Module - 2
NETWORK THEOREMS

Objectives:

1. To familiarize the basic laws, source transformations, theorems and the methods of analysing electrical circuits.
2. To explain the use of network theorems and the concept of resonance.
3. To familiarize the analysis of three-phase circuits, two port networks and networks with non-sinusoidal inputs.
4. To explain the importance of initial conditions, their evaluation and transient analysis of R-L and R-C circuits.
5. To impart basic knowledge on network analysis using Laplace transforms.

Many electric circuits are complex, but it is an engineer's goal to reduce their complexity to analyze them easily. In the previous chapters, we have mastered the ability to solve networks containing independent and dependent sources making use of either mesh or nodal analysis. In this chapter, we will introduce new techniques to strengthen our armoury to solve complicated networks. Also, these new techniques in many cases do provide insight into the circuit's operation that cannot be obtained from mesh or nodal analysis. Most often, we are interested only in the detailed performance of an isolated portion of a complex circuit. If we can model the remainder of the circuit with a simple equivalent network, then our task of analysis gets greatly reduced and simplified. For example, the function of many circuits is to deliver maximum power to load such as an audio speaker in a stereo system. Here, we develop the required relationship betweeen a load resistor and a fixed series resistor which can represent the remaining portion of the circuit. Two of the theorems that we present in this chapter will permit us to do just that.

3.1 Superposition theorem

The principle of superposition is applicable only for linear systems. The concept of superposition can be explained mathematically by the following response and excitation principle :

$$i_1 \rightarrow v_1$$
$$i_2 \rightarrow v_2$$
then,
$$i_1 + i_2 \rightarrow v_1 + v_2$$

The quantity to the left of the arrow indicates the excitation and to the right, the system response. Thus, we can state that a device, if excited by a current i_1 will produce a response v_1. Similarly, an excitation i_2 will cause a response v_2. Then if we use an excitation $i_1 + i_2$, we will find a response $v_1 + v_2$.

The principle of superposition has the ability to reduce a complicated problem to several easier problems each containing only a single independent source.

Superposition theorem states that,

In any linear circuit containing multiple independent sources, the current or voltage at any point in the network may be calculated as algebraic sum of the individual contributions of each source acting alone.

When determining the contribution due to a particular independent source, we disable all the remaining independent sources. That is, all the remaining voltage sources are made zero by replacing them with short circuits, and all remaining current sources are made zero by replacing them with open circuits. Also, it is important to note that if a dependent source is present, it must remain active (unaltered) during the process of superposition.

Action Plan:

(i) In a circuit comprising of many independent sources, only one source is allowed to be active in the circuit, the rest are deactivated (turned off).

(ii) To deactivate a voltage source, replace it with a short circuit, and to deactivate a current source, replace it with an open circuit.

(iii) The response obtained by applying each source, one at a time, are then added algebraically to obtain a solution.

Limitations: Superposition is a fundamental property of linear equations and, therefore, can be applied to any effect that is linearly related to the cause. That is, we want to point out that, superposition principle applies only to the current and voltage in a linear circuit but it cannot be used to determine power because power is a non-linear function.

EXAMPLE 3.1

Find the current in the $6\ \Omega$ resistor using the principle of superposition for the circuit of Fig. 3.1.

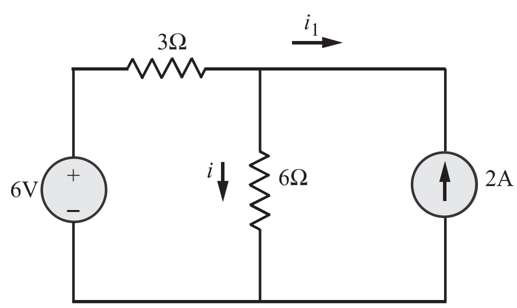


Figure 3.1

SOLUTION

As a first step, set the current source to zero. That is, the current source appears as an open circuit as shown in Fig. 3.2.

$$i_1 = \frac{6}{3+6} = \frac{6}{9}\text{A}$$

As a next step, set the voltage to zero by replacing it with a short circuit as shown in Fig. 3.3.

$$i_2 = \frac{2 \times 3}{3+6} = \frac{6}{9}\text{A}$$

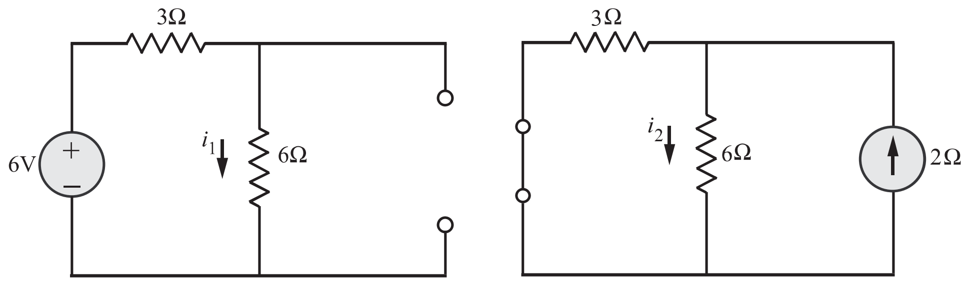


Figure 3.2 Figure 3.3

The total current i is then the sum of i_1 and i_2

$$i = i_1 + i_2 = \mathbf{\frac{12}{9}A}$$

EXAMPLE 3.2

Find i_o in the network shown in Fig. 3.4 using superposition.

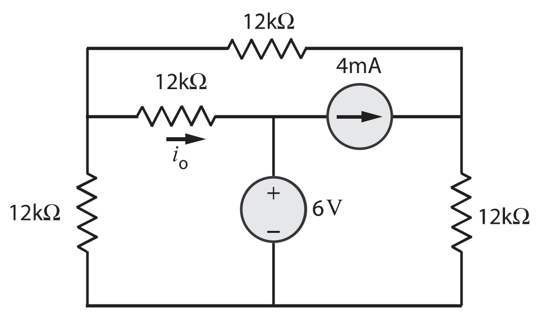


Figure 3.4

SOLUTION

As a first step, set the current source to zero. That is, the current source appears as an open circuit as shown in Fig. 3.5.

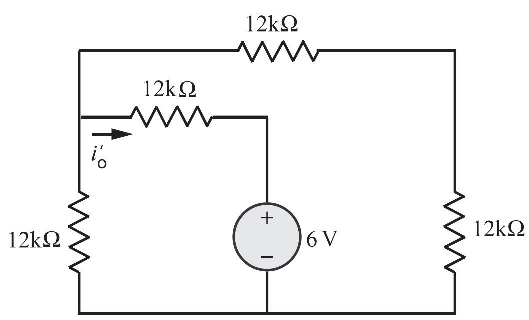
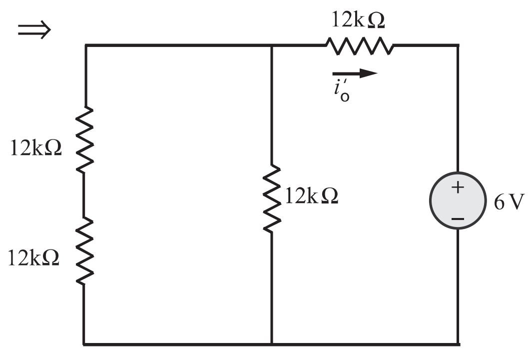


Figure 3.5

⇒



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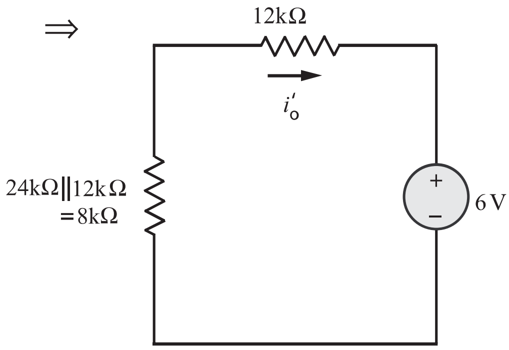


$$i_o' = \frac{-6}{(8+12)\times 10^3} = -0.3 \text{ mA}$$

As a second step, set the voltage source to zero. This means the voltage source in Fig. 3.4 is replaced by a short circuit as shown in Figs. 3.6 and 3.6(a). Using current division principle,

$$i_A = \frac{iR_2}{R_1 + R_2}$$

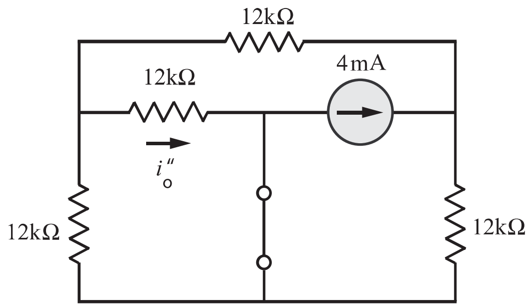


Figure 3.6

where $\quad R_1 = (12 \text{ k}\Omega || 12 \text{ k}\Omega) + 12 \text{ k}\Omega$

$\quad\quad = 6 \text{ k}\Omega + 12 \text{ k}\Omega$

$\quad\quad = 18 \text{ k}\Omega$

and $\quad R_2 = 12 \text{ k}\Omega$

$$\Rightarrow \quad i_A = \frac{4\times 10^{-3} \times 12 \times 10^3}{(12+18)\times 10^3}$$

$\quad\quad = 1.6 \text{ mA}$

Again applying the current division principle,

$$i_o'' = \frac{i_A \times 12}{12+12} = 0.8 \text{ mA}$$

Thus, $\quad i_o = i_o' + i_o'' = -0.3 + 0.8 = \mathbf{0.5\ mA}$

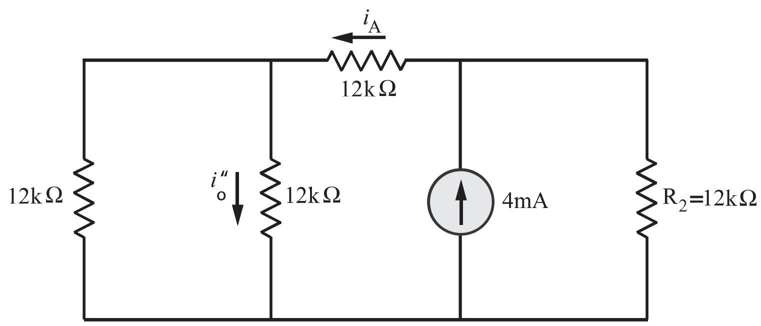


Figure 3.6(a)

EXAMPLE 3.3

Use superposition to find i_o in the circuit shown in Fig. 3.7.

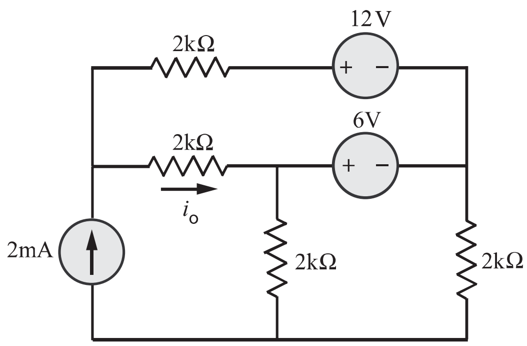


Figure 3.7

SOLUTION

As a first step, keep only the 12 V source active and rest of the sources are deactivated. That is, 2 mA current source is opened and 6 V voltage source is shorted as shown in Fig. 3.8.

$$
\begin{aligned}
i_o' &= \frac{12}{(2+2)\times 10^3} \\
&= 3\ \text{mA}
\end{aligned}
$$

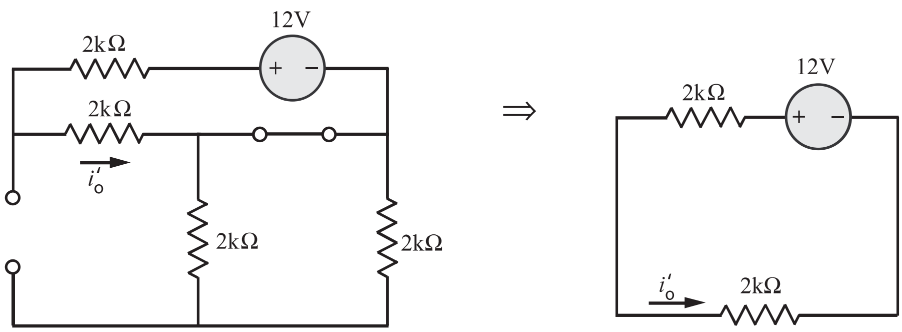


Figure 3.8

As a second step, keep only 6 V source active. Deactivate rest of the sources, resulting in a circuit diagram as shown in Fig. 3.9.

Applying KVL clockwise to the upper loop, we get

$$-2 \times 10^3 i_o'' - 2 \times 10^3 i_o'' - 6 = 0$$

$$\Rightarrow \qquad i_o'' = \frac{-6}{4 \times 10^3} = -1.5 \text{ mA}$$

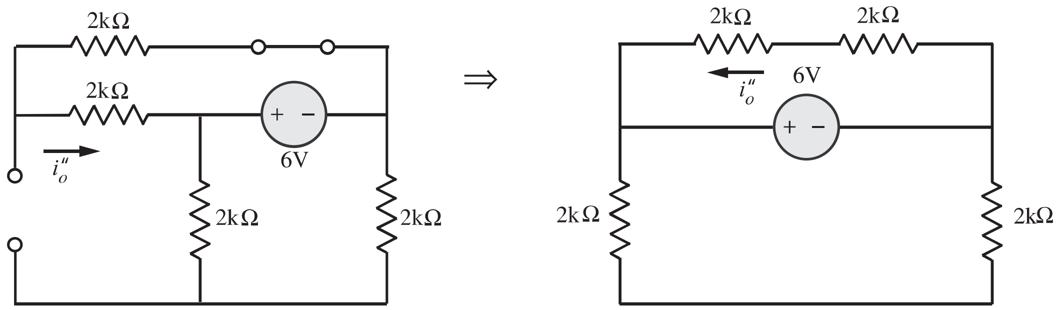


Figure 3.9

As a final step, deactivate all the independent voltage sources and keep only 2 mA current source active as shown in Fig. 3.10.

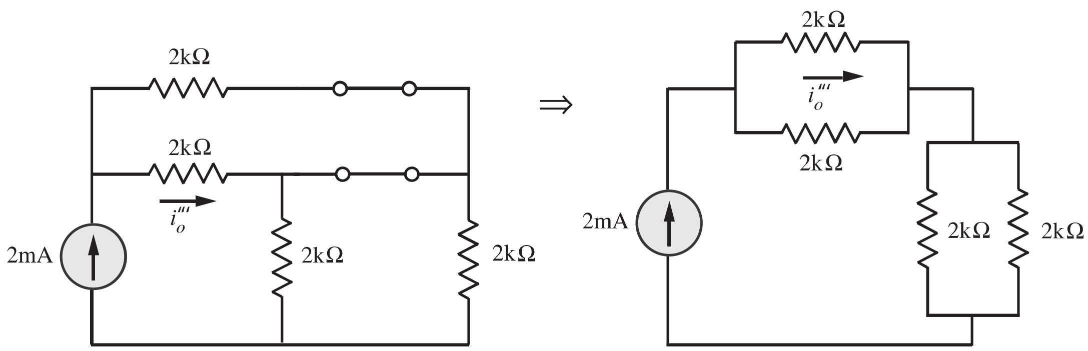


Figure 3.10

Current of 2 mA splits equally.

Hence, $\qquad i_o''' = 1\text{mA}$

Applying the superposition principle, we find that

$$\begin{aligned} i_o &= i_o' + i_o'' + i_o''' \\ &= 3 - 1.5 + 1 \\ &= \mathbf{2.5\ mA} \end{aligned}$$

EXAMPLE 3.4

Find the current i for the circuit of Fig. 3.11.

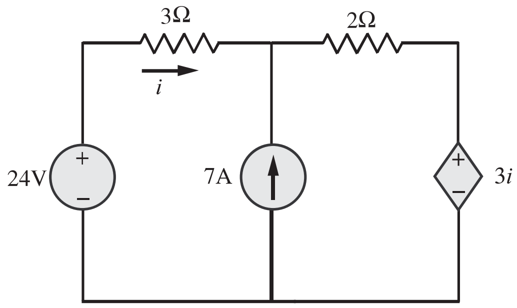


Figure 3.11

SOLUTION

We need to find the current i due to the two independent sources.

As a first step in the analysis, we will find the current resulting from the independent voltage source. The current source is deactivated and we have the circuit as shown as Fig. 3.12.

Applying *KVL* clockwise around loop shown in Fig. 3.12, we find that

$$5i_1 + 3i_1 - 24 = 0$$

$$\Rightarrow \qquad i_1 = \frac{24}{8} = 3\text{A}$$

As a second step, we set the voltage source to zero and determine the current i_2 due to the current source. For this condition, refer to Fig. 3.13 for analysis.

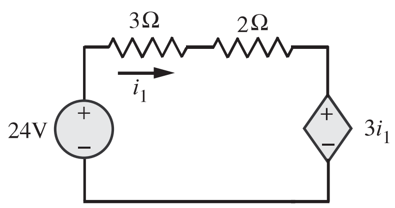


Figure 3.12

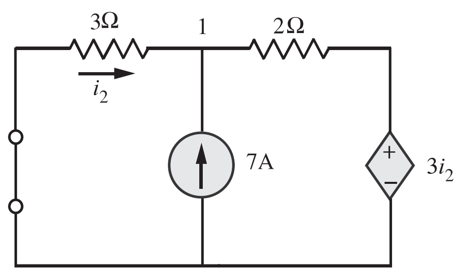


Figure 3.13

Applying KCL at node 1, we get

$$i_2 + 7 = \frac{v_1 - 3i_2}{2} \tag{3.1}$$

Noting that $\qquad -i_2 = \dfrac{v_1 - 0}{3}$

we get, $\qquad v_1 = -3i_2 \qquad (3.2)$

Making use of equation (3.2) in equation (3.1) leads to

$$i_2 + 7 = \frac{-3i_2 - 3i_2}{2}$$

$$\Rightarrow \qquad i_2 = -\frac{7}{4}\text{A}$$

Thus, the total current

$$\begin{aligned} i &= i_1 + i_2 \\ &= 3 - \frac{7}{4}\ \text{A} = \mathbf{\frac{5}{4}A} \end{aligned}$$

EXAMPLE 3.5

For the circuit shown in Fig. 3.14, find the terminal voltage V_{ab} using superposition principle.

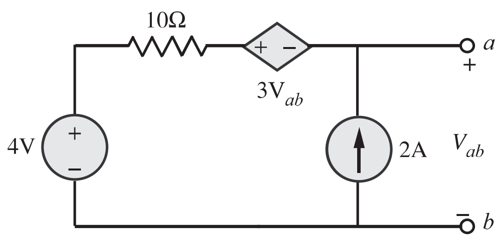


Figure 3.14

SOLUTION

As a first step in the analysis, deactivate the independent current source. This results in a circuit diagram as shown in Fig. 3.15.

Applying KVL clockwise gives

$$-4 + 10 \times 0 + 3V_{ab_1} + V_{ab_1} = 0$$

$$\Rightarrow \qquad 4V_{ab_1} = 4$$

$$\Rightarrow \qquad V_{ab_1} = 1\text{V}$$

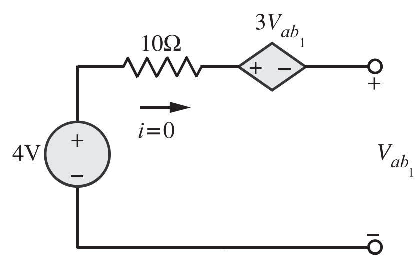


Figure 3.15

Next step in the analysis is to deactivate the independent voltage source, resulting in a circuit diagram as shown in Fig. 3.16.

Applying KVL gives

$$-10 \times 2 + 3V_{ab_2} + V_{ab_2} = 0$$

$$\Rightarrow \qquad 4V_{ab_2} = 20$$

$$\Rightarrow \qquad V_{ab_2} = 5\text{V}$$

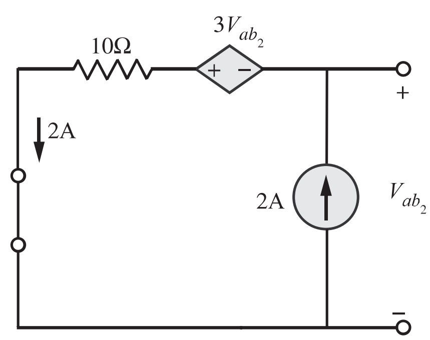


Figure 3.16

According to superposition principle,

$$\begin{aligned} V_{ab} &= V_{ab_1} + V_{ab_2} \\ &= 1 + 5 = \mathbf{6V} \end{aligned}$$

EXAMPLE 3.6

Use the principle of superposition to solve for v_x in the circuit of Fig. 3.17.

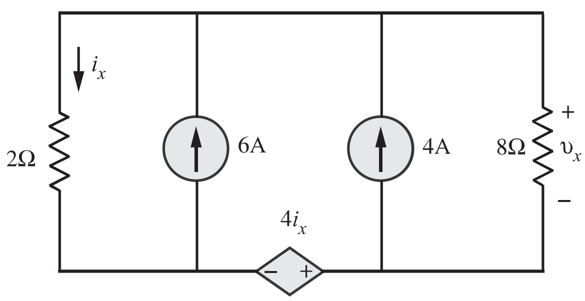


Figure 3.17

SOLUTION

According to the principle of superposition,

$$v_x = v_{x_1} + v_{x_2}$$

where v_{x_1} is produced by 6A source alone in the circuit and v_{x_2} is produced solely by 4A current source.

To find v_{x_1}, deactivate the 4A current source. This results in a circuit diagram as shown in Fig. 3.18.

KCL at node x_1 :

$$\frac{v_{x_1}}{2} + \frac{v_{x_1} - 4i_{x_1}}{8} = 6$$

But $$i_{x_1} = \frac{v_{x_1}}{2}$$

Hence, $$\frac{v_{x_1}}{2} + \frac{v_{x_1} - 4\frac{v_{x_1}}{2}}{8} = 6$$

$$\Rightarrow \quad \frac{v_{x_1}}{2} + \frac{v_{x_1} - 2v_{x_1}}{8} = 6$$

$$\Rightarrow \quad 4v_{x_1} + v_{x_1} - 2v_{x_1} = 48$$

$$\Rightarrow \quad v_{x_1} = \frac{48}{3} = 16\text{V}$$

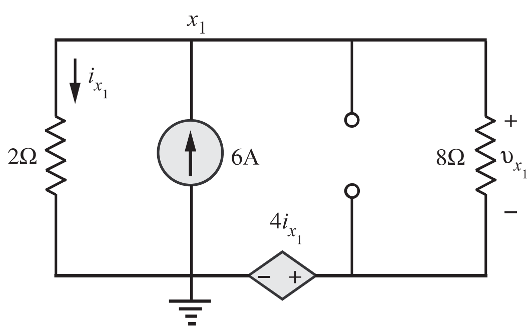


Figure 3.18

To find v_{x2}, deactivate the 6A current source, resulting in a circuit diagram as shown in Fig. 3.19.

KCL at node x_2 :

$$\frac{v_{x2}}{8} + \frac{v_{x2} - (-4i_{x2})}{2} = 4$$

$$\Rightarrow \quad \frac{v_{x2}}{8} + \frac{v_{x2} + 4i_{x2}}{2} = 4 \tag{3.3}$$

Applying KVL along dotted path, we get

$$v_{x2} + 4i_{x2} - 2i_{x2} = 0$$

$$\Rightarrow \quad v_{x2} = -2i_{x2} \quad \text{or} \quad i_{x2} = \frac{-v_{x2}}{2} \tag{3.4}$$

Substituting equation (3.4) in equation (3.3), we get

$$\frac{v_{x2}}{8} + \frac{v_{x2} + 4\left(\frac{-v_{x2}}{2}\right)}{2} = 4$$

$$\Rightarrow \quad \frac{v_{x2}}{8} + \frac{v_{x2} - 2v_{x2}}{2} = 4$$

$$\Rightarrow \quad \frac{v_{x2}}{8} - \frac{v_{x2}}{2} = 4$$

$$\Rightarrow \quad v_{x2} - 4v_{x2} = 32$$

$$\Rightarrow \quad v_{x2} = -\frac{32}{3}\text{V}$$

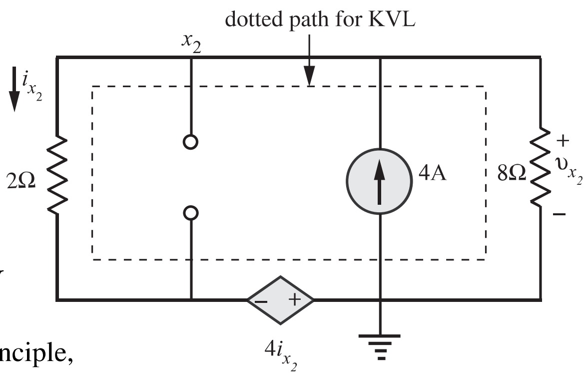


Figure 3.19

Hence, according to the superposition principle,

$$v_x = v_{x1} + v_{x2}$$
$$= 16 - \frac{32}{2} = \mathbf{5.33V}$$

EXAMPLE 3.7

Which of the source in Fig. 3.20 contributes most of the power dissipated in the 2 Ω resistor ? The least ? What is the power dissipated in 2 Ω resistor ?

