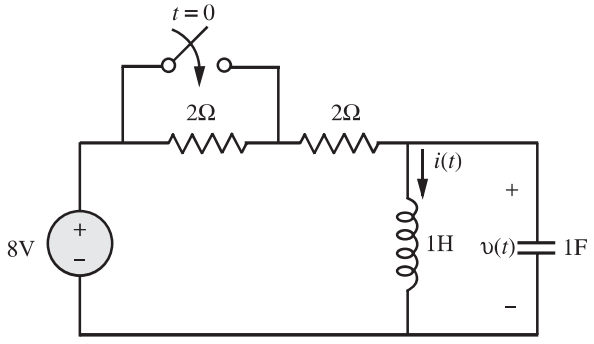

Figure R.P. 5.23

SOLUTION

The switch is open at $t = 0^-$ and closed at $t = 0^+$. Let us assume that at $t = 0^-$, the circuit is in steady state. The circuit at $t = 0^-$ is shown in Fig. R.P. 5.23(a).

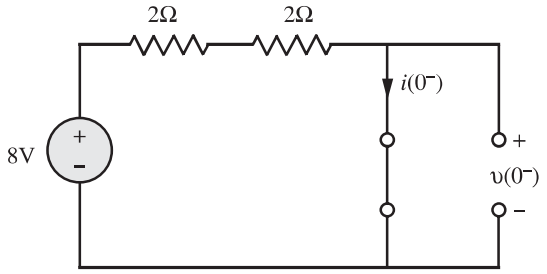

Figure R.P. 5.23(a)

Referring to Fig. R.P. 5.23(a), we get

$$i\left(0^{-}\right)=\frac{8}{2+2}=2\text{A}$$
$$v\left(0^{-}\right)=0$$

From switching principles, we know that the current through an inductor and the voltage across a capacitor cannot change instantaneously. Therefore,

$$i\left(0\right)=i\left(0^{+}\right)=i\left(0^{-}\right)=2\text{A}$$

and

$$v\left(0\right)=v\left(0^{+}\right)=v\left(0^{-}\right)=0\text{V}$$

We shall solve this probelm using nodal technique. Hence, in the frequency domain, we will use the parallel models for the capacitor and inductor because the parallel models contain current sources rather than voltage sources. The frequency domain circuit is shown in Fig. R.P. 5.23(b).

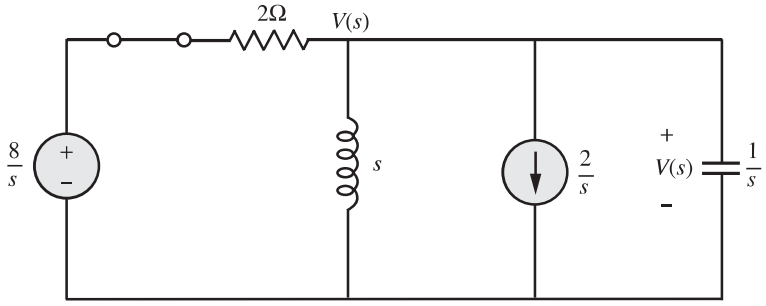

Figure R.P.5.23(b)

KCL at node $V(s)$:

$$\frac{V(s)-\frac{8}{s}}{2}+\frac{V(s)}{s}+\frac{V(s)}{\frac{1}{s}}+\frac{2}{s}=0$$

$$\Rightarrow \qquad V\left(s\right)\left[\frac{1}{2}+\frac{1}{s}+s\right]=\frac{4}{s}-\frac{2}{s}$$

$$\Rightarrow \qquad V\left(s\right)\left[\frac{s+2+2s^{2}}{2s}\right]=\frac{2}{s}$$

$$\Rightarrow \qquad V(s) = \frac{4}{2s^2 + s + 2}$$

$$= \frac{2}{s^2 + 0.5s + 1}$$

$$= \frac{2}{s^2 + 0.5s + (0.25)^2 - (0.25)^2 + 1}$$

$$= \frac{2}{(s + 0.25)^2 + (0.96824)^2}$$

$$= \frac{2}{0.96824} \times \frac{0.96824}{(s + 0.25)^2 + (0.96824)^2}$$

$$= 2.066 \times \frac{0.96824}{(s + 0.25)^2 + (0.96824)^2}$$

We know that,

$$\mathcal{L}^{-1}\left\{\frac{a}{(s+b)^2 + a^2}\right\} = e^{-bt} \sin at\, u(t)$$