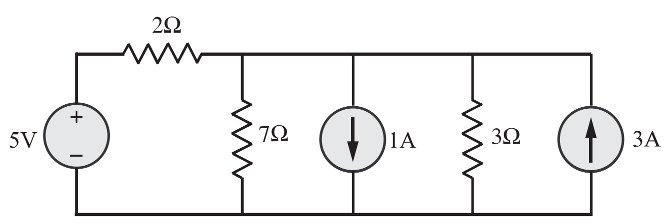


Figure 3.20

SOLUTION

The Superposition theorem cannot be used to identify the individual contribution of each source to the power dissipated in the resistor. However, the superposition theorem can be used to find the total power dissipated in the 2 Ω resistor.

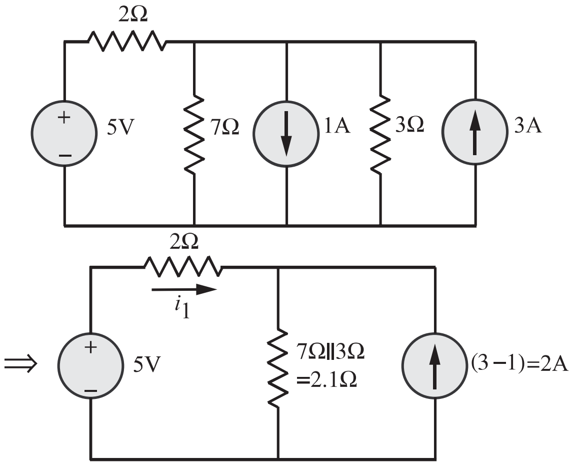


Figure 3.21

According to the superposition principle,

$$i_1 = i_1' + i_2'$$

where i_1' = Contribution to i_1 from 5V source alone.
and i_2' = Contribution to i_1 from 2A source alone.

Let us first find i_1'. This needs the deactivation of 2A source. Refer to Fig. 3.22.

$$i_1' = \frac{5}{2 + 2.1} = 1.22\text{A}$$

Similarly to find i_2' we have to disable the 5V source by shorting it.

Referring to Fig. 3.23, we find that

$$i_2' = \frac{-2 \times 2.1}{2 + 2.1} = -1.024 \text{ A}$$

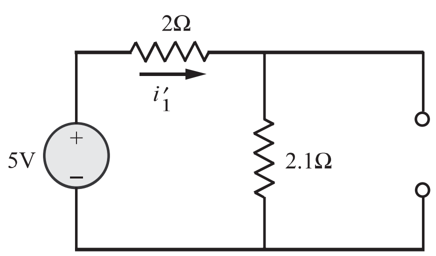


Figure 3.22

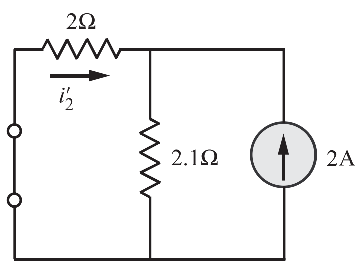


Figure 3.23

Total current,

$$\begin{aligned} i_1 &= i_1' + i_2' \\ &= 1.22 - 1.024 \\ &= 0.196 \text{ A} \end{aligned}$$

Thus, $$\begin{aligned} P_{2\Omega} &= (0.196)^2 \times 2 \\ &= 0.0768 \text{ Watts} \\ &= \mathbf{76.8} \text{ mW} \end{aligned}$$

EXAMPLE 3.8

Find the voltage V_1 using the superposition principle. Refer the circuit shown in Fig.3.24.

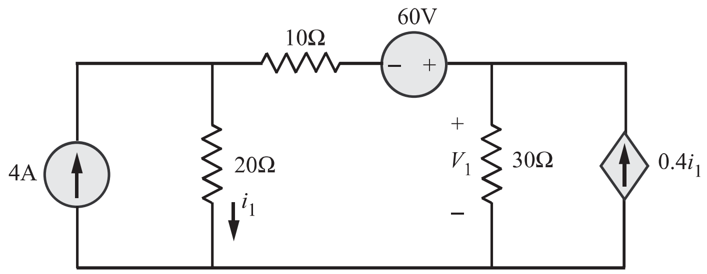


Figure 3.24

SOLUTION

According to the superposition principle,

$$V_1 = V_1' + V_1''$$

where V_1' is the contribution from 60V source alone and V_1'' is the contribution from 4A current source alone.

To find V_1', the 4A current source is opened, resulting in a circuit as shown in Fig. 3.25.

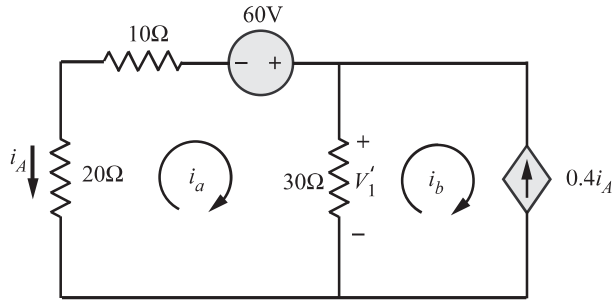


Figure 3.25

Applying KVL to the left mesh:

$$30i_a - 60 + 30\,(i_a - i_b) = 0 \tag{3.5}$$

Also

$$i_b = -0.4i_A$$
$$= -0.4\,(-i_a) = 0.4i_a \tag{3.6}$$

Substituting equation (3.6) in equation (3.5), we get

$$30i_a - 60 + 30i_a - 30 \times 0.4i_a = 0$$

$$\Rightarrow \quad i_a = \frac{60}{48} = 1.25\text{A}$$

$$i_b = 0.4i_a = 0.4 \times 1.25$$
$$= 0.5\text{A}$$

Hence,

$$V_1' = (i_a - i_b) \times 30$$
$$= 22.5 \text{ V}$$

To find, V_1'', the 60V source is shorted as shown in Fig. 3.26.

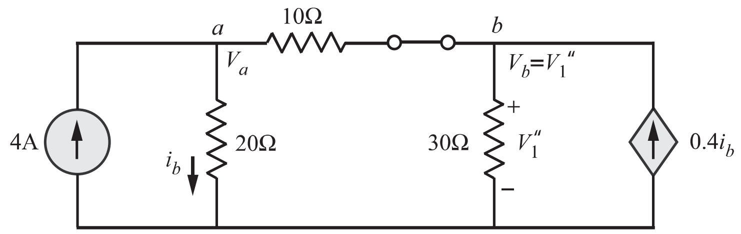


Figure 3.26

Applying KCL at node a:

$$\frac{V_a}{20} + \frac{V_a - V_1''}{10} = 4$$

$$\Rightarrow \quad 30V_a - 20V_1'' = 800 \tag{3.7}$$

Applying KCL at node b:

$$\frac{V_1''}{30} + \frac{V_1'' - V_a}{10} = 0.4i_b$$

Also,

$$V_a = 20i_a \quad \Rightarrow \quad i_b = \frac{V_a}{20}$$

Hence,

$$\frac{V_1''}{30} + \frac{V_1'' - V_a}{10} = \frac{0.4V_a}{20}$$

$$\Rightarrow \quad -7.2V_a + 8V_1'' = 0 \tag{3.8}$$

Solving the equations (3.7) and (3.8), we find that

$$V_1'' = 60\text{V}$$

Hence
$$\begin{aligned} V_1 &= V_1' + V_1'' \\ &= 22.5 + 60 = \mathbf{82.5V} \end{aligned}$$

EXAMPLE 3.9

(a) Refer to the circuit shown in Fig. 3.27. Before the 10 mA current source is attached to terminals $x - y$, the current i_a is found to be 1.5 mA. Use the superposition theorem to find the value of i_a after the current source is connected.
(b) Verify your solution by finding i_a, when all the three sources are acting simultaneously.

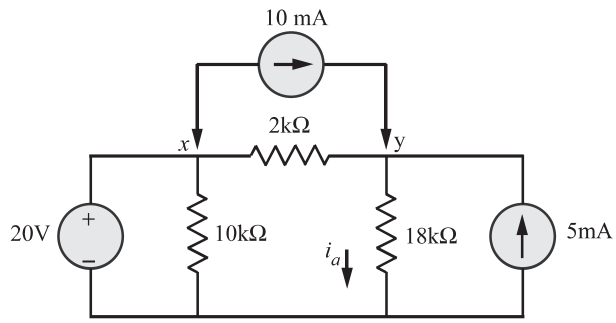


Figure 3.27

SOLUTION

According to the principle of superposition,

$$i_a = i_{a_1} + i_{a_2} + i_{a_3}$$

where i_{a_1}, i_{a_2} and i_{a_3} are the contributions to i_a from 20V source, 5 mA source and 10 mA source respectively.

As per the statement of the problem,

$$i_{a_1} + i_{a_2} = 1.5 \text{ mA}$$

To find i_{a_3}, deactivate 20V source and the 5 mA source. The resulting circuit diagram is shown in Fig 3.28.

$$i_{a_3} = \frac{10\text{mA} \times 2\text{k}}{18\text{k} + 2\text{k}} = 1 \text{ mA}$$

Hence, total current

$$\begin{aligned} i_a &= i_{a_1} + i_{a_2} + i_{a_3} \\ &= 1.5 + 1 = 2.5 \textbf{ mA} \end{aligned}$$

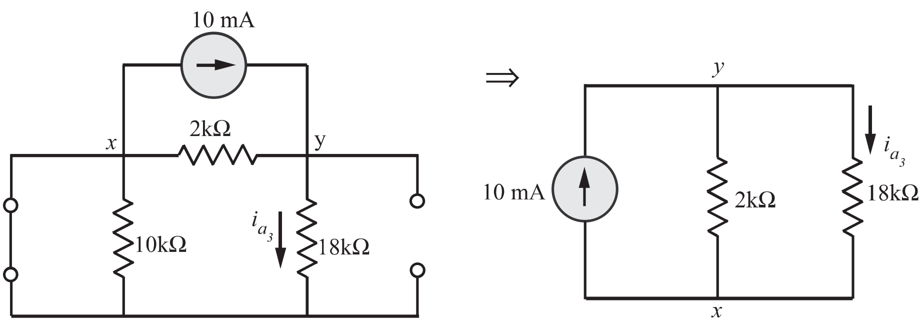


Figure 3.28

(b) Refer to Fig. 3.29
KCL at node y:

$$\frac{V_y}{18 \times 10^3} + \frac{V_y - 20}{2 \times 10^3} = (10+5) \times 10^{-3}$$

Solving, we get $\quad V_y = 45\text{V}.$

$$\text{Hence,} \quad i_a = \frac{V_y}{18 \times 10^3} = \frac{45}{18 \times 10^3} = 2.5 \text{ mA}$$

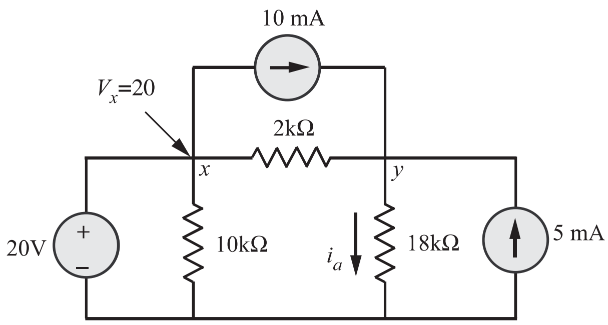


Figure 3.29

3.2 Thevenin's theorem

In section 3.1, we saw that the analysis of a circuit may be greatly reduced by the use of superposition principle. The main objective of Thevenin's theorem is to reduce some portion of a circuit to an equivalent source and a single element. This reduced equivalent circuit connected to the remaining part of the circuit will allow us to find the desired current or voltage. Thevenin's theorem is based on circuit equivalence. A circuit equivalent to another circuit exhibits identical characteristics at identical terminals.

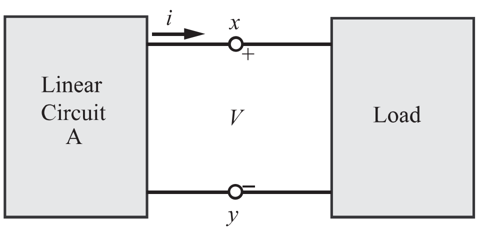


Figure 3.30 A Linear two terminal network

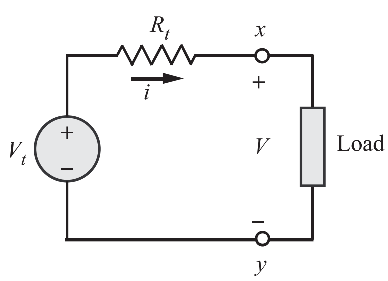


Figure 3.31 The Thevenin's equivalent circuit

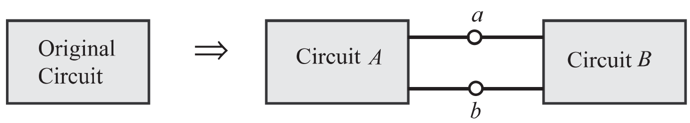
According to Thevenin's theorem, the linear circuit of Fig. 3.30 can be replaced by the one shown in Fig. 3.31 (The load resistor may be a single resistor or another circuit). The circuit to the left of the terminals $x - y$ in Fig. 3.31 is known as the Thevenin's equivalent circuit.

The Thevenin's theorem may be stated as follows:

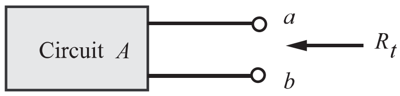
A linear two–terminal circuit can be replaced by an equivalent circuit consisting of a voltage source V_t in series with a resistor R_t, Where V_t is the open–circuit voltage at the terminals and R_t is the input or equivalent resistance at the terminals when the independent sources are turned off or R_t is the ratio of open–circuit voltage to the short–circuit current at the terminal pair.

Action plan for using Thevenin's theorem :

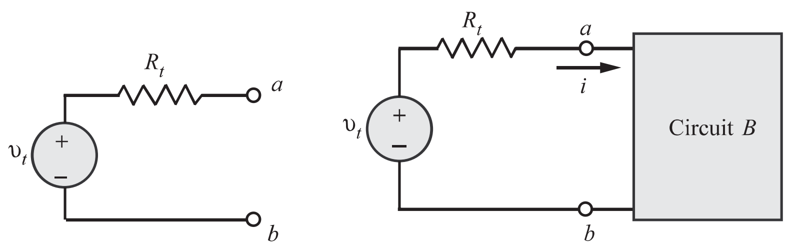
1. Divide the original circuit into circuit A and circuit B.



   In general, circuit B is the load which may be linear or non-linear. Circuit A is the balance of the original network exclusive of load and must be linear. In general, circuit A may contain independent sources, dependent sources and resistors or other linear elements.



2. Separate the circuit A from circuit B.
3. Replace circuit A with its Thevenin's equivalent.
4. Reconnect circuit B and determine the variable of interest (e.g. current 'i' or voltage 'v').



Procedure for finding R_t:

Three different types of circuits may be encountered in determining the resistance, R_t :

(i) If the circuit contains only independent sources and resistors, deactivate the sources and find R_t by circuit reduction technique. Independent current sources, are deactivated by opening them while independent voltage sources are deactivated by shorting them.

(ii) If the circuit contains resistors, dependent and independent sources, follow the instructions described below:

(a) Determine the open circuit voltage v_{oc} with the sources activated.

(b) Find the short circuit current i_{sc} when a short circuit is applied to the terminals $a - b$

(c) $R_t = \dfrac{v_{oc}}{i_{sc}}$

(iii) If the circuit contains resistors and only dependent sources, then

(a) $v_{oc} = 0$ (since there is no energy source)

(b) Connect 1A current source to terminals $a - b$ and determine v_{ab}.

(c) $R_t = \dfrac{v_{ab}}{1}$

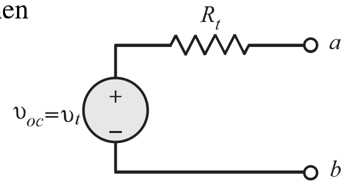


Figure 3.32

For all the cases discussed above, the Thevenin's equivalent circuit is as shown in Fig. 3.32.

EXAMPLE 3.10

Using the Thevenin's theorem, find the current i through $R = 2\ \Omega$. Refer Fig. 3.33.

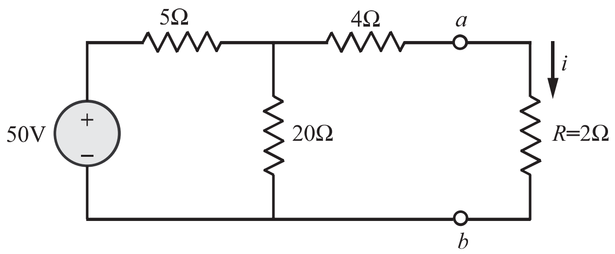


Figure 3.33

SOLUTION

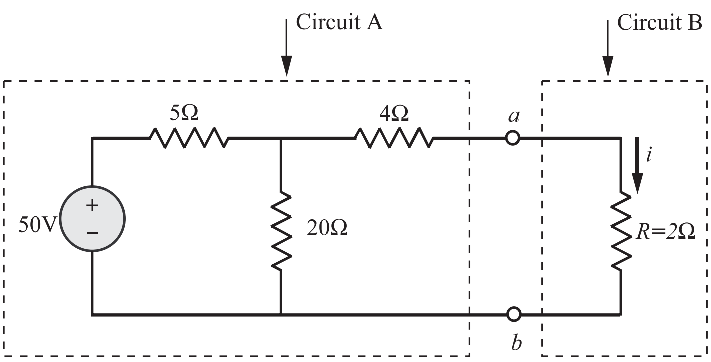


Figure 3.34

Since we are interested in the current i through R, the resistor R is identified as circuit B and the remainder as circuit A. After removing the circuit B, circuit A is as shown in Fig. 3.35.

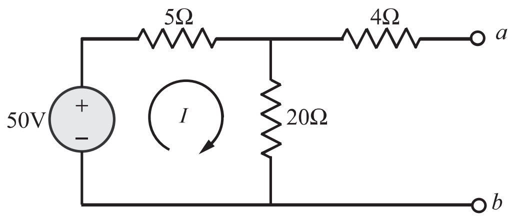


Figure 3.35

To find R_t, we have to deactivate the independent voltage source. Accordingly, we get the circuit in Fig. 3.36.

$$R_t = (5\ \Omega || 20\ \Omega) + 4\ \Omega$$
$$= \frac{5 \times 20}{5 + 20} + 4 = 8\ \Omega$$

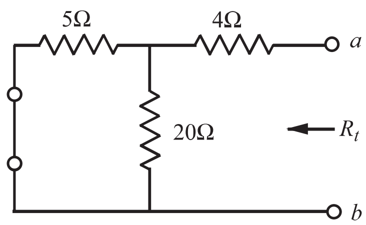


Figure 3.36

Referring to Fig. 3.35,

$$-50 + 25I = 0 \quad \Rightarrow \quad I = 2\text{A}$$

Hence $\qquad V_{ab} = V_{oc} = 20(I) = 40\text{V}$

Thus, we get the Thevenin's equivalent circuit which is as shown in Fig.3.37.

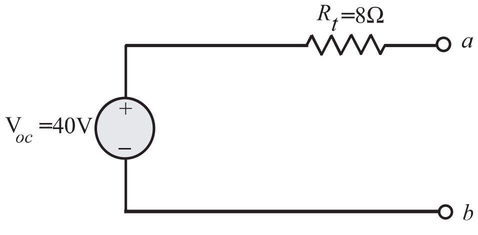


Figure 3.37

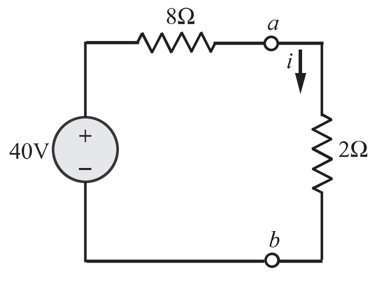


Figure 3.38

Reconnecting the circuit B to the Thevenin's equivalent circuit as shown in Fig. 3.38, we get

$$i = \frac{40}{2 + 8} = \mathbf{4A}$$

EXAMPLE 3.11

(a) Find the Thevenin's equivalent circuit with respect to terminals $a - b$ for the circuit shown in Fig. 3.39 by finding the open-circuit voltage and the short–circuit current.

(b) Solve the Thevenin resistance by removing the independent sources. Compare your result with the Thevenin resistance found in part (a).

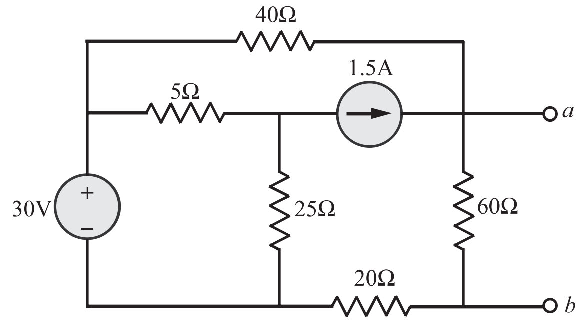


Figure 3.39

SOLUTION

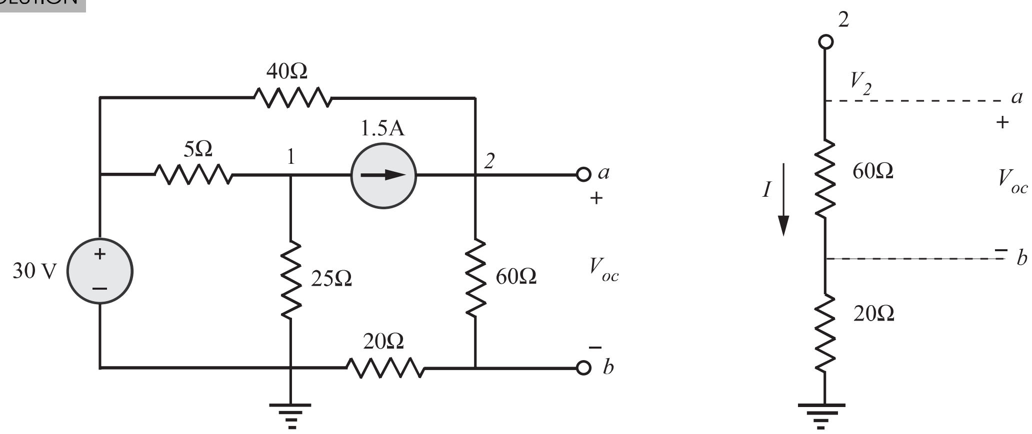


Figure 3.40

(a) *To find V_{oc} :*

Apply KCL at node 2 :

$$\frac{V_2}{60+20} + \frac{V_2 - 30}{40} - 1.5 = 0$$

$$\Rightarrow \qquad V_2 = 60 \text{ Volts}$$

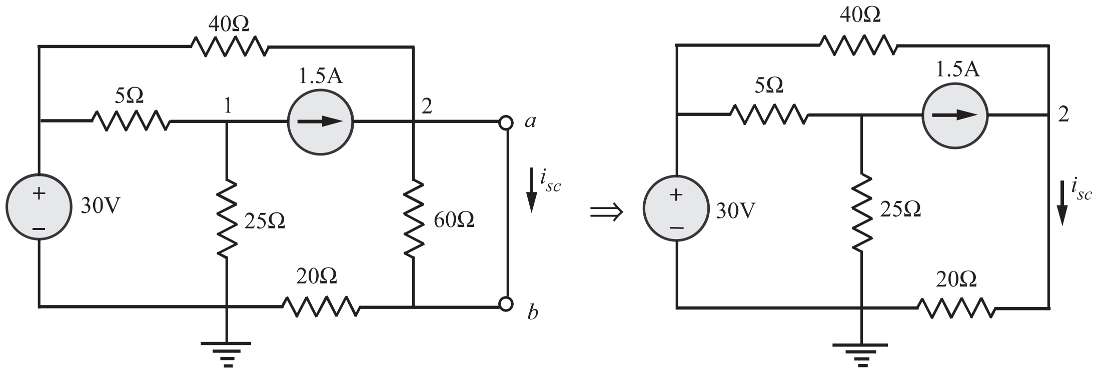
Hence,

$$V_{oc} = I \times 60$$

$$= \left[\frac{V_2 - 0}{60 + 20}\right] \times 60$$

$$= 60 \times \frac{60}{80} = 45 \text{ V}$$

To find i_{sc} :



Applying KCL at node 2:

$$\frac{V_2}{20} + \frac{V_2 - 30}{40} - 1.5 = 0$$

$$\Rightarrow \qquad V_2 = 30\text{V}$$

$$i_{sc} = \frac{V_2}{20} = 1.5\text{A}$$

Therefore,

$$R_t = \frac{V_{oc}}{i_{sc}} = \frac{45}{1.5}$$
$$= 30\ \Omega$$

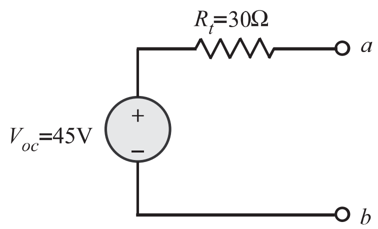
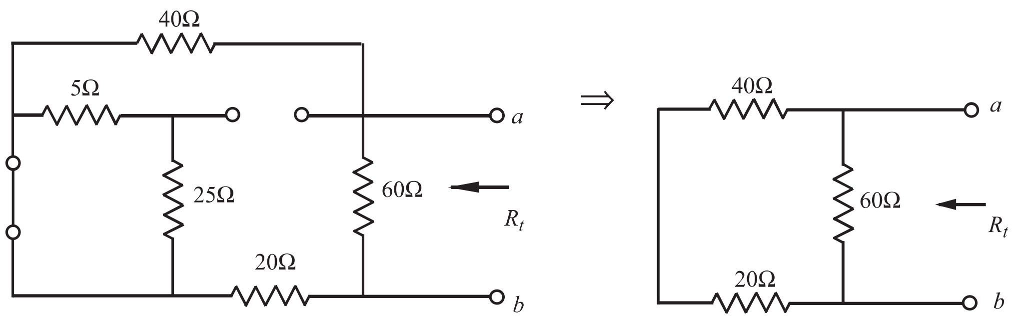


Figure 3.40 (a)

The Thevenin equivalent circuit with respect to the terminals $a-b$ is as shown in Fig. 3.40(a).
(b) Let us now find Thevenin resistance R_t by deactivating all the independent sources,



$$R_t = 60\ \Omega || (40 + 20)\ \Omega$$
$$= \frac{60}{2} = 30\ \Omega \text{ (verified)}$$

It is seen that, if only independent sources are present, it is easy to find R_t by deactivating all the independent sources.

EXAMPLE 3.12

Find the Thevenin equivalent for the circuit shown in Fig. 3.41 with respect to terminals $a - b$.

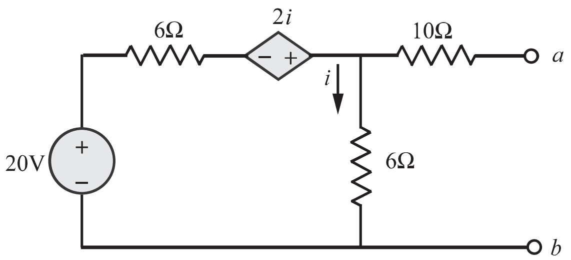


Figure 3.41

SOLUTION

To find $V_{oc} = V_{ab}$:

Applying KVL around the mesh of Fig. 3.42, we get

$$-20 + 6i - 2i + 6i = 0$$
$$\Rightarrow \qquad i = 2\text{A}$$

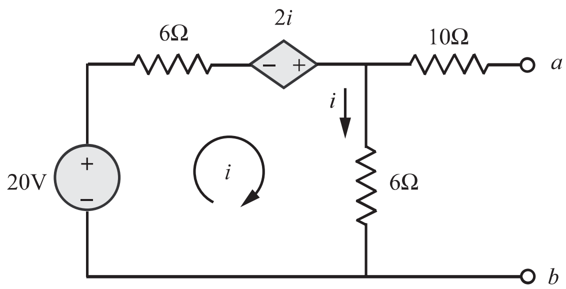


Figure 3.42

Since there is no current flowing in 10 Ω resistor, $V_{oc} = 6i = 12$ V

To find R_t*:* (Refer Fig. 3.43)

Since both dependent and independent sources are present, Thevenin resistance is found using the relation,

$$R_t = \frac{v_{oc}}{i_{sc}}$$

Applying KVL clockwise for mesh 1 :

$$-20 + 6i_1 - 2i + 6(i_1 - i_2) = 0$$
$$\Rightarrow \qquad 12i_1 - 6i_2 = 20 + 2i$$

Since $i = i_1 - i_2$, we get

$$12i_1 - 6i_2 = 20 + 2(i_1 - i_2)$$
$$\Rightarrow \qquad 10i_1 - 4i_2 = 20$$

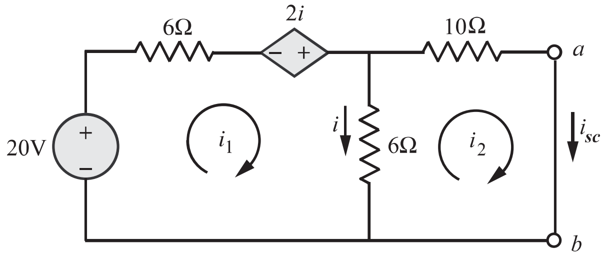


Figure 3.43

Applying KVL clockwise for mesh 2 :

$$10i_2 + 6(i_2 - i_1) = 0$$
$$\Rightarrow \qquad -6i_1 + 16i_2 = 0$$

Solving the above two mesh equations, we get

$$i_2 = \frac{120}{136}\text{A} \quad \Rightarrow \quad i_{sc} = i_2 = \frac{120}{136}\text{A}$$

$$R_t = \frac{v_{oc}}{i_{sc}} = \frac{12}{\frac{120}{136}} = 13.6\ \Omega$$

EXAMPLE 3.13

Find V_o in the circuit of Fig. 3.44 using Thevenin's theorem.

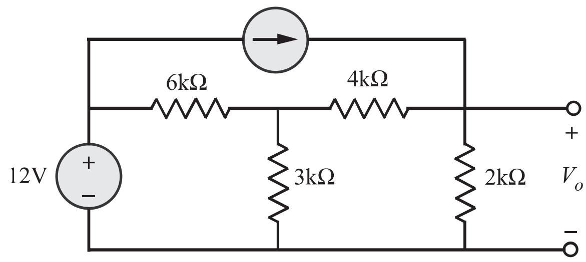


Figure 3.44

SOLUTION

To find V_{oc} :

Since we are interested in the voltage across 2 kΩ resistor, it is removed from the circuit of Fig. 3.44 and so the circuit becomes as shown in Fig. 3.45.

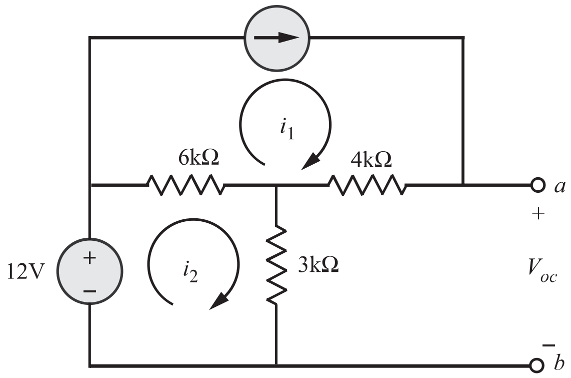


Figure 3.45

By inspection, $i_1 = 4$ mA

Applying KVL to mesh 2 :

$$-12 + 6 \times 10^3\,(i_2 - i_1) + 3 \times 10^3 i_2 = 0$$

$$\Rightarrow \quad -12 + 6 \times 10^3\left(i_2 - 4 \times 10^{-3}\right) + 3 \times 10^3 i_2 = 0$$

Solving, we get $\qquad i_2 = 4$ mA

Applying KVL to the path 4 kΩ $\rightarrow$ a–b $\rightarrow$ 3 kΩ, we get

$$-4 \times 10^3 i_1 + V_{oc} - 3 \times 10^3 i_2 = 0$$

$$\Rightarrow \qquad V_{oc} = 4 \times 10^3 i_1 + 3 \times 10^3 i_2$$

$$= 4 \times 10^3 \times 4 \times 10^{-3} + 3 \times 10^3 \times 4 \times 10^{-3} = 28\text{V}$$

To find R_t :

Deactivating all the independent sources, we get the circuit diagram shown in Fig. 3.46.

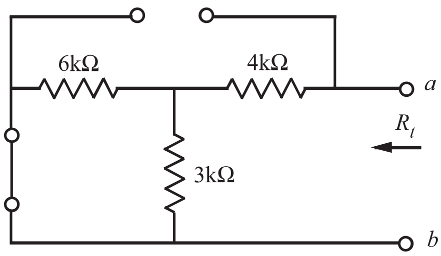


Figure 3.46

$$R_t = R_{ab} = 4\text{ k}\Omega + (6\text{ k}\Omega || 3\text{ k}\Omega) = 6\text{ k}\Omega$$

Hence, the Thevenin equivalent circuit is as shown in Fig. 3.47.

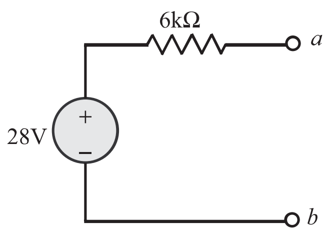


Figure 3.47

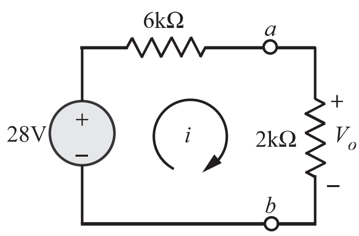


Figure 3.48

If we connect the 2 kΩ resistor to this equivalent network, we obtain the circuit of Fig. 3.48.

$$V_o = i\left(2 \times 10^3\right)$$

$$= \frac{28}{(6+2) \times 10^3} \times 2 \times 10^3 = 7\text{V}$$

EXAMPLE 3.14

The wheatstone bridge in the circuit shown in Fig. 3.49 (a) is balanced when $R_2 = 1200\ \Omega$. If the galvanometer has a resistance of 30 Ω, how much current will be detected by it when the bridge is unbalanced by setting R_2 to 1204 Ω ?

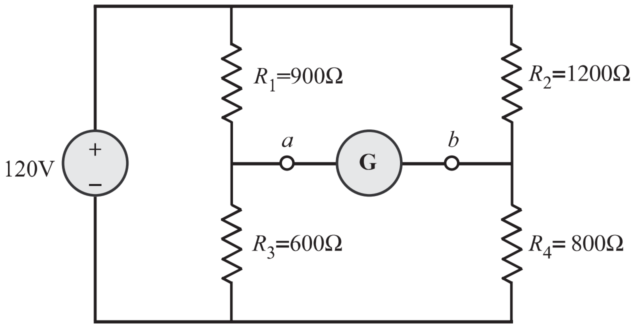


Figure 3.49(a)

SOLUTION

To find V_{oc} :

We are interested in the galavanometer current. Hence, it is removed from the circuit of Fig. 3.49 (a) to find V_{oc} and we get the circuit shown in Fig. 3.49 (b).

$$i_1 = \frac{120}{900+600} = \frac{120}{1500}\text{A}$$
$$i_2 = \frac{120}{1204+800} = \frac{120}{2004}\text{A}$$

Applying KVL clockwise along the path $1204\Omega \rightarrow\ b-a \rightarrow 900\ \ \Omega$, we get

$$1204i_2 - V_t - 900i_1 = 0$$
$$\Rightarrow \quad V_t = 1204i_2 - 900i_1$$
$$= 1204 \times \frac{120}{2004} - 900 \times \frac{120}{1500}$$
$$= 95.8\ \text{mV}$$

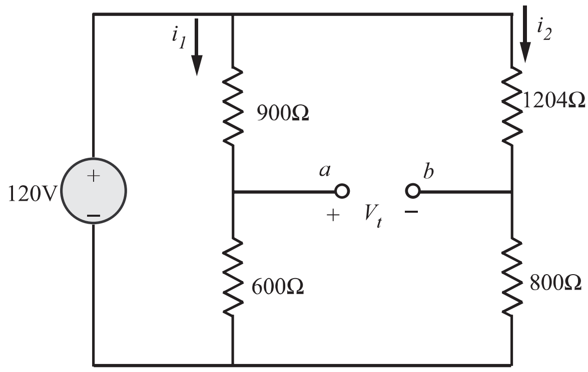


Figure 3.49(b)

To find R_t :

Deactivate all the independent sources and look into the terminals $a-b$ to determine the Thevenin's resistance.

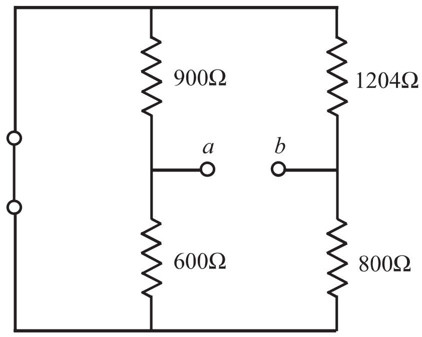


Figure 3.49(c)

⇒

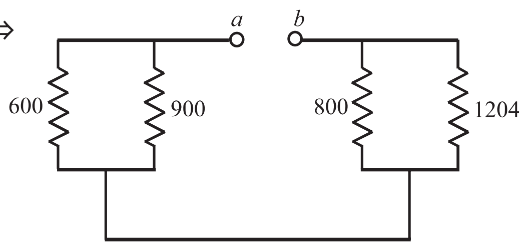


Figure 3.49(d)

$$R_t = R_{ab} = 600||900 + 800||1204$$
$$= \frac{900 \times 600}{1500} + \frac{1204 \times 800}{2004}$$
$$= 840.64\ \Omega$$

Hence, the Thevenin equivalent circuit consists of the 95.8 mV source in series with 840.64Ω resistor. If we connect 30Ω resistor (galvanometer resistance) to this equivalent network, we obtain the circuit in Fig. 3.50.

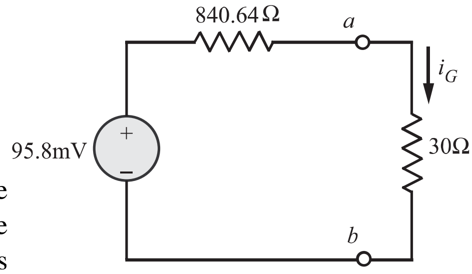


Figure 3.50

$$i_G = \frac{95.8 \times 10^{-3}}{840.64 + 30\ \Omega} \quad = 110.03\ \mu\text{A}$$

EXAMPLE 3.15

For the circuit shown in Fig. 3.51, find the Thevenin's equivalent circuit between terminals a and b.

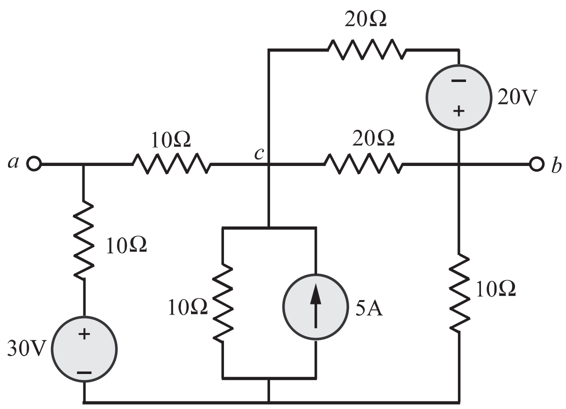


Figure 3.51

SOLUTION

With ab shorted, let $I_{sc} = I$. The circuit after transforming voltage sources into their equivalent current sources is as shown in Fig 3.52. Writing node equations for this circuit,

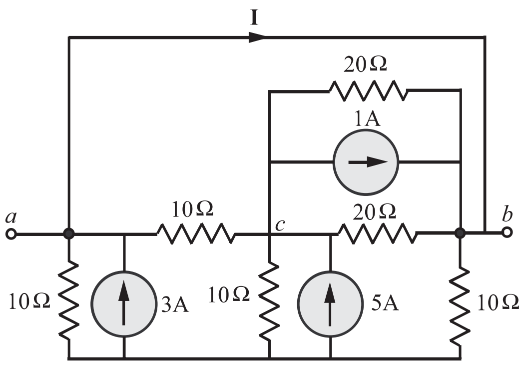


Figure 3.52

At a : $\quad 0.2V_a - 0.1\ V_c + I = 3$

At c : $\quad -0.1V_a + 0.3\ V_c - 0.1\ V_b = 4$

At b : $\quad -0.1V_c + 0.2\ V_b - I = 1$

As the terminals a and b are shorted $V_a = V_b$ and the above equations become

$$0.2V_a - 0.1\,V_c + I = 3$$
$$-0.2V_a + 0.3\,V_c = 4$$
$$0.2V_a - 0.1\,V_c - 1 = 1$$

Solving the above equations, we get the short circuit current, $I = I_{sc} = 1$ A.

Next let us open circuit the terminals a and b and this makes $I = 0$. And the node equations written earlier are modified to

$$0.2V_a - 0.1\,V_c = 3$$
$$-\,0.1V_a + 0.3\,V_c - 0.1\,V_b = 4$$
$$-0.1V_c + 0.2\,V_b = 1$$

Solving the above equations, we get

$$V_a = 30\text{V and } V_b = 20\text{V}$$

Hence, $V_{ab} = 30 - 20 = 10\text{ V} = V_{oc} = V_t$

Therefore $R_t = \dfrac{V_{oc}}{I_{sc}} = \dfrac{10}{1} = 10\Omega$

The Thevenin's equivalent is as shown in Fig 3.53

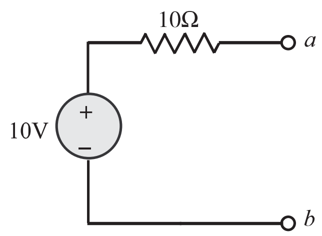


Figure 3.53

EXAMPLE 3.16

Refer to the circuit shown in Fig. 3.54. Find the Thevenin equivalent circuit at the terminals $a-b$.

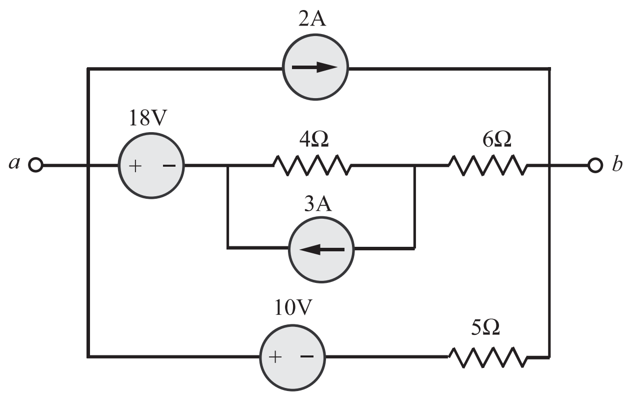


Figure 3.54

SOLUTION

To begin with let us transform 3 A current source and 10 V voltage source. This results in a network as shown in Fig. 3.55 (a) and further reduced to Fig. 3.55 (b).

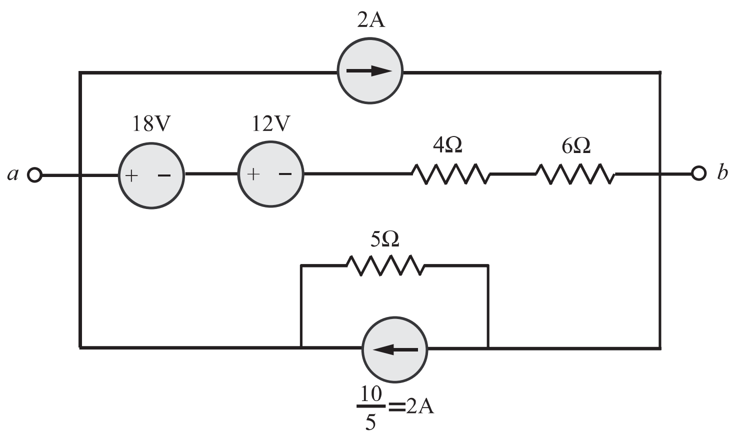


Figure 3.55(a)

Again transform the 30 V source and following the reduction procedure step by step from Fig. 3.55 (b) to 3.55 (d), we get the Thevenin's equivalent circuit as shown in Fig. 3.56.

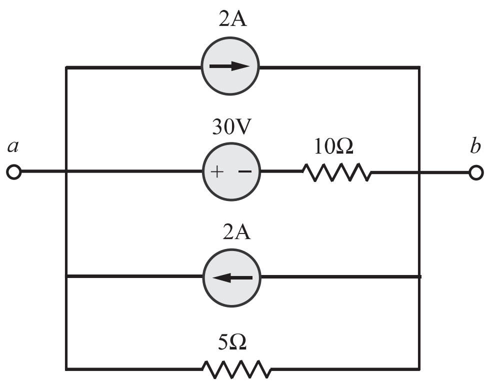


Figure 3.55(b)

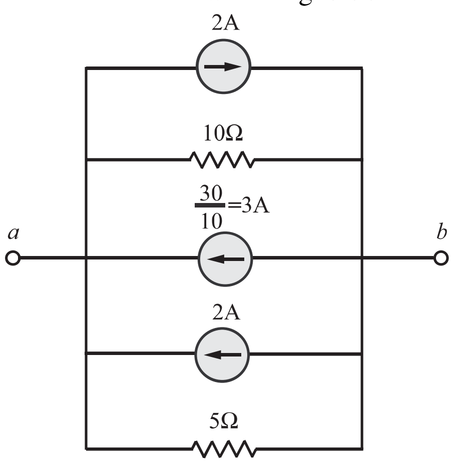


Figure 3.55(c)

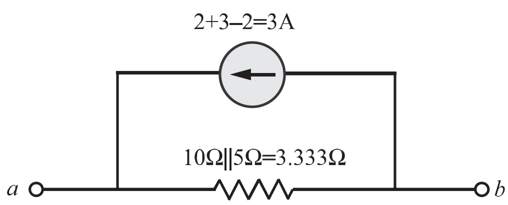


Figure 3.55(d)

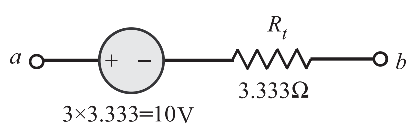


Figure 3.56 Thevenin equivalent circuit

EXAMPLE 3.17

Find the Thevenin equivalent circuit as seen from the terminals $a - b$. Refer the circuit diagram shown in Fig. 3.57.

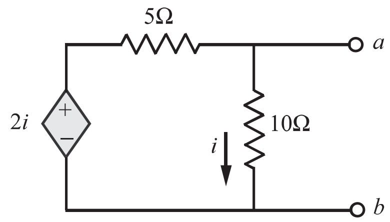


Figure 3.57

SOLUTION

Since the circuit has no independent sources, $i = 0$ when the terminals $a - b$ are open. Therefore, $V_{oc} = 0$.

The onus is now to find R_t. Since $V_{oc} = 0$ and $i_{sc} = 0$, R_t cannot be determined from $R_t = \frac{V_{oc}}{i_{sc}}$. Hence, we choose to connect a source of 1 A at the terminals $a - b$ as shown in Fig. 3.58. Then, after finding V_{ab}, the Thevenin resistance is,

$$R_t = \frac{V_{ab}}{1}$$

KCL at node a :

$$\frac{V_a - 2i}{5} + \frac{V_a}{10} - 1 = 0$$

Also,

$$i = \frac{V_a}{10}$$

Hence,

$$\frac{V_a - 2\left(\frac{V_a}{10}\right)}{5} + \frac{V_a}{10} - 1 = 0$$

$$\Rightarrow \quad V_a = \frac{50}{13}\text{V}$$

Hence,

$$R_t = \frac{V_a}{1} = \frac{50}{13}\ \Omega$$

Alternatively one could find R_t by connecting a 1V source at the terminals $a-b$ and then find the current from b to a. Then $R_t = \frac{1}{i_{ba}}$. The concept of finding R_t by connecting a 1A source between the terminals $a - b$ may also be used for circuits containing independent sources. Then set all the independent sources to zero and use 1A source at the terminals $a - b$ to find V_{ab} and hence, $R_t = \frac{V_{ab}}{1}$.