Hence,

$$\boldsymbol{v(t) = 2.066e^{-0.25t} \sin(0.96824t)\, u(t)\ \text{Volts}}$$

R.P 5.24

Find the impulse response of the circuit shown in Fig. R.P. 5.24.

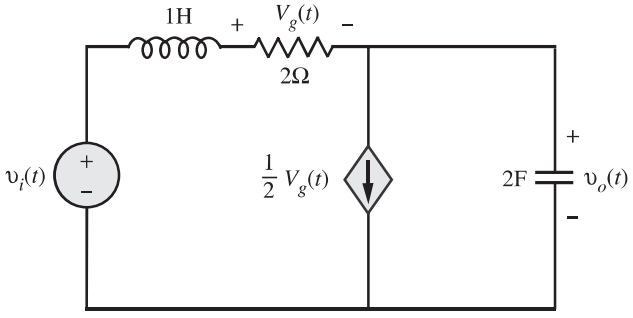

Figure R.P. 5.24

SOLUTION

The frequency domain representation of the circuit is shown in Fig. R.P. 5.24(a) by assuming that all initial conditions to be zero.

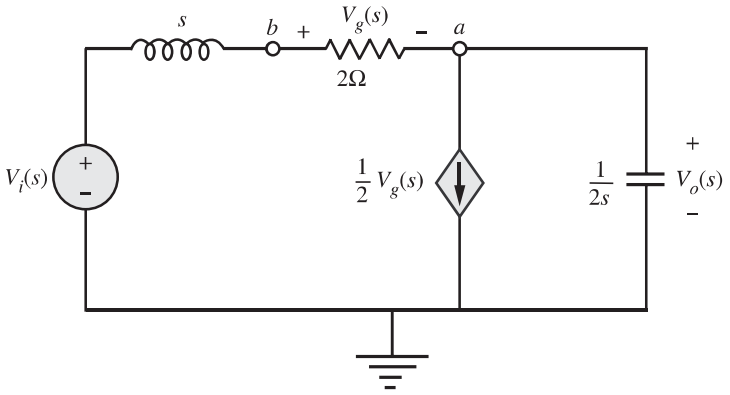

Figure R.P. 5.24(a)

KCL at node a:

$$\frac{V_a(s) - V_b(s)}{2} + \frac{1}{2}V_g(s) + \frac{V_a(s)}{\frac{1}{2s}} = 0$$

$$\Rightarrow \quad \frac{V_a(s) - V_b(s)}{2} + \frac{V_a(s) - V_b(s)}{2} + 2sV_a(s) = 0$$

$$\Rightarrow \quad V_a(s)\left[\frac{1}{2} + \frac{1}{2} + 2s\right] - V_b(s)\left[\frac{1}{2} + \frac{1}{2}\right] = 0$$

$$\Rightarrow \quad V_a(s)[1 + 2s] - V_b(s) = 0$$

KCL at node b:

$$\frac{V_b(s) - V_i(s)}{s} + \frac{V_b(s) - V_a(s)}{2} = 0$$

$$\Rightarrow \quad V_a(s)\left[\frac{-1}{2}\right] + V_b(s)\left[\frac{1}{s} + \frac{1}{2}\right] = \frac{V_i(s)}{s}$$

$$\Rightarrow \quad \frac{-V_a(s)}{2} + \frac{(2+s)}{2s}V_b(s) = \frac{V_i(s)}{s}$$

$$\Rightarrow \quad -sV_a(s) + (2+s)V_b(s) = 2V_i(s)$$

Putting the above nodal equations in matrix form, we get

$$\begin{bmatrix} 1+2s & -1 \\ -s & 2+s \end{bmatrix} \begin{bmatrix} V_a(s) \\ V_b(s) \end{bmatrix} = \begin{bmatrix} 0 \\ 2V_i(s) \end{bmatrix}$$