For the present problem, the Thevenin equivalent circuit as seen between the terminals $a - b$ is shown in Fig. 3.58 (a).

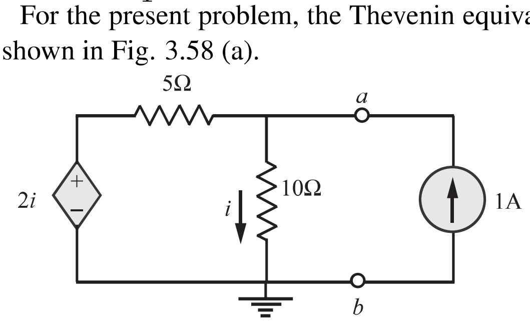


Figure 3.58

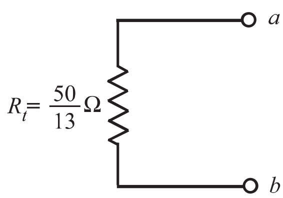


Figure 3.58 (a)

EXAMPLE 3.18

Determine the Thevenin equivalent circuit between the terminals $a - b$ for the circuit of Fig. 3.59.

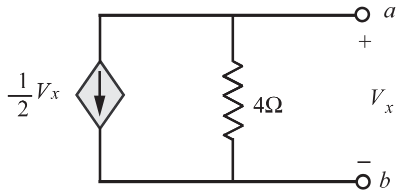


Figure 3.59

SOLUTION

As there are no independent sources in the circuit, we get $V_{oc} = V_t = 0$.

To find R_t, connect a 1V source to the terminals $a - b$ and measure the current I that flows from b to a. (Refer Fig. 3.60 a).

$$R_t = \frac{1}{I}\ \Omega$$

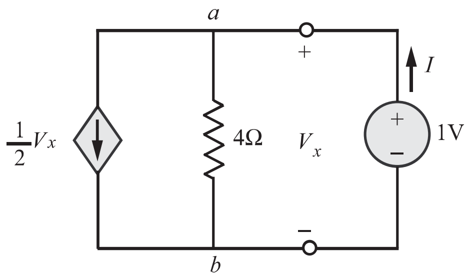


Figure 3.60(a)

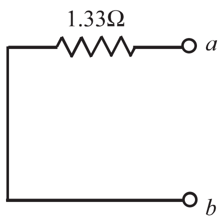


Figure 3.60(b)

Applying KCL at node a:

$$I = 0.5V_x + \frac{V_x}{4}$$

Since, $V_x = 1\text{V}$

we get, $I = 0.5 + \frac{1}{4} = 0.75\ \text{A}$

Hence, $R_t = \frac{1}{0.75} = 1.33\ \Omega$

The Thevenin equivalent circuit is shown in 3.60(b).

Alternatively, sticking to our strategy, let us connect 1A current source between the terminals $a - b$ and then measure V_{ab} (Fig. 3.60 (c)). Consequently, $R_t = \frac{V_{ab}}{1} = V_{ab}\ \Omega$.

Applying KCL at node a:

$$0.5V_x + \frac{V_x}{4} = 1 \Rightarrow V_x = 1.33\text{V}$$

$$\text{Hence} \quad R_t = \frac{V_{ab}}{1} = \frac{V_x}{1} = 1.33\ \Omega$$

The corresponding Thevenin equivalent circuit is same as shown in Fig. 3.60(b)

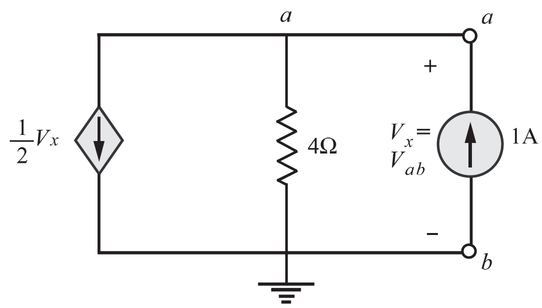


Figure 3.60(c)

3.3 Norton's theorem

An American engineer, E.L. Norton at Bell Telephone Laboratories, proposed a theorem similar to Thevenin's theorem.

Norton's theorem states that a linear two-terminal network can be replaced by an equivalent circuit consisting of a current source i_N in parallel with resistor R_N, where i_N is the short-circuit current through the terminals and R_N is the input or equivalent resistance at the terminals when the independent sources are turned off. If one does not wish to turn off the independent sources, then R_N is the ratio of open circuit voltage to short–circuit current at the terminal pair.

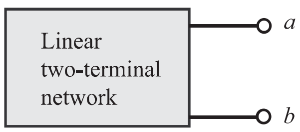


Figure 3.61(a) Original circuit

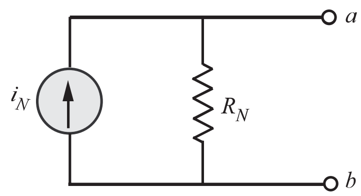


Figure 3.61(b) Norton's equivalent circuit

Figure 3.61(b) shows Norton's equivalent circuit as seen from the terminals $a - b$ of the original circuit shown in Fig. 3.61(a). Since this is the dual of the Thevenin circuit, it is clear that $R_N = R_t$ and $i_N = \frac{v_{oc}}{R_t}$. In fact, source transformation of Thevenin equivalent circuit leads to Norton's equivalent circuit.

Procedure for finding Norton's equivalent circuit:

(1) If the network contains resistors and independent sources, follow the instructions below:

(a) Deactivate the sources and find R_N by circuit reduction techniques.

(b) Find i_N with sources activated.

(2) If the network contains resistors, independent and dependent sources, follow the steps given below:

(a) Determine the short-circuit current i_N with all sources activated.

(b) Find the open-circuit voltage v_{oc}.

(c) $R_t = R_N = \dfrac{v_{oc}}{i_N}$

(3) If the network contains only resistors and dependent sources, follow the procedure described below:

(a) Note that $i_N = 0$.

(b) Connect 1A current source to the terminals $a - b$ and find v_{ab}.

(c) $R_t = \dfrac{v_{ab}}{1}$

Note: Also, since $v_t = v_{oc}$ and $i_N = i_{sc}$

$$R_t = \frac{v_{oc}}{i_{sc}} = R_N$$

The open–circuit and short–circuit test are sufficient to find any Thevenin or Norton equivalent.

3.3.1 PROOF OF THEVENIN'S AND NORTON'S THEOREMS

The principle of superposition is employed to provide the proof of Thevenin's and Norton's theorems.

Derivation of Thevenin's theorem:

Let us consider a linear circuit having two accessible terminals $x - y$ and excited by an external current source i. The linear circuit is made up of resistors, dependent and independent sources. For the sake of simplified analysis, let us assume that the linear circuit contains only two independent voltage sources v_1 and v_2 and two independent current sources i_1 and i_2. The terminal voltage v may be obtained, by applying the principle of superposition. That is, v is made up of contributions due to the external source and independent sources within the linear network.

Hence,
$$v = a_0 i + a_1 v_1 + a_2 v_2 + a_3 i_1 + a_4 i_2 \tag{3.9}$$
$$= a_0 i + b_0 \tag{3.10}$$

where
$$b_0 = a_1 v_1 + a_2 v_2 + a_3 i_1 + a_4 i_2$$
$= $ contribution to the terminal voltage v by independent sources within the linear network.

Let us now evaluate the values of constants a_0 and b_0.

(i) When the terminals x and y are open–circuited, $i = 0$ and $v = v_{oc} = v_t$. Making use of this fact in equation 3.10, we find that $b_0 = v_t$.

(ii) When all the internal sources are deactivated, $b_0 = 0$. This enforces equation 3.10 to become

$$v = a_0 i = R_t i \Rightarrow a_0 = R_t$$

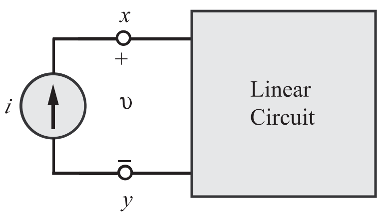


Figure 3.62 Current-driven circuit

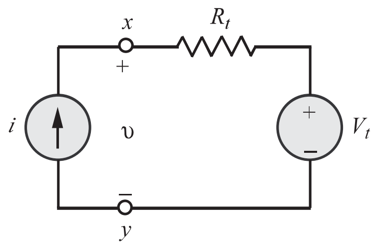


Figure 3.63 Thevenin's equivalent circuit of Fig. 3.62

where R_t is the equivalent resistance of the linear network as viewed from the terminals $x - y$. Also, a_0 must be R_t in order to obey the ohm's law. Substuting the values of a_0 and b_0 in equation 3.10, we find that

$$v = R_t i + v_1$$

which expresses the voltage-current relationship at terminals $x - y$ of the circuit in Fig. 3.63. Thus, the two circuits of Fig. 3.62 and 3.63 are equivalent.

Derivation of Norton's theorem:

Let us now assume that the linear circuit described earlier is driven by a voltage source v as shown in Fig. 3.64.

The current flowing into the circuit can be obtained by superposition as

$$i = c_0 v + d_0 \tag{3.11}$$

where $c_0 v$ is the contribution to i due to the external voltage source v and d_0 contains the contributions to i due to all independent sources within the linear circuit. The constants c_0 and d_0 are determined as follows :

(i) When terminals $x - y$ are short-circuited, $v = 0$ and $i = -i_{sc}$. Hence from equation (3.11), we find that $i = d_0 = -i_{sc}$, where i_{sc} is the short-circuit current flowing out of terminal x, which is same as Norton current i_N

Thus, $$d_0 = -i_N$$

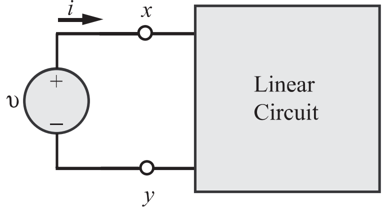


Figure 3.64
Voltage-driven circuit

(ii) Let all the independent sources within the linear network be turned off, that is $d_0 = 0$. Then, equation (3.11) becomes

$$i = c_0 v$$

For dimensional validity, c_0 must have the dimension of conductance. This enforces $c_0 = \frac{1}{R_t}$ where R_t is the equivalent resistance of the linear network as seen from the terminals $x - y$. Thus, equation (3.11) becomes

$$\begin{aligned} i &= \frac{1}{R_t}v - i_{sc} \\ &= \frac{1}{R_t}v - i_N \end{aligned}$$

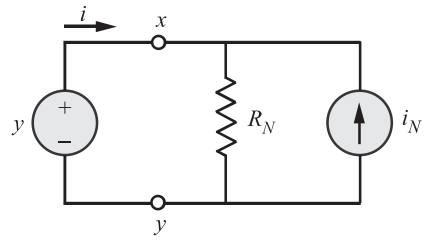


Figure 3.65 Norton's equivalent of voltage driven circuit

This expresses the voltage-current relationship at the terminals $x - y$ of the circuit in Fig. (3.65), validating that the two circuits of Figs. 3.64 and 3.65 are equivalents.

EXAMPLE 3.19

Find the Norton equivalent for the circuit of Fig. 3.66.

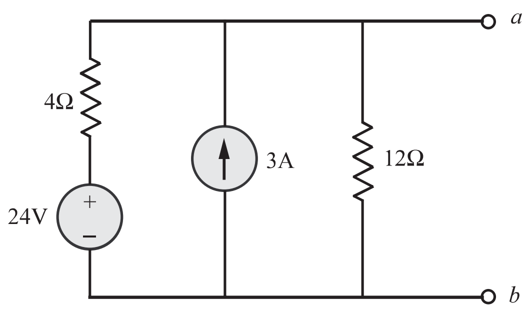


Figure 3.66

SOLUTION

As a first step, short the terminals $a - b$. This results in a circuit diagram as shown in Fig. 3.67. *Applying KCL at node a*, we get

$$\frac{0-24}{4} - 3 + i_{sc} = 0$$
$$\Rightarrow i_{sc} = 9\text{A}$$

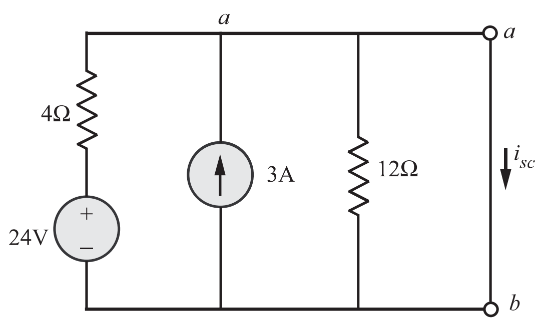


Figure 3.67

To find R_N, deactivate all the independent sources, resulting in a circuit diagram as shown in Fig. 3.68 (a). We find R_N in the same way as R_t in the Thevenin equivalent circuit.

$$R_N = \frac{4 \times 12}{4 + 12} = 3\,\Omega$$

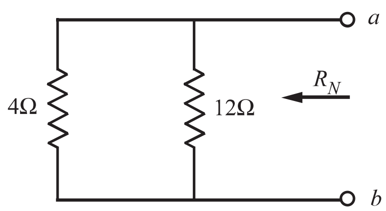


Figure 3.68(a)

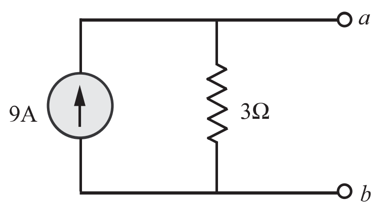


Figure 3.68(b)

Thus, we obtain Nortion equivalent circuit as shown in Fig. 3.68(b).

EXAMPLE 3.20

Refer the circuit shown in Fig. 3.69. Find the value of i_b using Norton equivalent circuit. Take $R = 667\ \Omega$.

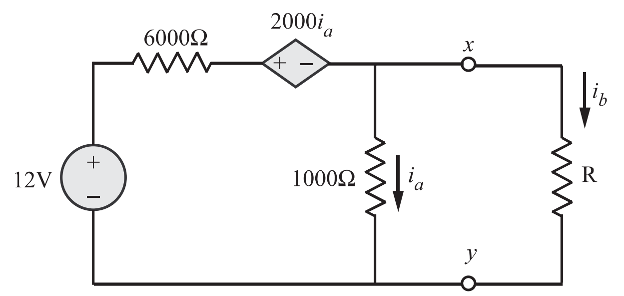


Figure 3.69

SOLUTION

Since we want the current flowing through R, remove R from the circuit of Fig. 3.69. The resulting circuit diagram is shown in Fig. 3.70.

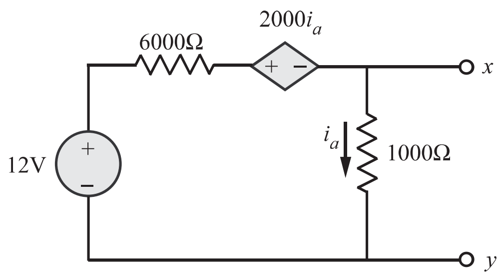


Figure 3.70

To find i_{ac} or i_N referring Fig 3.70(a) :

$$i_a = \frac{0}{1000} = 0\text{A}$$
$$i_{sc} = \frac{12}{6000}\text{A} = 2\text{ mA}$$

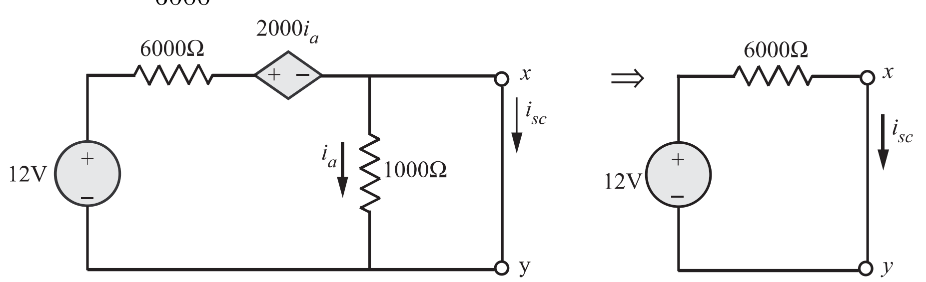


Figure 3.70(a)

To find R_N :
The procedure for finding R_N is same that of R_t in the Thevenin equivalent circuit.

$$R_t = R_N = \frac{v_{oc}}{i_{sc}}$$

To find v_{oc}, make use of the circuit diagram shown in Fig. 3.71. Do not deactivate any source.
Applying KVL clockwise, we get

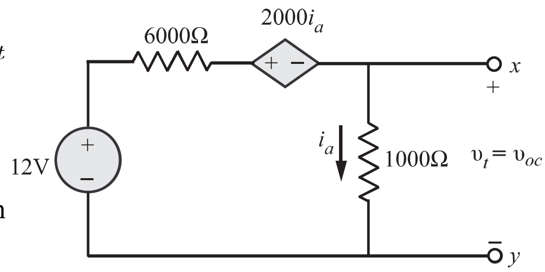


Figure 3.71

$$-12 + 6000i_a + 2000i_a + 1000i_a = 0$$

$$\Rightarrow \qquad i_a = \frac{4}{3000}\text{A}$$

$$\Rightarrow \qquad v_{\text{oc}} = i_a \times 1000 = \frac{4}{3}\text{V}$$

$$\text{Therefore,} \qquad R_N = \frac{v_{oc}}{i_{sc}} = \frac{\frac{4}{3}}{2 \times 10^{-3}} = 667\ \Omega$$

The Norton equivalent circuit along with resistor R is as shown below:

$$i_b = \frac{i_{sc}}{2} = \frac{2\text{mA}}{2} = 1\text{mA}$$

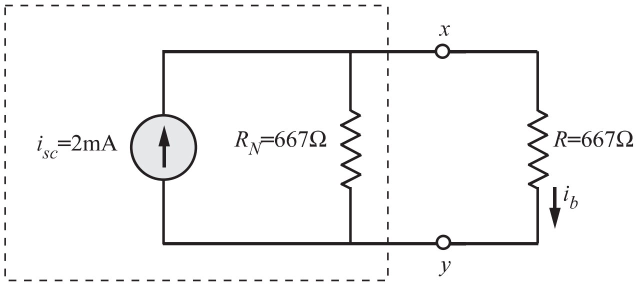


Figure : Norton equivalent circuit with load R

EXAMPLE 3.21

Find I_o in the network of Fig. 3.72 using Norton's theorem.

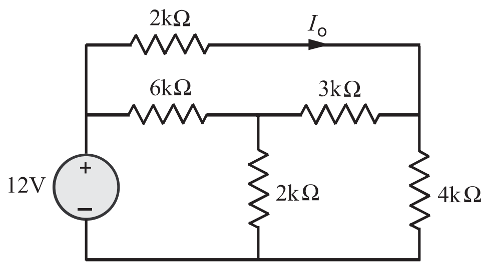


Figure 3.72

SOLUTION

We are interested in I_o, hence the 2 kΩ resistor is removed from the circuit diagram of Fig. 3.72. The resulting circuit diagram is shown in Fig. 3.73(a).

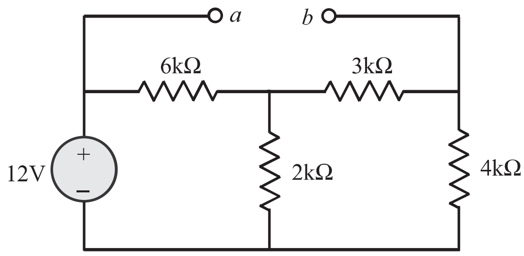


Figure 3.73(a)

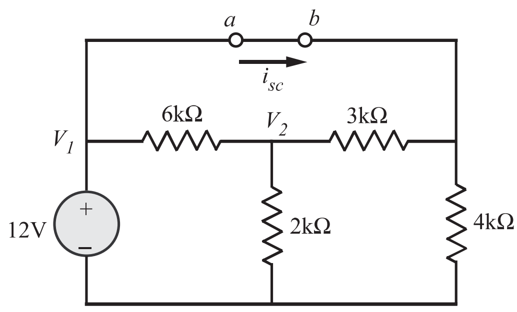


Figure 3.73(b)

To find i_N or i_{sc}:

Refer Fig. 3.73(b). By inspection, $V_1 = 12$ V

Applying KCL at node V_2 :

$$\frac{V_2 - V_1}{6\ \mathrm{k}\Omega} + \frac{V_2}{2\ \mathrm{k}\Omega} + \frac{V_2 - V_1}{3\ \mathrm{k}\Omega} = 0$$

Substituting $V_1 = 12$ V and solving, we get

$$V_2 = 6\mathrm{V}$$
$$i_{sc} = \frac{V_1 - V_2}{3\ \mathrm{k}\Omega} + \frac{V_1}{4\ \mathrm{k}\Omega} = 5\ \mathrm{mA}$$

To find R_N:

Deactivate all the independent sources (refer Fig. 3.73(c)).

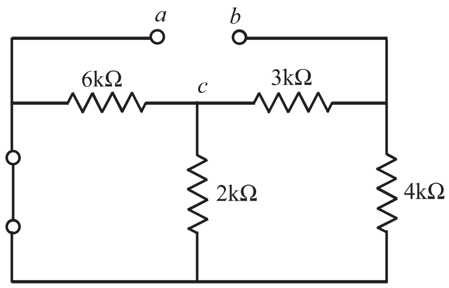


Figure 3.73(c)

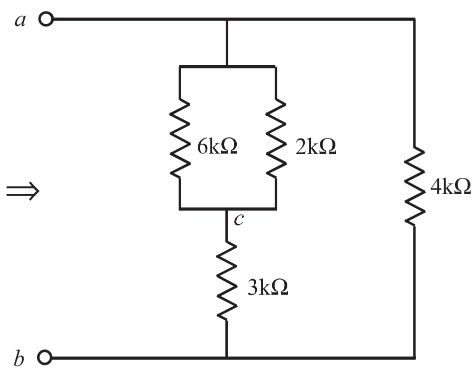


Figure 3.73(d)

Referring to Fig. 3.73 (d), we get

$$R_N = R_{ab} = 4\,\text{k}\Omega||\,[3\,\text{k}\Omega + (6\,\text{k}\Omega||2\,\text{k}\Omega)] = 2.12\,\text{k}\Omega$$

Hence, the Norton equivalent circuit along with 2 kΩ resistor is as shown in Fig. 3.73(e).

$$I_o = \frac{i_{sc} \times R_N}{R + R_N} = 2.57\text{mA}$$

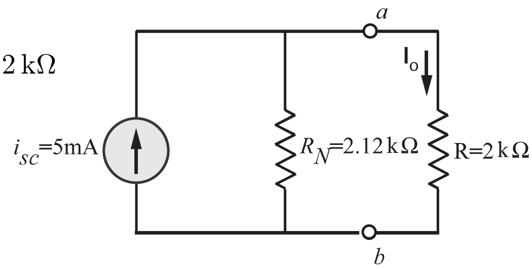


Figure 3.73(e)

EXAMPLE 3.22

Find V_o in the circuit of Fig. 3. 74.

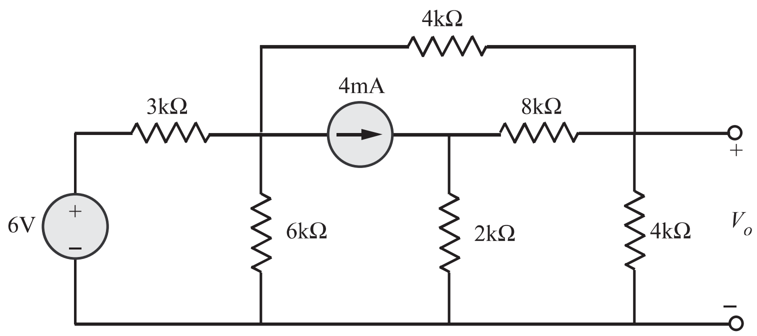


Figure 3.74

SOLUTION

Since we are interested in V_o, the voltage across 4 kΩ resistor, remove this resistance from the circuit. This results in a circuit diagram as shown in Fig. 3.75.

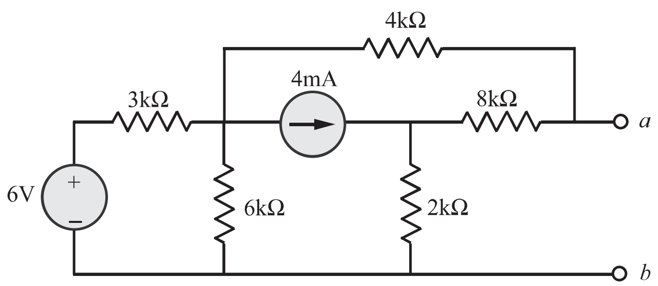
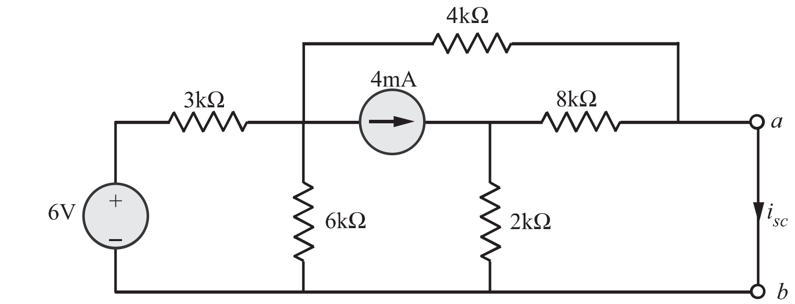
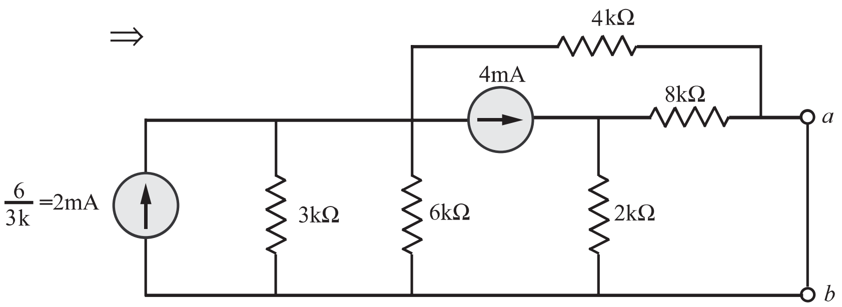


Figure 3.75

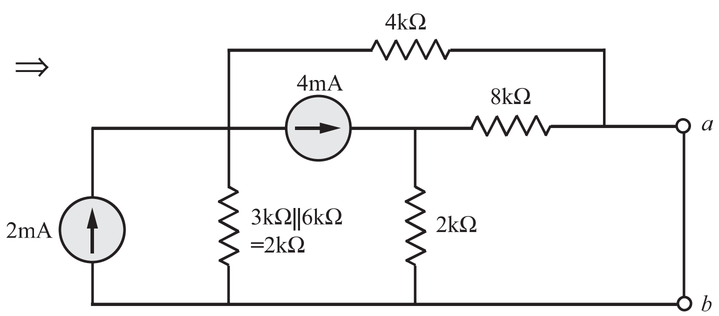
To find i_{sc}, short the terminals $a - b$:



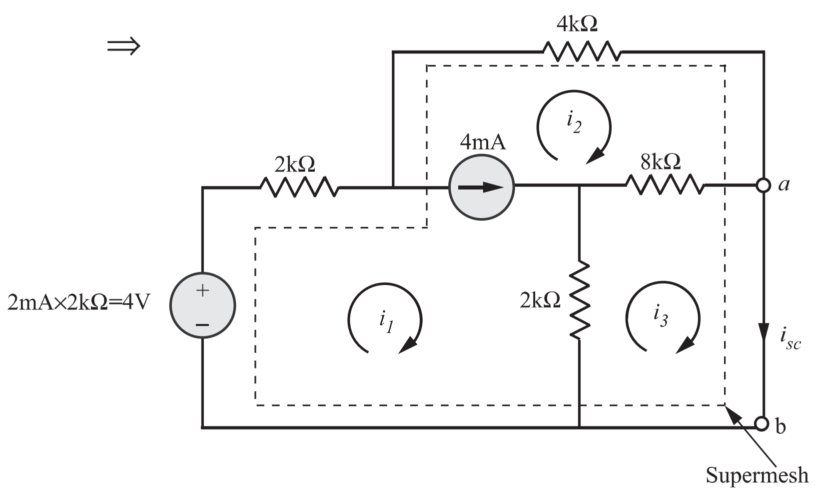
$\Rightarrow$



$\Rightarrow$



$\Rightarrow$



Constraint equation :

$$i_1 - i_2 = 4\text{mA} \tag{3.12}$$

KVL around supermesh :

$$-4 + 2 \times 10^3 i_1 + 4 \times 10^3 i_2 = 0 \tag{3.13}$$

KVL around mesh 3 :

$$8 \times 10^3(i_3 - i_2) + 2 \times 10^3(i_3 - i_1) = 0$$

Since $i_3 = i_{sc}$, the above equation becomes,

$$8 \times 10^3(i_{sc} - i_2) + 2 \times 10^3(i_{sc} - i_1) = 0 \tag{3.14}$$

Solving equations (3.12), (3.13) and (3.14) simultaneously, we get $i_{sc} = 0.1333$ mA.

To find R_N:

Deactivate all the sources in Fig. 3.75. This yields a circuit diagram as shown in Fig. 3.76.

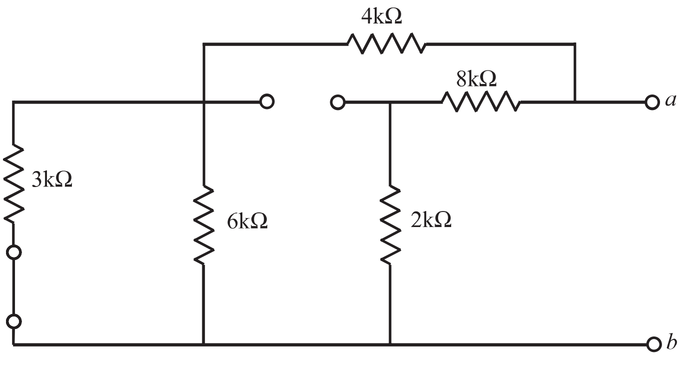
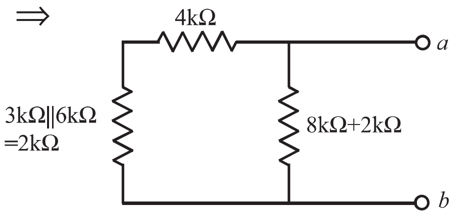
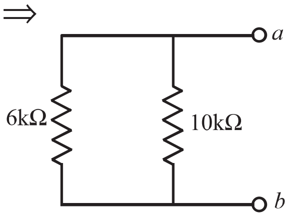


Figure 3.76

$\Rightarrow$



$\Rightarrow$



$$R_N = 6\ \text{k}\Omega || 10\ \text{k}\Omega$$
$$= \frac{6 \times 10}{6 + 10} = 3.75\ \text{k}\Omega$$

Hence, the Norton equivalent circuit is as shown in Fig 3.76 (a).

To the Norton equivalent circuit, now connect the 4 kΩ resistor that was removed earlier to get the network shown in Fig. 3.76(b).

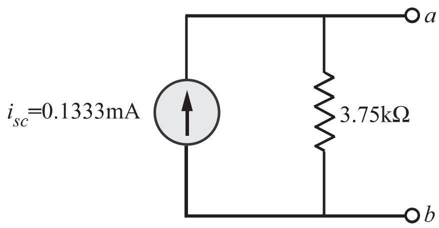


Figure 3.76(a)

$$V_o = i_{sc}\,(R_N||R) = i_{sc}\frac{R_N R}{R_N + R} = 258\text{ mV}$$

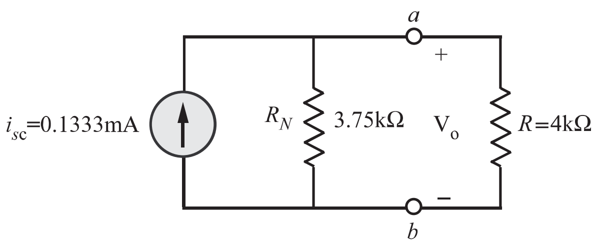


Figure 3.76(b) Norton equivalent circuit with $R = 4\text{ k}\Omega$

EXAMPLE 3.23

Find the Norton equivalent to the left of the terminals $a - b$ for the circuit of Fig. 3.77.

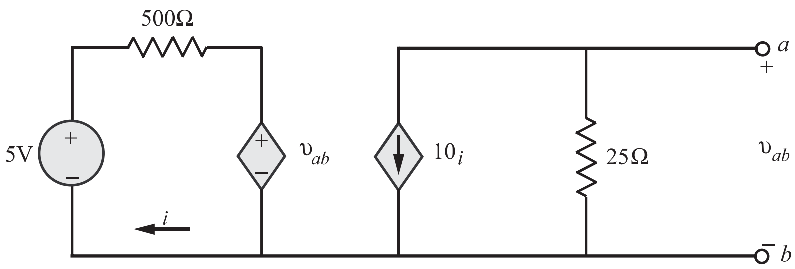
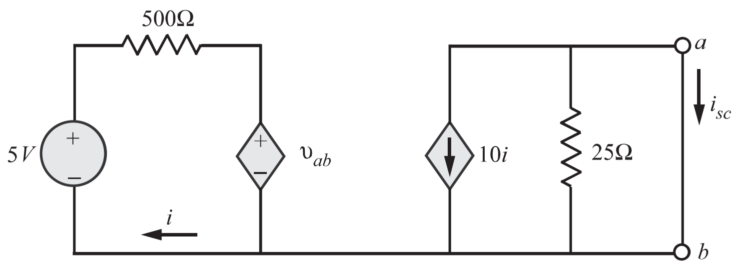


Figure 3.77

SOLUTION

To find i_{sc}:

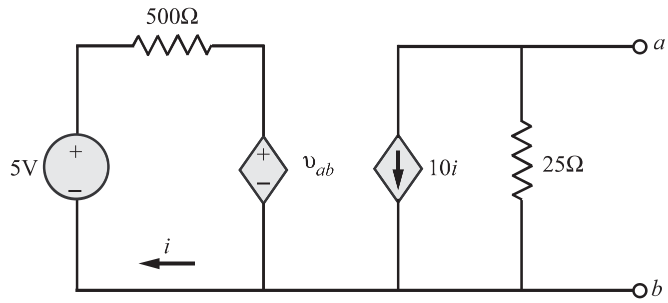


Note that $v_{ab} = 0$ when the terminals $a - b$ are short-circuited.

Then $$i = \frac{5}{500} = 10\text{ mA}$$

Therefore, for the right–hand portion of the circuit, $i_{sc} = -10i = -100$ mA.

To find R_N or R_t :



Writing the KVL equations for the left-hand mesh, we get

$$-5 + 500i + v_{ab} = 0 \tag{3.15}$$

Also for the right-hand mesh, we get

$$v_{ab} = -25(10i) = -250i$$

Therefore $\qquad i = \dfrac{-v_{ab}}{250}$

Substituting i into the mesh equation (3.15), we get

$$-5 + 500\left(\frac{-v_{ab}}{250}\right) + v_{ab} = 0$$

$$\Rightarrow \qquad v_{ab} = -5 \text{ V}$$

$$R_N = R_t \triangleq \frac{v_{oc}}{i_{sc}} = \frac{v_{ab}}{i_{sc}} = \frac{-5}{-0.1} = 50\ \Omega$$

The Norton equivalent circuit is shown in Fig 3.77 (a).

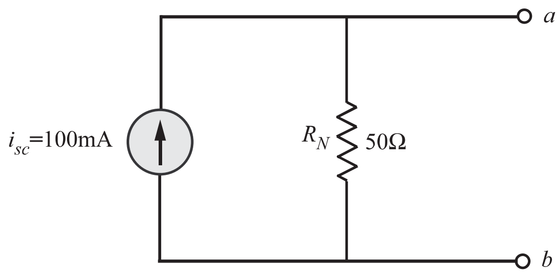


Figure 3.77 (a)

EXAMPLE 3.24

Find the Norton equivalent of the network shown in Fig. 3.78.

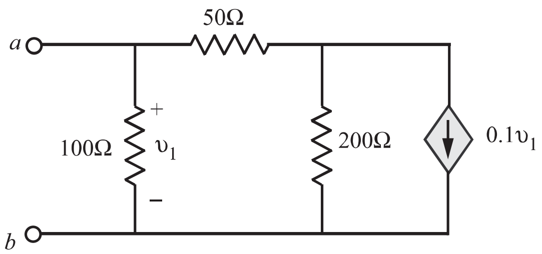


Figure 3.78

SOLUTION

Since there are no independent sources present in the network of Fig. 3.78, $i_N = i_{sc} = 0$.

To find R_N, we inject a current of 1A between the terminals $a - b$. This is illustrated in Fig. 3.79.

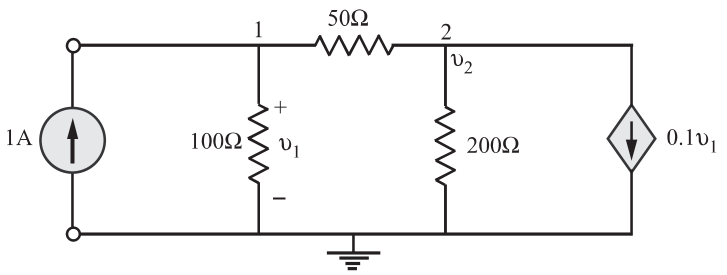


Figure 3.79

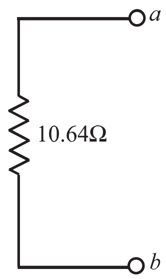


Figure 3.79(a) Norton equivalent circuit

KCL at node 1:

$$1 = \frac{v_1}{100} + \frac{v_1 - v_2}{50}$$

$$\Rightarrow \qquad 0.03v_1 - 0.02v_2 = 1$$

KCL at node 2:

$$\frac{v_2}{200} + \frac{v_2 - v_1}{50} + 0.1v_1 = 0$$

$$\Rightarrow \qquad 0.08v_1 + 0.025v_2 = 0$$

Solving the above two nodal equations, we get

$$v_1 = 10.64 \text{ volts} \quad \Rightarrow \quad v_{oc} = 10.64 \text{ volts}$$

Hence,

$$R_N = R_t = \frac{v_{oc}}{1} = \frac{10.64}{1} = 10.64\ \Omega$$

Norton equivalent circuit for the network shown in Fig. 3.78 is as shown in Fig. 3.79(a).

EXAMPLE 3.25

Find the Thevenin and Norton equivalent circuits for the network shown in Fig. 3.80 (a).

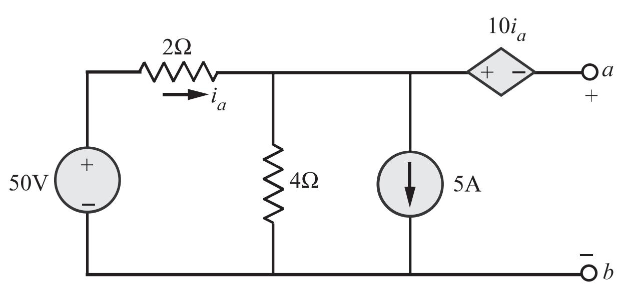


Figure 3.80(a)

SOLUTION

To find V_{oc} :

Performing source transformation on 5A current source, we get the circuit shown in Fig. 3.80 (b).

Applying KVL around Left mesh :

$$-50 + 2i_a - 20 + 4i_a = 0$$

$$\Rightarrow \qquad i_a = \frac{70}{6} \text{A}$$

Applying KVL around right mesh:

$$20 + 10i_a + V_{oc} - 4i_a = 0$$

$$\Rightarrow \qquad V_{oc} = -90 \text{ V}$$

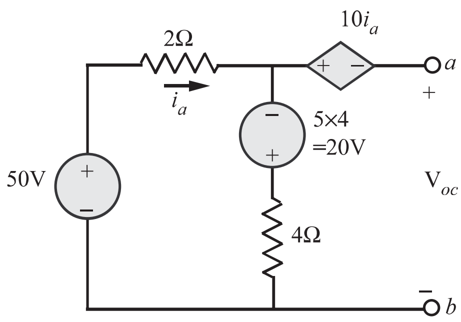


Figure 3.80(b)

To find i_{sc}(referring Fig 3.80 (c)) :

KVL around Left mesh :

$$-50 + 2i_a - 20 + 4(i_a - i_{sc}) = 0$$

$$\Rightarrow \qquad 6i_a - 4i_{sc} = 70$$

KVL around right mesh :

$$4(i_{sc} - i_a) + 20 + 10i_a = 0$$

$$\Rightarrow \qquad 6i_a + 4i_{sc} = -20$$

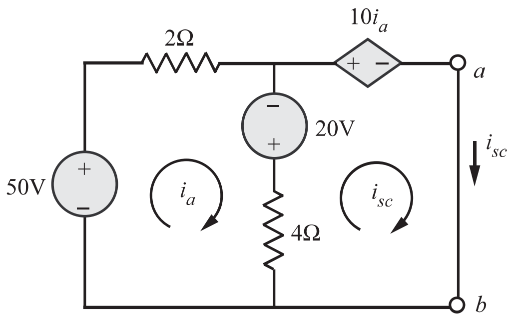
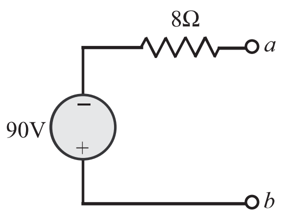


Figure 3.80(c)

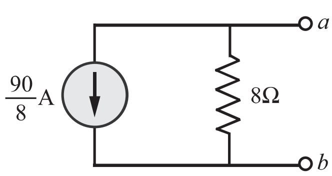
Solving the two mesh equations simultaneously, we get $i_{sc} = -11.25$ A

Hence, $R_t = R_N = \dfrac{v_{oc}}{i_{sc}} = \dfrac{-90}{-11.25} = 8\ \Omega$

Performing source transformation on Thevenin equivalent circuit, we get the norton equivalent circuit (both are shown below).



Thevenin equivalent circuit



Norton equivalent circuit

EXAMPLE 3.26

If an 8 kΩ load is connected to the terminals of the network in Fig. 3.81, $V_{AB} = 16$ V. If a 2 kΩ load is connected to the terminals, $V_{AB} = 8$V. Find V_{AB} if a 20 kΩ load is connected across the terminals.

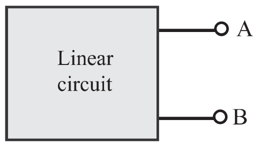
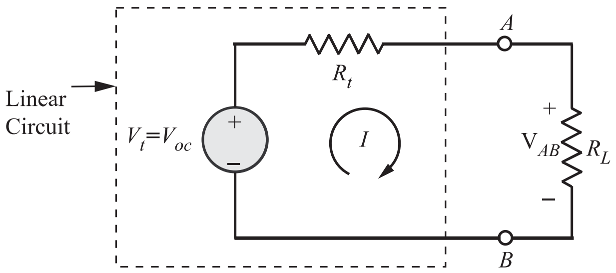


Figure 3.81

SOLUTION



Applying KVL around the mesh, we get $(R_t + R_L)\, I = V_{oc}$

If $\quad R_L = 2\text{ k}\Omega,\; I = 10\text{ mA} \Rightarrow V_{oc} = 20 + 0.01R_t$

If $\quad R_L = 10\text{ k}\Omega,\; I = 6\text{ mA} \Rightarrow V_{oc} = 60 + 0.006R_t$

Solving, we get $V_{oc} = 120$ V, $R_t = 10$ kΩ.

If $$R_L = 20\text{ k}\Omega,\; I = \frac{V_{oc}}{(R_L + R_t)} = \frac{120}{(20 \times 10^3 + 10 \times 10^3)} = 4\text{ mA}$$

3.4 Maximum Power Transfer Theorem

In circuit analysis, we are some times interested in determining the maximum power that a circuit can supply to the load. Consider the linear circuit A as shown in Fig. 3.82.

Circuit A is replaced by its Thevenin equivalent circuit as seen from a and b (Fig 3.83).

We wish to find the value of the load R_L such that the maximum power is delivered to it.

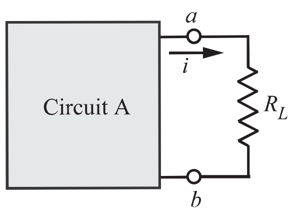


Figure 3.82 Circuit A with load R_L

The power that is delivered to the load is given by

$$p = i^2 R_L = \left[\frac{V_t}{R_t + R_L}\right]^2 R_L \tag{3.16}$$

Assuming that V_t and R_t are fixed for a given source, the maximum power is a function of R_L. In order to determine the value of R_L that maximizes p, we differentiate p with respect to R_L and equate the derivative to zero.

$$\frac{dp}{dR_L} = V_t^2 \left[\frac{(R_t + R_L)^2 - 2(R_t + R_L)}{(R_L + R_t)^2} \right] = 0$$

which yields $$R_L = R_t \tag{3.17}$$

To confirm that equation (3.17) is a maximum, it should be shown that $\frac{d^2p}{dR_L^2} < 0$. Hence, maximum power is transferred to the load when R_L is equal to the Thevenin equivalent resistance R_t. The maximum power transferred to the load is obtained by substituting $R_L = R_t$ in equation 3.16.

Accordingly,

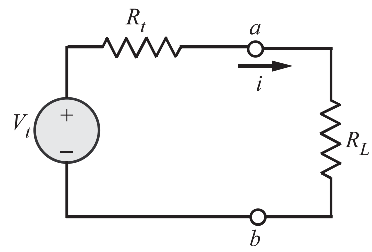


Figure 3.83 Thevenin equivalent circuit is substituted for circuit A

$$P_{\max} = \frac{V_t^2 R_L}{(2R_L)^2} = \frac{V_t^2}{4R_L}$$

The maximum power transfer theorem states that the maximum power delivered by a source represented by its Thevenin equivalent circuit is attained when the load R_L is equal to the Thevenin resistance R_t.

EXAMPLE 3.27

Find the load R_L that will result in maximum power delivered to the load for the circuit of Fig. 3.84. Also determine the maximum power $P_{\max}$.

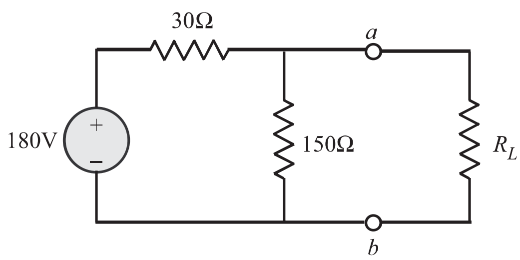


Figure 3.84

SOLUTION

Disconnect the load resistor R_L. This results in a circuit diagram as shown in Fig. 3.85(a).

Next let us determine the Thevenin equivalent circuit as seen from $a - b$.

$$i = \frac{180}{150 + 30} = 1\text{A}$$
$$V_{oc} = V_t = 150 \times i = 150 \text{ V}$$

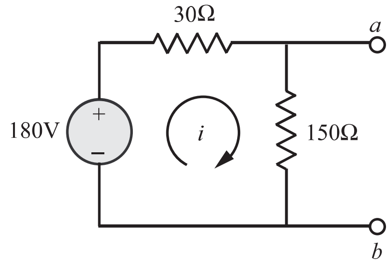


Figure 3.85(a)

To find R_t, deactivate the 180 V source. This results in the circuit diagram of Fig. 3.85(b).

$$R_t = R_{ab} = 30 \ \Omega || 150 \ \Omega$$
$$= \frac{30 \times 150}{30 + 150} = 25 \ \Omega$$

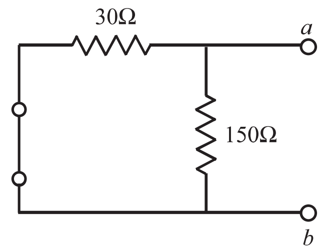


Figure 3.85(b)

The Thevenin equivalent circuit connected to the load resistor is shown in Fig. 3.86.
Maximum power transfer is obtained when $R_L = R_t = 25 \ \Omega$.
Then the maximum power is

$$P_{\max} = \frac{V_t^2}{4R_L} = \frac{(150)^2}{4 \times 25}$$
$$= 2.25 \text{ Watts}$$

The Thevenin source V_t actually provides a total power of

$$P_t = 150 \times i$$
$$= 150 \times \left(\frac{150}{25 + 25}\right)$$
$$= 450 \text{ Watts}$$

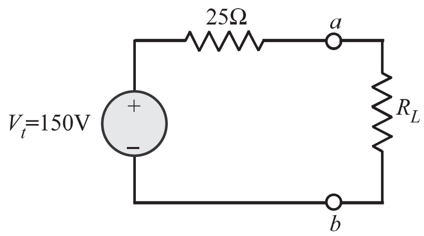


Figure 3.86

Thus, we note that one-half the power is dissipated in R_L.

EXAMPLE 3.28

Refer to the circuit shown in Fig. 3.87. Find the value of R_L for maximum power transfer. Also find the maximum power transferred to R_L.