Solving, we get

$$V_a(s) = \frac{2V_i(s)}{2 + s + 4s + 2s^2 - s}$$

$$\Rightarrow \quad \frac{V_a(s)}{V_i(s)} = \frac{1}{(s+1)^2}$$

Given $\quad v_i(t) = \delta(t) \quad \Rightarrow \quad V_i(s) = 1$

Hence, $$\frac{V_a(s)}{1} = \frac{1}{(s+1)^2}$$

$$\Rightarrow \quad V_a(s) = \frac{1}{(s+1)^2}$$

Taking inverse Laplace transform, we get

$$\boldsymbol{v_a(t) = h(t) = te^{-t}u(t)}$$

R.P 5.25

Find the convolution of $h(t) = t$ and $f(t) = e^{-\propto t}$ for $t > 0$, using the inverse transform of $H(s)F(s)$.

SOLUTION

$$h(t) * f(t) = \mathscr{L}^{-1}\{H(s)F(s)\}$$

where $$H(s) = \mathscr{L}\{h(t)\} = \frac{1}{s^2}$$

$$F(s) = \mathscr{L}\{f(t)\} = \frac{1}{s+\alpha}$$

Hence, $$H(s)F(s) = \frac{1}{s^2(s+\alpha)}$$

$$= \frac{K_1}{s} + \frac{K_2}{s^2} + \frac{K_3}{s+\alpha}$$

Solving the partial fractions yields

$$K_1 = -\frac{1}{\alpha^2}, \quad K_2 = \frac{1}{\alpha}, \quad K_3 = \frac{1}{\alpha^2}$$

Hence, $$H(s)F(s) = \frac{-1}{\alpha^2}\left(\frac{1}{s}\right) + \frac{1}{\alpha}\left(\frac{1}{s^2}\right) + \frac{1}{\alpha^2}\left(\frac{1}{s+\alpha}\right)$$

$$\Rightarrow \quad h(t) * f(t) = \mathscr{L}^{-1}\{H(s)F(s)\}$$

$$= -\frac{1}{\alpha^2}u(t) + \frac{1}{\alpha}tu(t) + \frac{1}{\alpha^2}e^{-\alpha t}u(t)$$

$$= \boldsymbol{\left[-\frac{1}{\alpha^2} + \frac{t}{\alpha} + \frac{1}{\alpha^2}e^{-\alpha t}\right]u(t)}$$

R.P 5.26

Consider a pulse of amplitude 5V for a duration of 4 seconds with its starting point $t = 0$. Find the convolution of this pulse with itself and draw the convolution $x(t) * x(t)$ versus time.

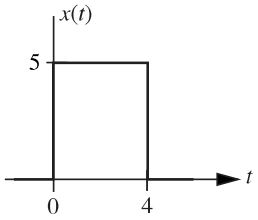

Figure R.P. 5.26(a)

SOLUTION

$$x(t) = 5u(t) - 5u(t-4)$$

$$\Rightarrow \quad X(s) = \frac{5}{s} - \frac{5}{s}e^{-4s}$$

Let $y(t) = x(t) * x(t)$

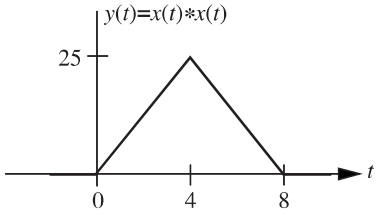

Figure R.P. 5.26(b)

Taking Laplace transform, we get

$$Y(s) = X(s)\,X(s)$$
$$= \frac{25}{s^2} - \frac{50}{s^2}e^{-4s} + \frac{25}{s^2}e^{-8s}$$