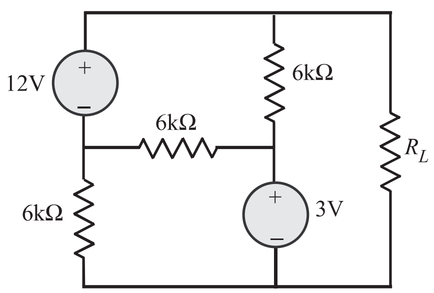


Figure 3.87

SOLUTION

Disconnecting R_L, results in a circuit diagram as shown in Fig. 3.88(a).

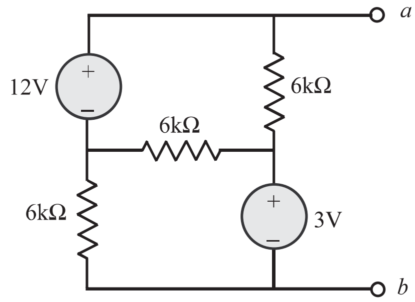


Figure 3.88(a)

To find R_t, deactivate all the independent voltage sources as in Fig. 3.88(b).

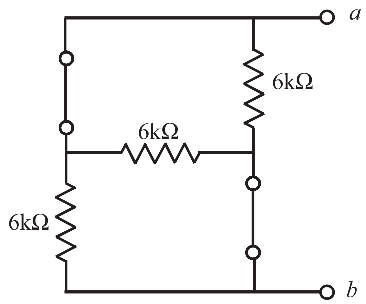


Figure 3.88(b)

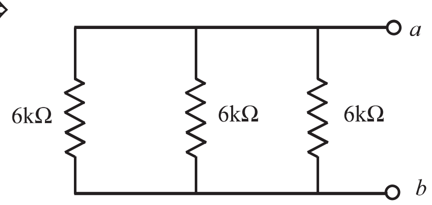


Figure 3.88(c)

$$
\begin{aligned}
R_t = R_{ab} &= 6\ \mathrm{k\Omega}||6\ \mathrm{k\Omega}||6\ \mathrm{k\Omega} \\
&= 2\ \mathrm{k\Omega}
\end{aligned}
$$

To find V_t :

Refer the Fig. 3.88(d).

Constraint equation :

$$V_3 - V_1 = 12\ \mathrm{V}$$

By inspection, $V_2 = 3$ V

KCL at supernode :

$$\frac{V_3 - V_2}{6\mathrm{k}} + \frac{V_1}{6\mathrm{k}} + \frac{V_1 - V_2}{6\mathrm{k}} = 0$$

$$\Rightarrow \qquad \frac{V_3 - 3}{6\mathrm{k}} + \frac{V_3 - 12}{6\mathrm{k}} + \frac{V_3 - 12 - 3}{6\mathrm{k}} = 0$$

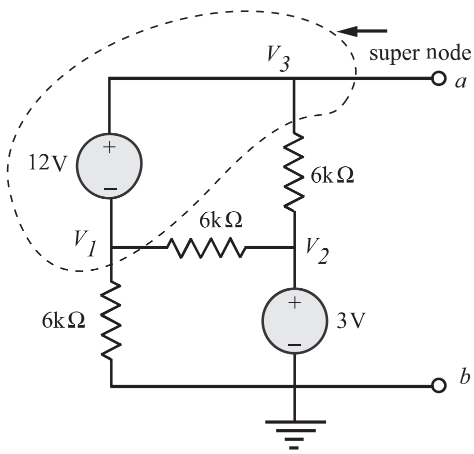


Figure 3.88(d)

$$\Rightarrow \quad V_3 - 3 + V_3 - 12 + V_3 - 15 = 0$$

$$\Rightarrow \quad 3V_3 = 30$$

$$\Rightarrow \quad V_3 = 10$$

$$\Rightarrow \quad V_t = V_{ab} = V_3 = 10 \text{ V}$$

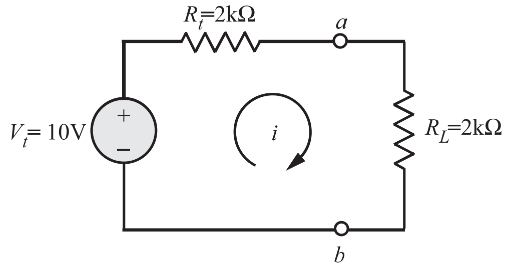


Figure 3.88(e)

The Thevenin equivalent circuit connected to the load resistor R_L is shown in Fig. 3.88(e).

$$\begin{aligned} P_{\max} &= i^2 R_L \\ &= \left[\frac{V_t}{2R_L}\right]^2 R_L \\ &= 12.5 \text{ mW} \end{aligned}$$

Alternate method :

It is possible to find $P_{\max}$, without finding the Thevenin equivalent circuit. However, we have to find R_t. For maximum power transfer, $R_L = R_t = 2$ kΩ. Insert the value of R_L in the original circuit given in Fig. 3.87. Then use any circuit reduction technique of your choice to find power dissipated in R_L.

Refer Fig. 3.88(f). By inspection we find that, $V_2 = 3$ V.

Constraint equation :

$$V_3 - V_1 = 12$$

$$\Rightarrow \quad V_1 = V_3 - 12$$

KCL at supernode :

$$\frac{V_3 - V_2}{6\text{k}} + \frac{V_1 - V_2}{6\text{k}} + \frac{V_3}{2\text{k}} + \frac{V_1}{6\text{k}} = 0$$

$$\Rightarrow \frac{V_3 - 3}{6\text{k}} + \frac{V_3 - 12 - 3}{6\text{k}} + \frac{V_3}{2\text{k}} + \frac{V_3 - 12}{6\text{k}} = 0$$

$$\Rightarrow \quad V_3 - 3 + V_3 - 15 + 3V_3 + V_3 - 12 = 0$$

$$\Rightarrow \quad 6V_3 = 30$$

$$\Rightarrow \quad V_3 = 5 \text{ V}$$

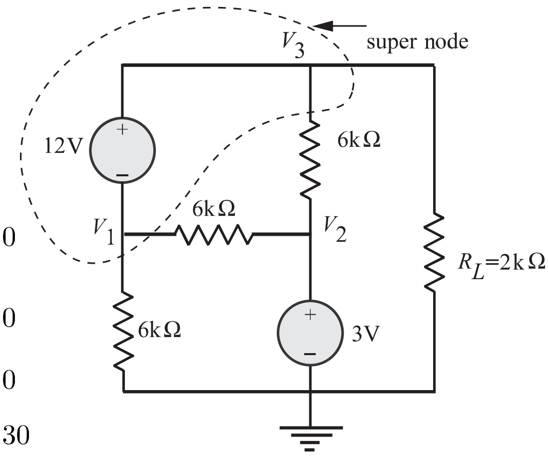


Figure 3.88(f)

Hence, $$P_{\max} = \frac{V_3^2}{R_L} = \frac{25}{2\text{k}} = \mathbf{12.5\ mW}$$

EXAMPLE 3.29

Find R_L for maximum power transfer and the maximum power that can be transferred in the network shown in Fig. 3.89.

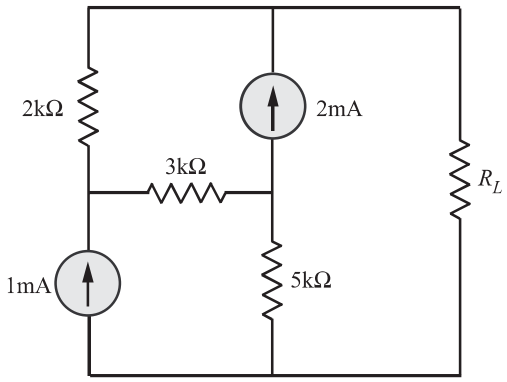


Figure 3.89

SOLUTION

Disconnect the load resistor R_L. This results in a circuit as shown in Fig. 3.89(a).

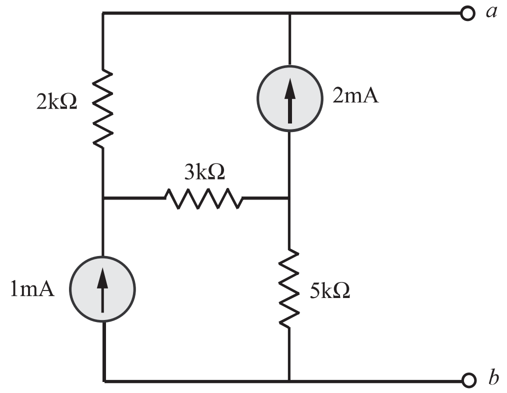


Figure 3.89(a)

To find R_t, let us deactivate all the independent sources, which results the circuit as shown in Fig. 3.89(b).

$$R_t = R_{ab} = 2\ \text{k}\Omega + 3\ \text{k}\Omega + 5\ \text{k}\Omega = 10\ \text{k}\Omega$$

For maximum power transfer $R_L = R_t$ = 10 kΩ.

Let us next find V_{oc} or V_t.

Refer Fig. 3.89 (c). By inspection, $i_1 = -2$ mA & $i_2 = 1$ mA.

Figure 3.89(b)

Applying KVL clockwise to the loop $5\text{ k}\Omega \to 3\text{ k}\Omega \to 2\text{ k}\Omega \to a-b$, we get

$$-5\text{k}\times i_2 + 3\text{k}\,(i_1 - i_2) + 2\text{k}\times i_1 + V_t = 0$$

$$\Rightarrow -5\times 10^3\left(1\times 10^{-3}\right) + 3\times 10^3\left(-2\times 10^{-3} - 1\times 10^{-3}\right) + 2\times 10^3\left(-2\times 10^{-3}\right) + V_t = 0$$

$$\Rightarrow \quad -5 - 9 - 4 + V_t = 0$$

$$\Rightarrow \quad V_t = 18\text{ V.}$$

The Thevenin equivalent circuit with load resistor R_L is as shown in Fig. 3.89 (d).

$$i = \frac{18}{(10+10)\times 10^3} = 0.9\text{ mA}$$

Then,

$$P_{\max} = P_L = (0.9\text{ mA})^2 \times 10\text{ k}\Omega$$
$$= 8.1\text{ mW}$$

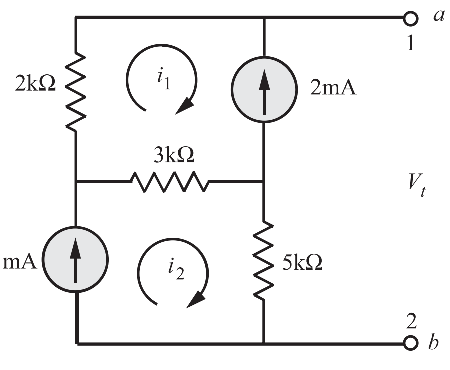


Figure 3.89(c)

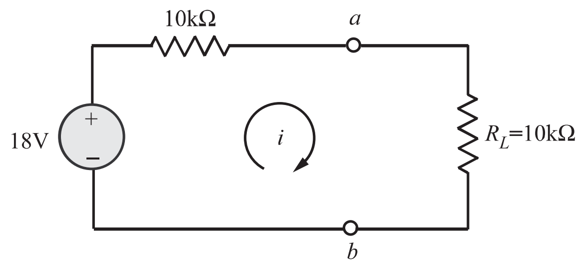


Figure 3.89(d)

EXAMPLE 3.30

Find the maximum power dissipated in R_L. Refer the circuit shown in Fig. 3.90.

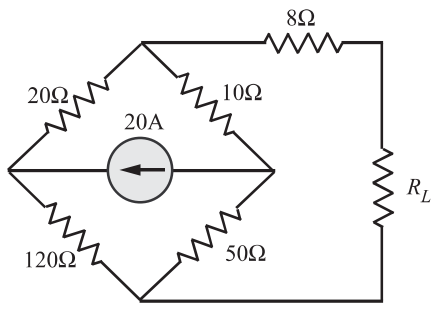


Figure 3.90

SOLUTION

Disconnecting the load resistor R_L from the original circuit results in a circuit diagram as shown in Fig. 3.91.

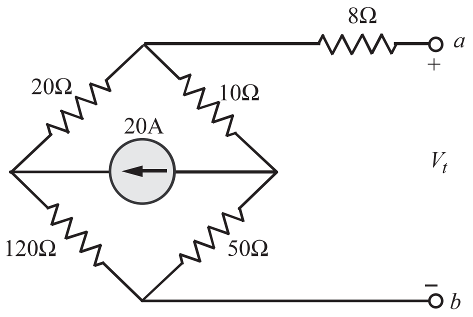


Figure 3.91

As a first step in the analysis, let us find R_t. While finding R_t, we have to deactivate all the independent sources. This results in a network as shown in Fig 3.91 (a) :

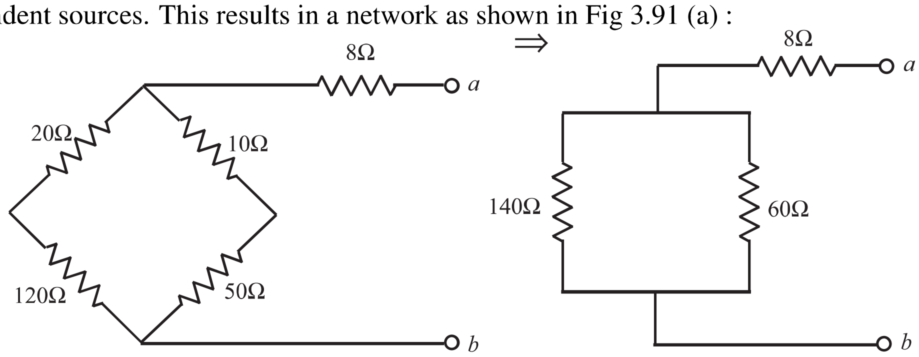


Figure 3.91(a)

$$R_t = R_{ab} = [140\ \Omega || 60\ \Omega] + 8\ \Omega$$
$$= \frac{140 \times 60}{140 + 60} + 8 = 50\ \Omega.$$

For maximum power transfer, $R_L = R_t = 50\ \Omega$. Next step in the analysis is to find V_t. Refer Fig 3.91(b), using the principle of current division,

$$i_1 = \frac{i \times R_2}{R_1 + R_2}$$
$$= \frac{20 \times 170}{170 + 30} = 17\ \text{A}$$
$$i_2 = \frac{i \times R_1}{R_1 + R_2} = \frac{20 \times 30}{170 + 30}$$
$$= \frac{600}{200} = 3\text{A}$$

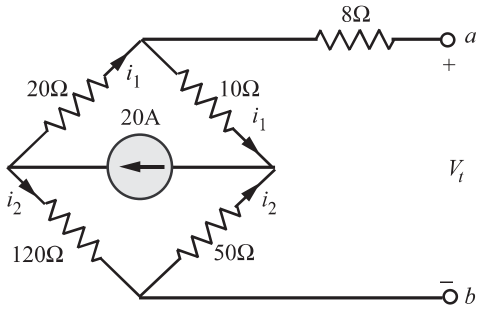


Figure 3.91(a)

Applying KVL clockwise to the loop comprising of 50 Ω → 10 Ω → 8 Ω → $a - b$, we get

$$50i_2 - 10i_1 + 8 \times 0 + V_t = 0$$
$$\Rightarrow 50(3) - 10\,(17) + V_t = 0$$
$$\Rightarrow \qquad V_t = 20 \text{ V}$$

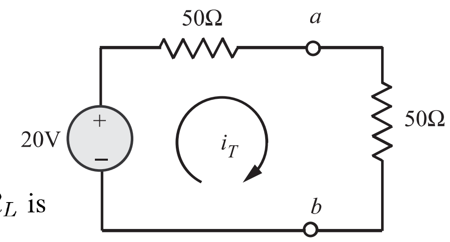


Figure 3.91(c)

The Thevenin equivalent circuit with load resistor R_L is as shown in Fig. 3.91(c).

$$i_T = \frac{20}{50 + 50} = 0.2\text{A}$$
$$P_{\max} = i_T^2 \times 50 = 0.04 \times 50 = \mathbf{2\ W}$$

EXAMPLE 3.31

Find the value of R_L for maximum power transfer in the circuit shown in Fig. 3.92. Also find $P_{\max}$.

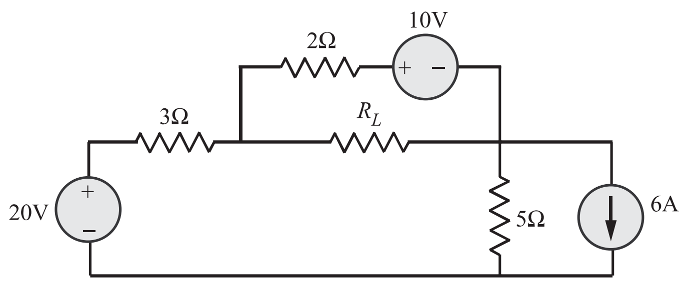


Figure 3.92

SOLUTION

Disconnecting R_L from the original circuit, we get the network shown in Fig. 3.93.

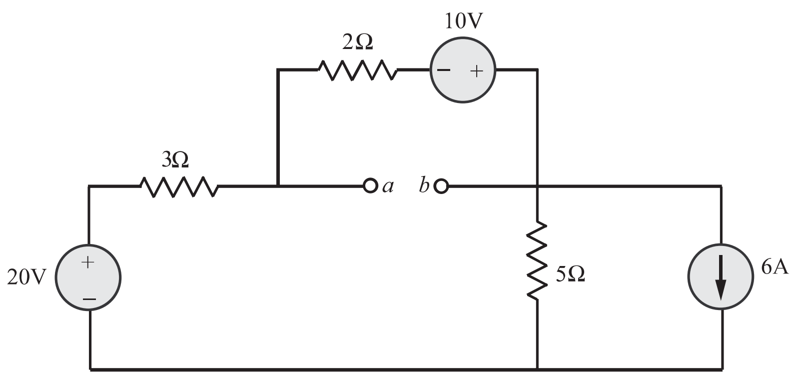


Figure 3.93

Let us draw the Thevenin equivalent circuit as seen from the terminals $a - b$ and then insert the value of $R_L = R_t$ between the terminals $a - b$. To find R_t, let us deactivate all independent sources which results in the circuit as shown in Fig. 3.94.

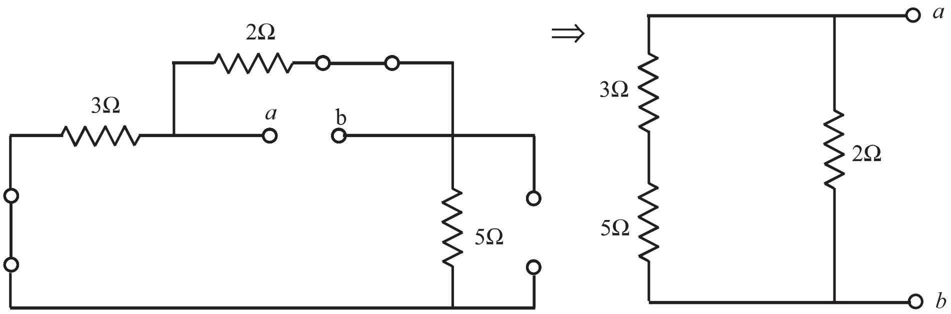


Figure 3.94

$$
\begin{aligned}
R_t &= R_{ab} \\
&= 8\ \Omega || 2\ \Omega \\
&= \frac{8 \times 2}{8+2} = 1.6\ \Omega
\end{aligned}
$$

Next step is to find V_{oc} or V_t.

By performing source transformation on the circuit shown in Fig. 3.93, we obtain the circuit shown in Fig. 3.95.

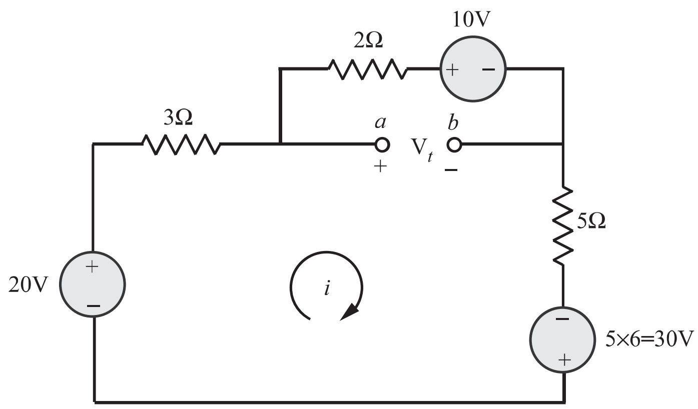


Figure 3.95

Applying KVL to the loop made up of 20 V → 3 Ω → 2 Ω → 10 V → 5 Ω → 30 V, we get

$$-20 + 10i - 10 - 30 = 0$$

$$\Rightarrow \quad i = \frac{60}{10} = 6\text{A}$$

Again applying KVL clockwise to the path $2\ \Omega \to 10\ \text{V} \to a - b$, we get

$$2i - 10 - V_t = 0$$

$$\Rightarrow \qquad V_t = 2i - 10$$

$$= 2(6) - 10 = 2\ \text{V}$$

The Thevenin equivalent circuit with load resistor R_L is as shown in Fig. 3.95 (a).

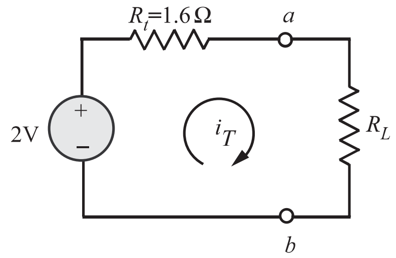


Figure 3.95(a) Thevenin equivalent circuit

$$P_{\max} = i_T^2 R_L$$

$$= \frac{V_t^2}{4R_t} = \mathbf{625}\ \text{mW}$$

EXAMPLE 3.32

Find the value of R_L for maximum power transfer. Hence find $P_{\max}$.

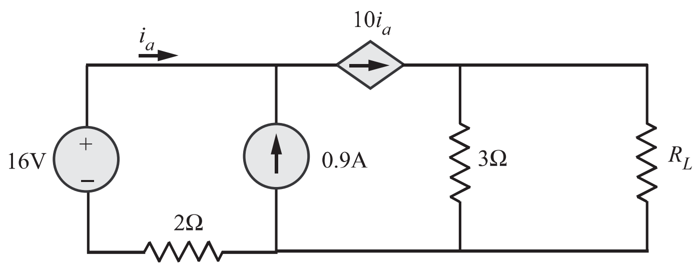


Figure 3.96

SOLUTION

Removing R_L from the original circuit gives us the circuit diagram shown in Fig. 3.97.

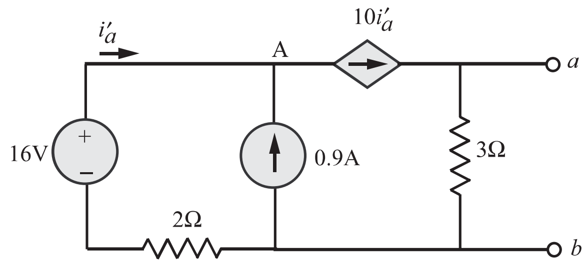


Figure 3.97

To find V_{oc} :

KCL at node A :

$$-i'_a - 0.9 + 10i'_a = 0$$

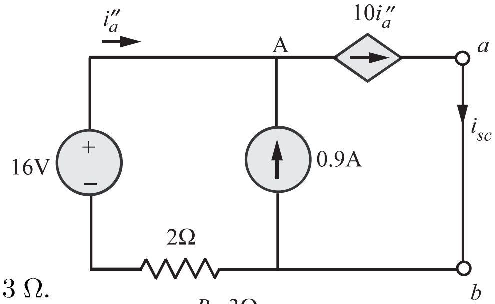
$$\Rightarrow \qquad i'_a = 0.1\ \text{A}$$

Hence, $\qquad V_{oc} = 3\left(10i'_a\right)$

$$= 3 \times 10 \times 0.1 = 3\ \text{V}$$

To find R_t, we need to compute i_{sc} with all independent sources activated.

KCL at node A:



$$-i_a'' - 0.9 + 10i_a'' = 0$$

$$\Rightarrow \qquad i_a'' = 0.1 \text{ A}$$

Hence $i_{sc} = 10i_a'' = 10 \times 0.1 = 1$ A

$$R_t = \frac{V_{oc}}{i_{sc}} = \frac{3}{1} = 3\ \Omega$$

Hence, for maximum power transfer $R_L = R_t = 3\ \Omega$. The Thevenin equivalent circuit with $R_L = 3\ \Omega$ inserted between the terminals $a-b$ gives the network shown in Fig. 3.97(a).

$$i_T = \frac{3}{3+3} = 0.5 \text{ A}$$

$$P_{\max} = i_T^2 R_L$$

$$= (0.5)^2 \times 3$$

$$= \mathbf{0.75} \text{ W}$$

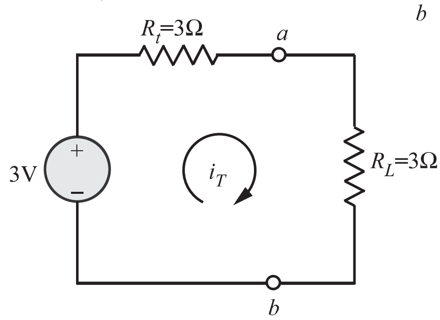


Figure 3.97(a)

EXAMPLE 3.33

Find the value of R_L in the network shown that will achieve maximum power transfer, and determine the value of the maximum power.

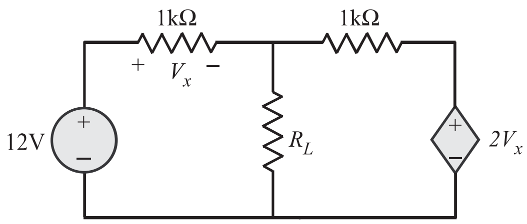


Figure 3.98(a)

SOLUTION

Removing R_L from the circuit of Fig. 3.98(a), we get the circuit of Fig 3.98(b).

Applying KVL clockwise we get

$-12 + 2 \times 10^3 i + 2V_x' = 0$

Also $\qquad V_x' = 1 \times 10^3 i$

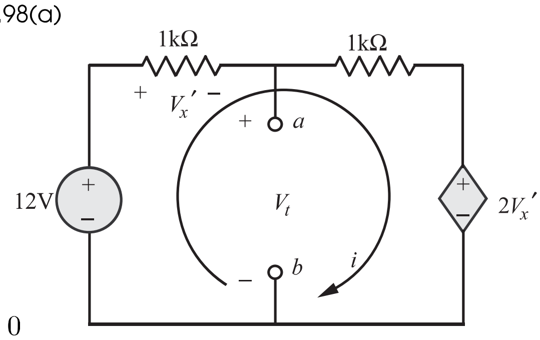


Figure 3.98(b)

Hence, $\quad -12 + 2 \times 10^3 i + 2\left(1 \times 10^3 i\right) = 0$

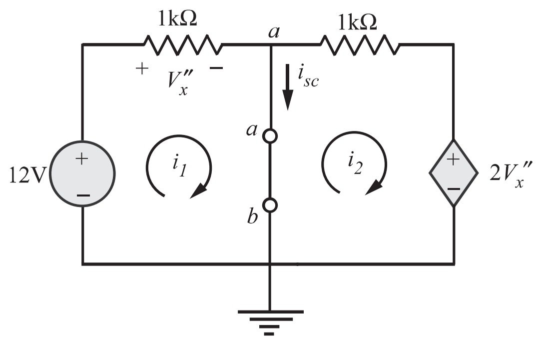
$$i = \frac{12}{4 \times 10^3} = 3 \text{ mA}$$

Applying KVL to loop $1\text{ k}\Omega \rightarrow 2V_x' \rightarrow b-a$, we get

$$1\times 10^3 i + 2V_x' - V_t = 0$$

$$\Rightarrow \quad V_t = 1\times 10^3 i + 2\left(1\times 10^3 i\right) = \left(1\times 10^3 + 2\times 10^3\right) i = 3\times 10^3\left(3\times 10^{-3}\right) = 9\text{ V}$$

To find R_t, we need to find i_{sc}. While finding i_{sc}, none of the independent sources must be deactivated.



Applying KVL to mesh 1:

$$-12 + V_x'' + 0 = 0$$

$$\Rightarrow \quad V_x'' = 12$$

$$\Rightarrow \quad 1\times 10^3 i_1 = 12 \;\Rightarrow\; i_1 = 12\text{ mA}$$

Applying KVL to mesh 2:

$$1\times 10^3 i_2 + 2V_x'' = 0$$

$$\Rightarrow \quad 1\times 10^3 i_2 = -24$$

$$i_2 = -24\text{ mA}$$

Applying KCL at node a:

$$i_{sc} = i_1 - i_2 = 12 + 24 = 36\text{ mA}$$

Hence,

$$R_t = \frac{V_t}{i_{sc}} = \frac{V_{oc}}{i_{sc}} = \frac{9}{36\times 10^{-3}} = 250\ \Omega$$

For maximum power transfer, $R_L = R_t = 250\ \Omega$. Thus, the Thevenin equivalent circuit with R_L is as shown in Fig 3.98 (c) :

$$i_T = \frac{9}{250+250} = \frac{9}{500}\text{A}$$

$$P_{\max} = i_T^2 \times 250 = \left(\frac{9}{500}\right)^2 \times 250 = \mathbf{81\ mW}$$

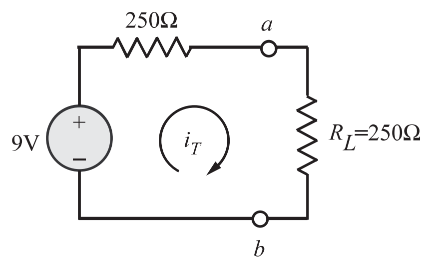


Figure 3.98 (c) Thevenin equivalent circuit

EXAMPLE 3.34

The variable resistor R_L in the circuit of Fig. 3.99 is adjusted untill it absorbs maximum power from the circuit.

(a) Find the value of R_L.

(b) Find the maximum power.

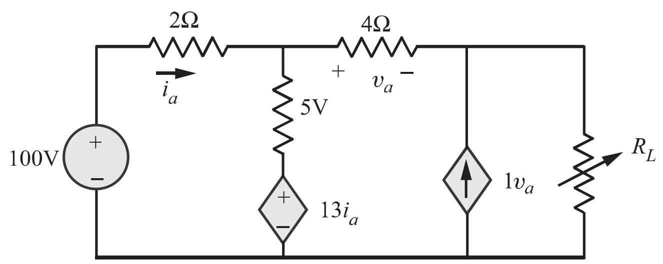


Figure 3.99

SOLUTION

Disconnecting the load resistor R_L from the original circuit, we get the circuit shown in Fig. 3.99(a).

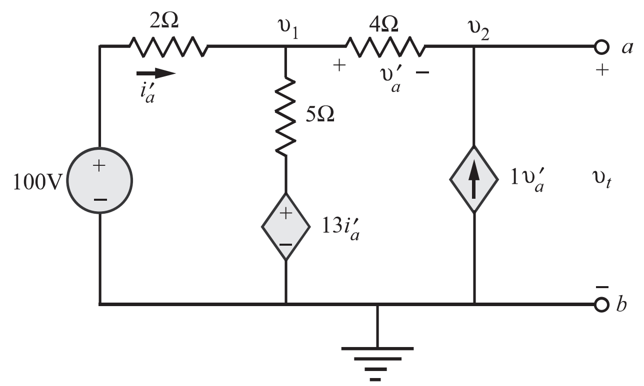


Figure 3.99(a)

KCL at node v_1 :

$$\frac{v_1-100}{2}+\frac{v_1-13i'_a}{5}+\frac{v_1-v_2}{4}=0 \tag{3.18}$$

Constraint equations :

$$i'_a=\frac{100-v_1}{2} \tag{3.19}$$

$$\frac{v_2-v_1}{4}=v'_a \qquad (applying\ KCL\ at\ v_2) \tag{3.20}$$

$$v'_a=v_1-v_2 \qquad (potential\ across\ 4\,\Omega) \tag{3.21}$$

From equations (3.20) and (3.21), we have

$$\frac{v_2 - v_1}{4} = v_1 - v_2$$

$$\Rightarrow \quad v_2 - v_1 = 4v_1 - 4v_2$$

$$\Rightarrow \quad 5v_1 - 5v_2 = 0$$

$$\Rightarrow \quad v_1 = v_2 \tag{3.22}$$

Making use of equations (3.19) and (3.22) in (3.18), we get

$$\frac{v_1 - 100}{2} + \frac{v_2 - 13\dfrac{(100 - v_1)}{2}}{5} + \frac{v_1 - v_1}{4} = 0$$

$$\Rightarrow \quad 5\,(v_1 - 100) + 2\left[v_1 - 13\frac{(100 - v_1)}{2}\right] = 0$$

$$\Rightarrow \quad 5v_1 - 500 + 2v_1 - 13 \times 100 + 13v_1 = 0$$

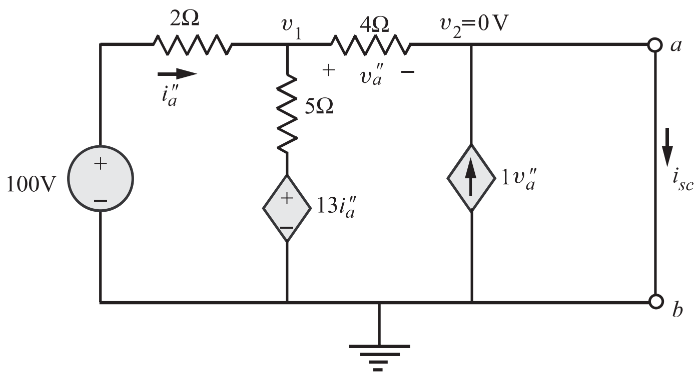
$$\Rightarrow \quad 20v_1 = 1800$$

$$\Rightarrow \quad v_1 = 90 \text{ Volts}$$

Hence, $\quad v_t = v_2 = v_1 = 90$ Volts

We know that, $\quad R_t = \dfrac{v_{oc}}{i_{sc}} = \dfrac{v_t}{i_{sc}}$

The short circuit current is calculated using the circuit shown below:



Here $\quad i_a'' = \dfrac{100 - v_1}{2}$

Applying KCL at node v_1 :

$$\frac{v_1 - 100}{2} + \frac{v_1 - 13i_a''}{5} + \frac{v_1 - 0}{4} = 0$$

$$\Rightarrow \quad \frac{v_1 - 100}{2} + \frac{v_1 - 13\dfrac{(100 - v_1)}{2}}{5} + \frac{v_1}{4} = 0$$

Solving we get $v_1 = 80$ volts $= v_a''$
Applying KCL at node a :

$$\frac{0 - v_1}{4} + i_{sc} = v_a''$$

$$\Rightarrow \qquad i_{sc} = \frac{v_1}{4} + v_a''$$

$$= \frac{80}{4} + 80 = 100 \text{ A}$$

Hence,

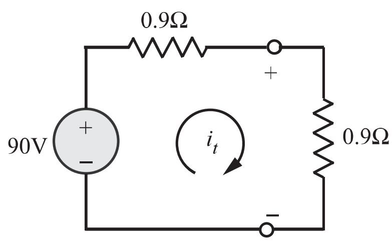
$$R_t = \frac{v_{oc}}{i_{sc}} = \frac{v_t}{i_{sc}}$$

$$= \frac{90}{100} = 0.9\ \Omega$$

Hence for maximum power transfer,

$$R_L = R_t = \mathbf{0.9\ \Omega}$$

The Thevenin equivalent circuit with $R_L = 0.9\ \Omega$ is as shown.



$$i_t = \frac{90}{0.9 + 0.9} = \frac{90}{1.8}$$

$$P_{\max} = i_t^2 \times 0.9$$

$$= \left(\frac{90}{1.8}\right)^2 \times 0.9 = \mathbf{2250\ W}$$

EXAMPLE 3.35

Refer to the circuit shown in Fig. 3.100 :

(a) Find the value of R_L for maximum power transfer.

(b) Find the maximum power that can be delivered to R_L.

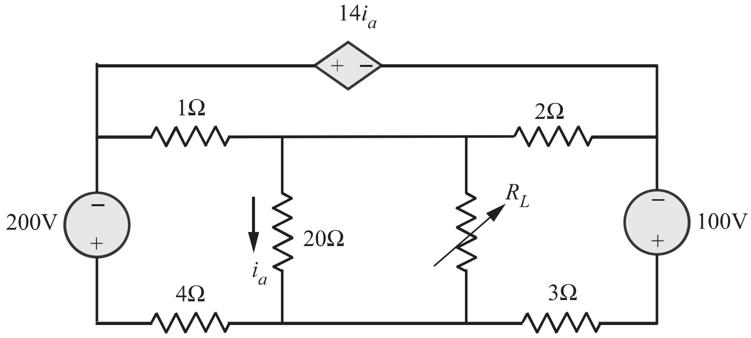


Figure 3.100

SOLUTION

Removing the load resistor R_L, we get the circuit diagram shown in Fig. 3.100(a). Let us proceed to find V_t.

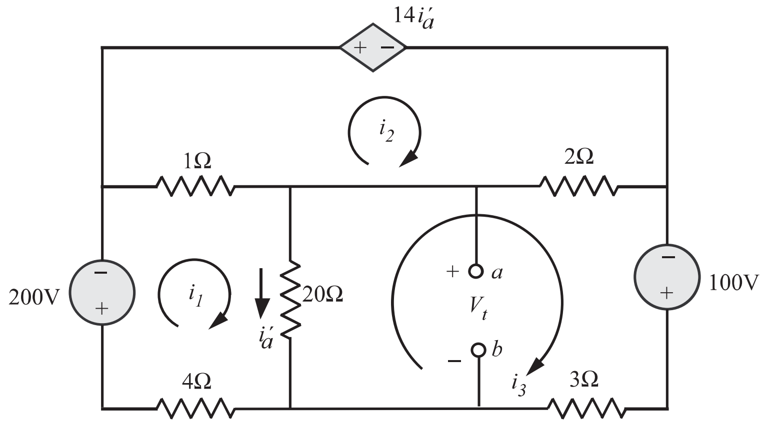


Figure 3.100(a)

Constraint equation :

$$i'_a = i_1 - i_3$$

KVL clockwise to mesh 1 :

$$200 + 1\,(i_1 - i_2) + 20\,(i_1 - i_3) + 4i_1 = 0$$

$$\Rightarrow \qquad 25i_1 - i_2 - 20i_3 = -200$$

KVL clockwise to mesh 2 :

$$14i'_a + 2\,(i_2 - i_3) + 1\,(i_2 - i_1) = 0$$

$$\Rightarrow \qquad 14\,(i_1 - i_3) + 2\,(i_2 - i_3) + 1\,(i_2 - i_1) = 0$$

$$\Rightarrow \qquad 13i_1 + 3i_2 - 16i_3 = 0$$

KVL clockwise to mesh 3 :

$$2\,(i_3 - i_2) - 100 + 3i_3 + 20\,(i_3 - i_1) = 0$$

$$\Rightarrow \qquad -20i_1 - 2i_2 + 25i_3 = 100$$

Solving the mesh equations, we get

$$i_1 = -2.5\text{A}, i_3 = 5\text{A}$$

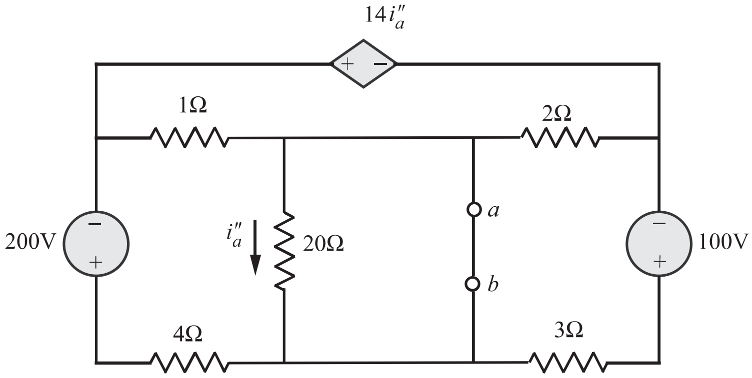
Applying KVL clockwise to the path comprising of $a - b \rightarrow 20\ \Omega$, we get

$$V_t - 20i'_a = 0$$

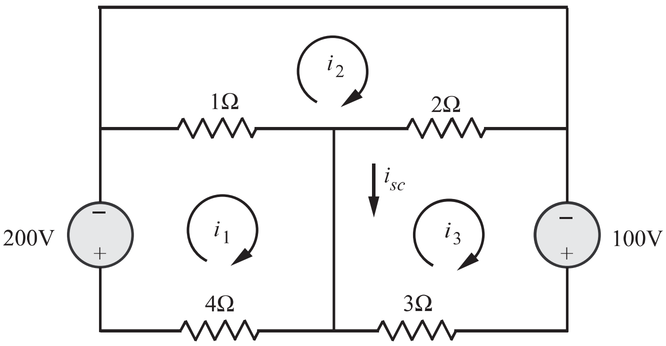
$$\begin{aligned}\Rightarrow \qquad V_t &= 20i'_a \\ &= 20\,(i_1 - i_3) \\ &= 20\,(-2.5 - 5) \\ &= -150\text{ V}\end{aligned}$$

Next step is to find R_t.

$$R_t = \frac{V_{oc}}{i_{sc}} = \frac{V_t}{i_{sc}}$$



When terminals $a - b$ are shorted, $i_a'' = 0$. Hence, $14\ i_a''$ is also zero.



KVL clockwise to mesh 1 :

$$200 + 1\,(i_1 - i_2) + 4i_1 = 0$$
$$\Rightarrow \quad 5i_1 - i_2 = -200$$

KVL clockwise to mesh 2 :

$$2\,(i_2 - i_3) + 1\,(i_2 - i_1) = 0$$
$$\Rightarrow \quad -i_1 + 3i_2 - 2i_3 = 0$$

KVL clockwise to mesh 3 :

$$-100 + 3i_3 + 2\,(i_3 - i_2) = 0$$
$$\Rightarrow \quad -2i_2 + 5i_3 = 100$$

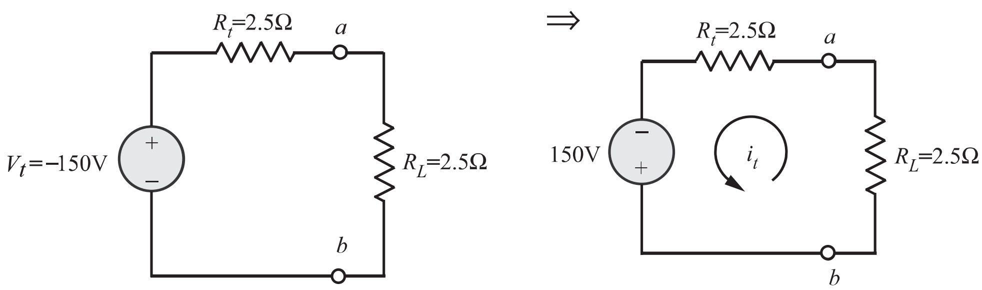
Solving the mesh equations, we find that

$$i_1 = -40\text{A},\ i_3 = 20\text{A},$$

$$\Rightarrow \qquad i_{sc} = i_1 - i_3 = -60\text{A}$$

$$R_t = \frac{V_t}{i_{sc}} = \frac{-150}{-60} = 2.5\ \Omega$$

For maximum power transfer, $R_L = R_t = 2.5\ \Omega$. The Thevenin equivalent circuit with R_L is as shown below :



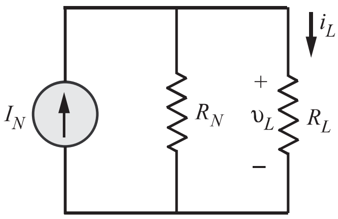
$$\begin{aligned} P_{\max} &= i_1^2 R_L \\ &= \left[\frac{150}{2.5+2.5}\right]^2 \times 2.5 \\ &= \mathbf{2250\ W} \end{aligned}$$

EXAMPLE 3.36

A practical current source provides 10 W to a $250\ \Omega$ load and 20 W to an $80\ \Omega$ load. A resistance R_L, with voltage v_L and current i_L, is connected to it. Find the values of R_L, v_L and i_L if (a) $v_L i_L$ is a maximum, (b) v_L is a maximum and (c) i_L is a maximum.

SOLUTION

Load current calculation:



$$10\text{W to } 250\ \Omega \text{ corresponds to } i_L = \sqrt{\frac{10}{250}}$$
$$= 200\text{ mA}$$
$$20\text{W to } 80\ \Omega \text{ corresponds to } i_L = \sqrt{\frac{20}{80}}$$
$$= 500\text{ mA}$$

Using the formula for division of current between two parallel branches :

$$i_2 = \frac{i \times R_1}{R_1 + R_2}$$

In the present context, $$0.2 = \frac{I_N R_N}{R_N + 250} \tag{3.23}$$

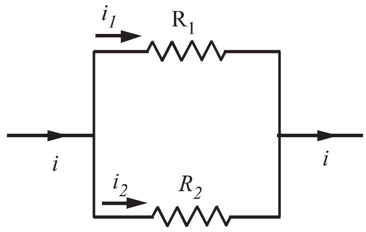
and $$0.5 = \frac{I_N R_N}{R_N + 80} \tag{3.24}$$

Solving equations (3.23) and (3.24), we get

$$I_N = 1.7 \text{ A}$$
$$R_N = 33.33\ \Omega$$

(a) If $v_L i_L$ is maximum,

$$
\begin{aligned}
R_L &= R_N = 33.33\ \Omega \\
i_L &= 1.7 \times \frac{33.33}{33.33 + 33.33} \\
&= 850 \text{ mA} \\
v_L &= i_L R_L = 850 \times 10^{-3} \times 33.33 \\
&= 28.33 \text{ V}
\end{aligned}
$$



(b) $v_L = I_N (R_N || R_L)$ is a maximum when $R_N || R_L$ is a maximum, which occurs when $R_L = \infty$.

Then, $i_L = 0$ and

$$
\begin{aligned}
v_L &= 1.7 \times R_N \\
&= 1.7 \times 33.33 \\
&= 56.66 \text{ V}
\end{aligned}
$$

(c) $i_L = \dfrac{I_N R_N}{R_N + R_L}$ is maxmimum when $R_L = 0\ \Omega$

$\Rightarrow \qquad i_L = 1.7\text{A}$ and $v_L = 0$ V

3.5 Sinusoidal steady state analysis using superposition, Thevenin and Norton equivalents

Circuits in the frequency domain with phasor currents and voltages and impedances are analogous to resistive circuits.

To begin with, let us consider the principle of superposition, which may be restated as follows :

For a linear circuit containing two or more independent sources, any circuit voltage or current may be calculated as the algebraic sum of all the individual currents or voltages caused by each independent source acting alone.

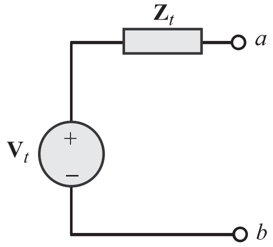


Figure 3.101 Thevenin equivalent circuit

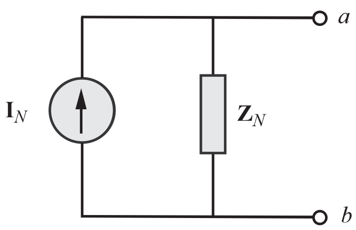


Figure 3.102 Norton equivalent circuit

The superposition principle is particularly useful if a circuit has two or more sources acting at different frequencies. The circuit will have one set of impedance values at one frequency and a different set of impedance values at another frequency. Phasor responses corresponding to different frequencies cannot be superposed; only their corresponding sinusoids can be superposed. That is, when frequencies differ, the principle of superposition applies to the summing of time domain components, not phasors. Within a component, problem corresponding to a single frequency, however phasors may be superposed.

Thevenin and Norton equivalents in phasor circuits are found exactly in the same manner as described earlier for resistive circuits, except for the subtitution of impedance $\mathbf{Z}$ in place of resistance R and subsequent use of complex arithmetic. The Thevenin and Norton equivalent circuits are shown in Fig. 3.101 and 3.102.

The Thevenin and Norton forms are equivalent if the relations

$$\text{(a) } \mathbf{Z}_t = \mathbf{Z}_N \qquad\qquad \text{(b) } \mathbf{V}_t = \mathbf{Z}_N\mathbf{I}_N$$

hold between the circuits.

A step by step procedure for finding the Thevenin equivalent circuit is as follows:

1. Identify a seperate circuit portion of a total circuit.
2. Find $\mathbf{V}_t = \mathbf{V}_{oc}$ at the terminals.
3. (a) If the circuit contains only impedances and independent sources, then deactivate all the independent sources and then find $\mathbf{Z}_t$ by using circuit reduction techniques.

   (b) If the circuit contains impedances, independent sources and dependent sources, then either short–circuit the terminals and determine $\mathbf{I}_{sc}$ from which

$$\mathbf{Z}_t = \frac{\mathbf{V}_{oc}}{\mathbf{I}_{sc}}$$

or deactivate the independent sources, connect a voltage or current source at the terminals, and determine both $\mathbf{V}$ and $\mathbf{I}$ at the terminals from which

$$\mathbf{Z}_t = \frac{\mathbf{V}}{\mathbf{I}}$$

A step by step procedure for finding Norton equivalent circuit is as follows:

(i) Identify a seperate circuit portion of the original circuit.