Taking inverse Laplace transform, we get

$$y(t) = 25tu(t) - 50(t-4)\,u(t-4) + 25(t-8)\,u(t-8)$$

Hence, $\boldsymbol{y(t) = 25r(t) - 50r(t-4) + 25r(t-8)}$

R.P 5.27

Show that $\mathcal{L}\left\{\dfrac{Kt^{r-1}e^{-at}}{(r-1)!}\right\} = \dfrac{K}{(s+a)^r}$

SOLUTION

Let $f(t) = 1$

then $F(s) = \dfrac{1}{s}$

Thus, $\dfrac{d^nF(s)}{ds^n} = \dfrac{(-1)^n\,n!}{s^{n-1}}$

We know that, $\mathcal{L}\{t^n f(t)\} = (-1)^n \dfrac{d^nF(s)}{ds^n}$

With $f(t) = 1$, we get

$$\mathcal{L}\{t^n\} = (-1)^n\left[\frac{(-1)^n\,n!}{s^{n+1}}\right]$$
$$= \frac{n!}{s^{n+1}}$$

Putting $n = r - 1$, we get

$$\mathcal{L}\{t^{r-1}\} = \frac{(r-1)!}{s^r}$$

and
$$\mathcal{L}\{t^{r-1}e^{-at}\} = \frac{(r-1)!}{(s+a)^r}$$

Therefore,
$$\frac{K}{(r-1)!}\mathcal{L}\{t^{r-1}e^{-at}\} = \frac{K}{(s+a)^r}$$

R.P 5.28

Tests conducted on a certain network revealed that the current was $i(t) = -2e^{-t} + 4e^{-3t}$ when a unit step voltage was suddenly applied to the input terminals of the network at $t = 0$. What voltage must be applied to get an output current of $i(t) = 2e^{-t}$ if the network remains unchanged?

SOLUTION

Given, $i(t) = -2e^{-t} + 4e^{-3t},\ t \geq 0$ when $v(t) = u(t)$

Hence,
$$I(s) = \frac{-2}{s+1} + \frac{4}{s+3}$$

and
$$V(s) = \frac{1}{s}$$

$$\text{System function} = H(s) = \frac{\text{Laplace transform of the output}}{\text{Laplace transform of the input}}$$

$$\Rightarrow \quad H(s) = \frac{I(s)}{V(s)}$$

$$= \frac{2s\,(s-1)}{(s+1)\,(s+3)}$$

We have to find $v(t)$ when $i(t) = 2e^{-t}$.

First we will find $V(s)$ when $I(s) = \dfrac{2}{s+1}$ using the relation $H(s) = \dfrac{I(s)}{V(s)}$.

Hence,
$$V\,(s) = \frac{I\,(s)}{H\,(s)}$$

$$= \frac{\dfrac{2}{s+1}}{\dfrac{2s\,(s-1)}{(s+1)\,(s+3)}}$$

$$= \frac{(s+3)}{s\,(s-1)}$$

$$= \frac{K_1}{s} + \frac{K_2}{s-1}$$

Using partial fractions, we find that $K_1 = -3$ and $K_2 = 4$

Hence, $$V(s) = \frac{-3}{s} + \frac{4}{s-1}$$

$\Rightarrow$ $$\boldsymbol{v(t) = -3u(t) + 4e^{t}u(t) \ \text{Volts}}$$

Outcomes:

1. Use Laplace transforms to investigate the behavior of electric circuits.
2. Ability to analyze RLC circuits and other circuits driven by standard input functions which include steps, ramps, impulses.
3. Be able to apply the Laplace transform technique to circuits which include initial conditions, and solve for the time domain response.
4. Be able to solve for the transfer function of a system response using Laplace transforms.
5. Be able to calculate the response of a linear system to a input signal which is a periodic & non-periodic pulse.

Resources:

1. http://en.wikibooks.org/wiki/Circuit_Theory/Laplace_Transform
2. http://www.ee.nmt.edu/~anders/courses/ee521f06/handout01.pdf
3. http://ocw.nthu.edu.tw/ocw/upload/12/244/12handout.pdf