(ii) Short the terminals after seperating a portion of the original circuit and find the current through the short circuit at the terminals, so that $\mathbf{I}_N = \mathbf{I}_{sc}$.

(iii) (a) If the circuit contains only impedances and independent sources, then deactivate all the independent sources and then find $\mathbf{Z}_N = \mathbf{Z}_t$ by using circuit reduction techniques.

(b) If the circuit contains impedances, independent sources and one or more dependent sources, find the open–circuit voltage at the terminals, $\mathbf{V}_{oc}$, so that $\mathbf{Z}_N = \mathbf{Z}_t = \dfrac{\mathbf{V}_{oc}}{\mathbf{I}_{sc}}$.

EXAMPLE 3.37

Find the Thevenin and Norton equivalent circuits at the terminals $a - b$ for the circuit in Fig. 3.103.

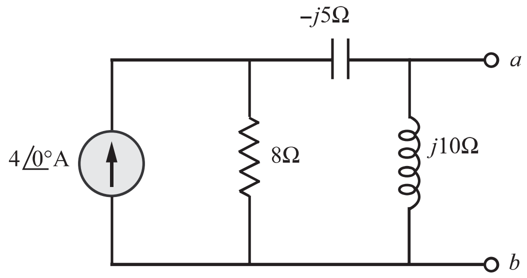
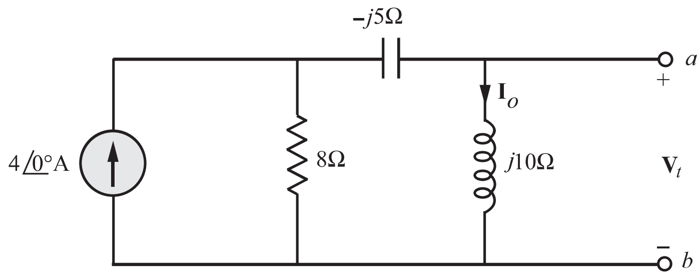


Figure 3.103

SOLUTION

As a first step in the analysis, let us find $\mathbf{V}_t$.



Using the principle of current division,

$$\mathbf{I}_o = \frac{8\,(4\,\underline{/0^\circ})}{8 + j10 - j5} = \frac{32}{8 + j5}$$

$$\mathbf{V}_t = \mathbf{I}_o(j10) = \frac{j320}{8 + j5} = 33.92\,\underline{/58^\circ}\ \ \text{V}$$

To find $\mathbf{Z}_t$, deactivate all the independent sources. This results in a circuit diagram as shown in Fig. 3.103 (a).

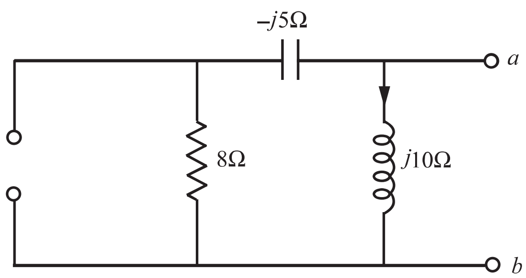


Figure 3.103(a)

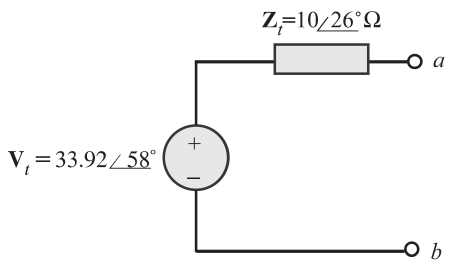


Figure 3.103(b) Thevenin equivalent circuit

$$
\begin{aligned}
\mathbf{Z}_t &= j10 \,||\, (8 - j5)\ \Omega \\
&= \frac{(j10)(8-j5)}{j10 + 8 - j5} \\
&= 10\,\underline{/26^\circ}\ \ \Omega
\end{aligned}
$$

The Thevenin equivalent circuit as viewed from the terminals $a - b$ is as shown in Fig 3.103(b). Performing source transformation on the Thevenin equivalent circuit, we get the Norton equivalent circuit.

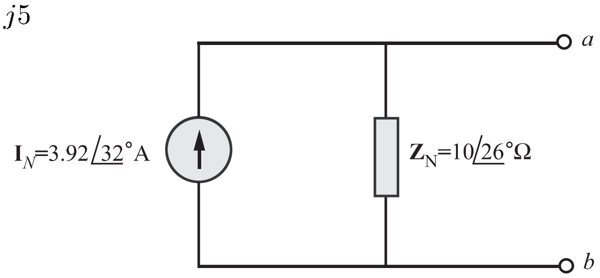


Figure : Norton equivalent circuit

$$
\begin{aligned}
\mathbf{I}_N &= \frac{\mathbf{V}_t}{\mathbf{Z}_t} = \frac{33.92\,\underline{/58^\circ}}{10\,\underline{/26^\circ}} \\
&= 3.392\,\underline{/32^\circ}\ \text{A} \\
\mathbf{Z}_N &= \mathbf{Z}_t = 10\,\underline{/26^\circ}\ \ \Omega
\end{aligned}
$$

EXAMPLE 3.38

Find v_o using Thevenin's theorem. Refer to the circuit shown in Fig. 3.104.

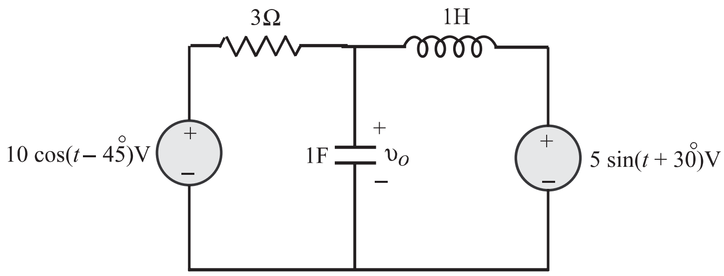


Figure 3.104

SOLUTION

Let us convert the circuit given in Fig. 3.104 into a frequency domain equiavalent or phasor circuit (shown in Fig. 3.105(a)). $\omega = 1$

$$
\begin{aligned}
10\cos(t - 45^\circ) &\quad\rightarrow\quad 10\,\underline{/-45^\circ}\ \ \text{V} \\
5\sin(t + 30^\circ) = 5\cos(t - 60^\circ) &\quad\rightarrow\quad 5\,\underline{/-60^\circ}\ \ \text{V}
\end{aligned}
$$

$$
\begin{aligned}
L = 1\text{H} &\rightarrow j\,\omega L = j \times 1 \times 1 = j1\ \Omega \\
C = 1\text{F} &\rightarrow \frac{1}{j\,\omega C} = \frac{1}{j \times 1 \times 1} = -j1\ \Omega
\end{aligned}
$$

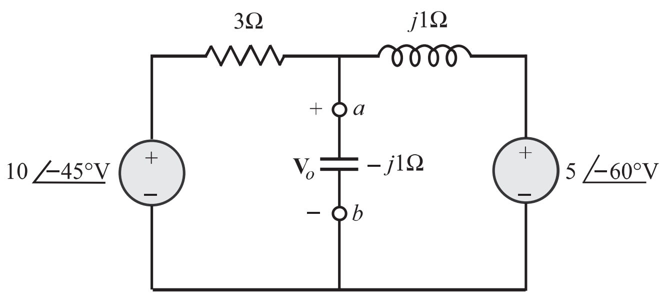


Figure 3.105(a)

Disconnecting the capicator from the original circuit, we get the circuit shown in Fig. 3.105(b). This circuit is used for finding $\mathbf{V}_t$.

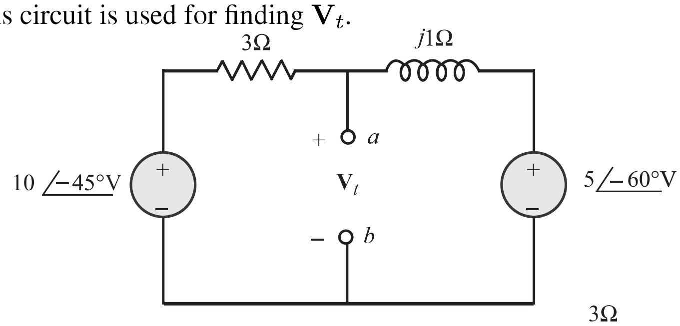


Figure 3.105(b)

KCL at node a :

$$\frac{\mathbf{V}_t - 10\,\underline{/-45^\circ}}{3} + \frac{\mathbf{V}_t - 5\,\underline{/-60^\circ}}{j1} = 0$$

Solving, $\qquad \mathbf{V}_t = 4.97\,\underline{/-40.54^\circ}$

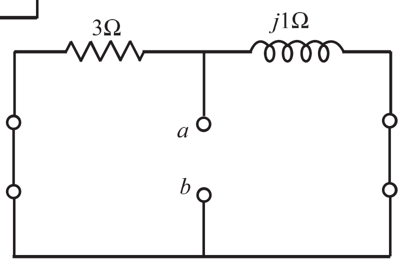


Figure 3.105(c)

To find $\mathbf{Z}_t$ deactivate all the independent sources in Fig. 3.105(b). This results in a network as shown in Fig. 3.105(c) :

$$\begin{aligned}\mathbf{Z}_t = \mathbf{Z}_{ab} &= 3\Omega || j1\ \Omega \\ &= \frac{j3}{3+j} = \frac{3}{10}(1+j3)\ \Omega\end{aligned}$$

The Thevenin equivalent circuit along with the capicator is as shown in Fig 3.105(d).

$$\begin{aligned}\mathbf{V}_o &= \frac{\mathbf{V}_t}{\mathbf{Z}_t - j1}(-j1) \\ &= \frac{4.97\,\underline{/-40.54^\circ}}{0.3(1+j3) - j1}(-j1) \\ &= 15.73\,\underline{/247.9^\circ}\ \ \mathrm{V}\end{aligned}$$

Hence, $\quad v_o = 15.73\cos(t + 247.9^\circ)\ \ \mathrm{V}$

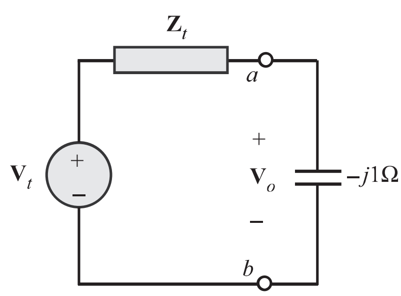


Figure 3.105(d) Thevenin equivalent circuit

EXAMPLE 3.39

Find the Thevenin equivalent circuit of the circuit shown in Fig. 3.106.

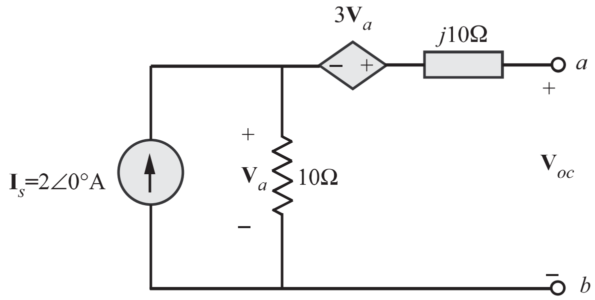


Figure 3.106

SOLUTION

Since terminals $a - b$ are open,

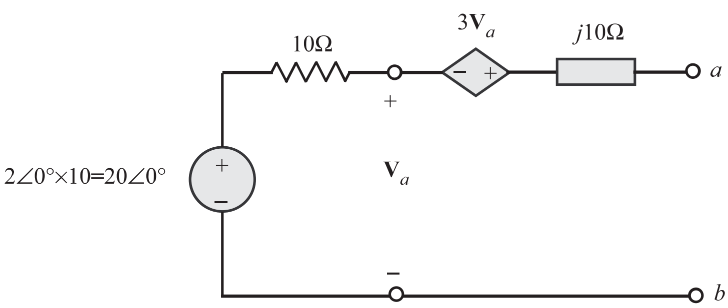
$$\mathbf{V}_a = \mathbf{I}_s \times 10$$
$$= 20\ \underline{/0^\circ}\ \ \text{V}$$

Applying KVL clockwise for the mesh on the right hand side of the circuit, we get

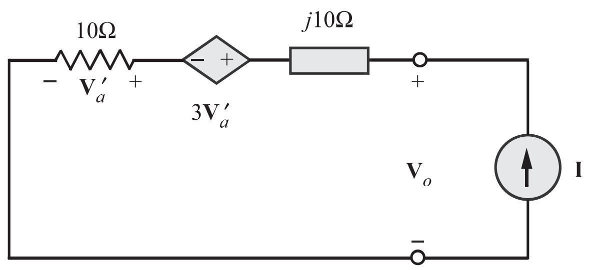
$$-3\mathbf{V}_a + 0\,(j10) + \mathbf{V}_{oc} - \mathbf{V}_a = 0$$

$$\mathbf{V}_{oc} = 4\mathbf{V}_a$$
$$= 80\ \underline{/0^\circ}\ \ \text{V}$$

Let us transform the current source with $10\ \Omega$ parallel resistance to a voltage source with $10\ \Omega$ series resistance as shown in figure below :



To find $\mathbf{Z}_t$, the independent voltage source is deactivated and a current source of $\mathbf{I}$ A is connected at the terminals as shown below :



Applying KVL clockwise we get,

$$-\mathbf{V}_a' - 3\mathbf{V}_a' - j10\mathbf{I} + \mathbf{V}_o = 0$$

$$\Rightarrow \quad -4\mathbf{V}_a' - j10\mathbf{I} + \mathbf{V}_o = 0$$

Since $\mathbf{V}_a' = 10\mathbf{I}$

we get $-40\mathbf{I} - j10\mathbf{I} = -\mathbf{V}_o$

Hence, $\mathbf{Z}_t = \dfrac{\mathbf{V}_o}{\mathbf{I}} = 40 + j10\Omega$

Hence the Thevenin equivalent circuit is as shown in Fig 3.106(a) :

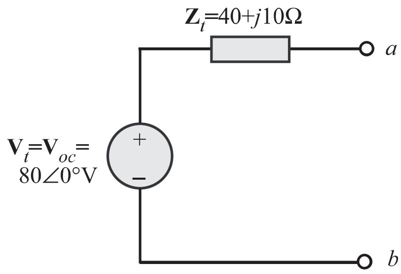


Figure 3.106(a)

EXAMPLE 3.40

Find the Thevenin and Norton equivalent circuits for the circuit shown in Fig. 3.107.

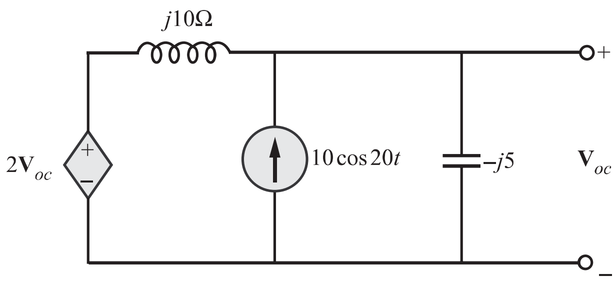


Figure 3.107

SOLUTION

The phasor equivalent circuit of Fig. 3.107 is shown in Fig. 3.108.

KCL at node a :

$$\frac{\mathbf{V}_{oc} - 2\mathbf{V}_{oc}}{j10} - 10 + \frac{\mathbf{V}_{oc}}{-j5} = 0$$

$$\Rightarrow \quad \mathbf{V}_{oc} = -j\frac{100}{3} = \frac{100}{3}\,\underline{/-90^\circ}\ \text{V}$$

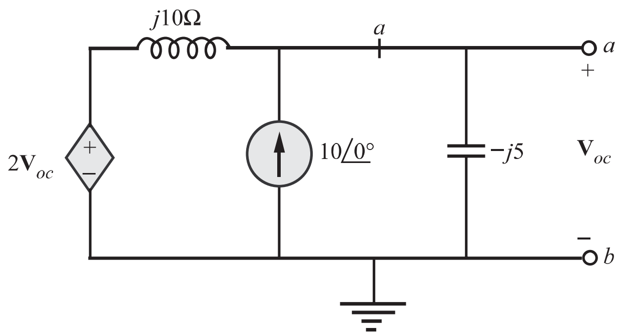


Figure 3.108

To find $\mathbf{I}_{sc}$, short the terminals $a - b$ of Fig. 3.108 as in Fig. 3.108(a).

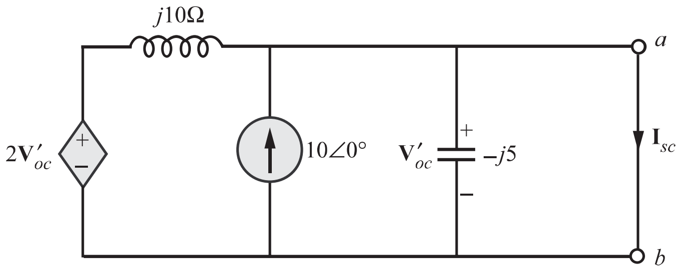


Figure 3.108 (a)

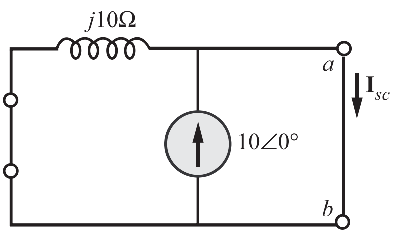


Figure 3.108 (b)

Since $\mathbf{V}_{oc} = 0$, the above circuit takes the form shown in Fig 3.108 (b).

$$\mathbf{I}_{sc} = 10 \underline{/0^\circ} \text{ A}$$

Hence,
$$\mathbf{Z}_t = \frac{\mathbf{V}_{oc}}{\mathbf{I}_{sc}} = \frac{\frac{100}{3} \underline{/-90^\circ}}{10 \underline{/0^\circ}} = \frac{10}{3} \underline{/-90^\circ} \ \Omega$$

The Thevenin equivalent and the Norton equivalent circuits are as shown below.

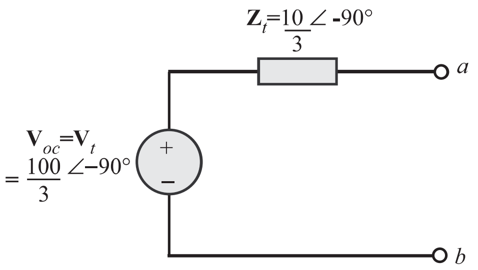


Figure Thevenin equivalent

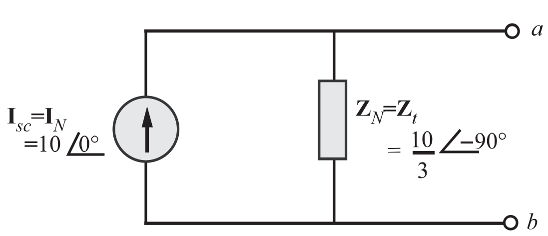


Figure Norton equivalent

EXAMPLE 3.41

Find the Thevenin and Norton equivalent circuits in frequency domain for the network shown in Fig. 3.109.

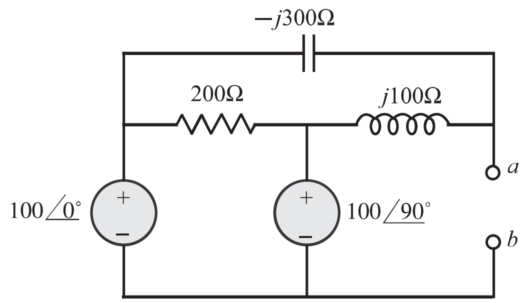
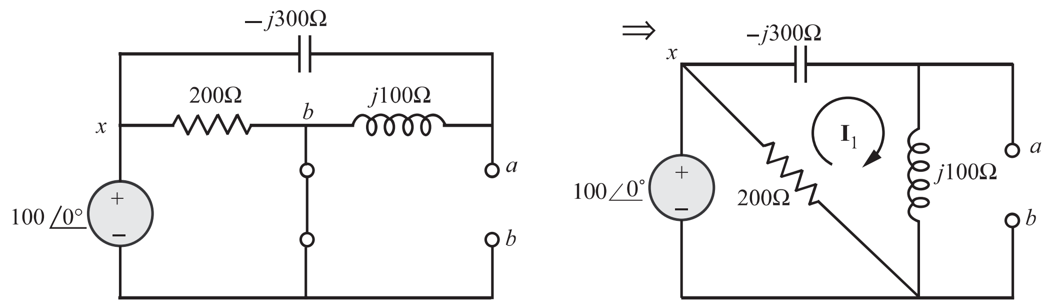


Figure 3.109

SOLUTION

Let us find $\mathbf{V}_t = \mathbf{V}_{ab}$ using superpostion theorem.

(i) $\mathbf{V}_{ab}$ due to $100\,\underline{/0^\circ}$

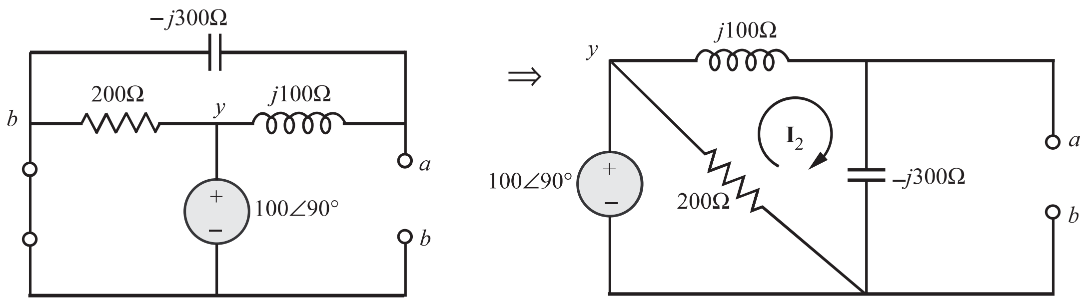


$$\mathbf{I}_1 = \frac{100\,\underline{/0^\circ}}{-j300 + j100} = \frac{100}{-j200}\text{A}$$

$$\mathbf{V}_{ab_1} = \mathbf{I}_1\,(j100)$$

$$= \frac{100}{-j200}\,(j100) = -50\,\underline{/0^\circ}\text{ Volts}$$

(ii) $\mathbf{V}_{ab}$ due to $100\,\underline{/90^\circ}$

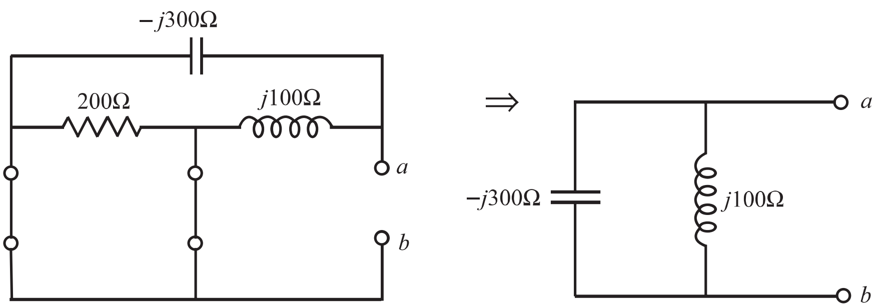


$$
\begin{aligned}
\mathbf{I}_2 &= \frac{100\,\underline{/90^\circ}}{j100 - j300} \\
\mathbf{V}_{ab_2} &= \mathbf{I}_2\,(-j300) \\
&= \frac{100\,\underline{/90^\circ}}{j100 - j300}\,(-j300) = j150\ \text{V}
\end{aligned}
$$

Hence,

$$
\begin{aligned}
\mathbf{V}_t &= \mathbf{V}_{ab_1} + \mathbf{V}_{ab_2} \\
&= -50 + j150 \\
&= 158.11\,\underline{/108.43^\circ}\ \ \text{V}
\end{aligned}
$$

To find $\mathbf{Z}_t$, deactivate all the independent sources.



$$
\begin{aligned}
\mathbf{Z}_t &= j100\ \Omega || -j300\ \Omega \\
&= \frac{j100(-j300)}{j100 - j300} = j150\ \Omega
\end{aligned}
$$

Hence the Thevenin equivalent circuit is as shown in Fig. 3.109(a). Performing source transformation on the Thevenin equivalent circuit, we get the Norton equivalent circuit.

$$
\begin{aligned}
\mathbf{I}_N &= \frac{\mathbf{V}_t}{\mathbf{Z}_t} = \frac{158.11\,\underline{/108.43^\circ}}{150\,\underline{/90^\circ}} = 1.054\,\underline{/18.43^\circ}\ \text{A} \\
\mathbf{Z}_N &= \mathbf{Z}_t = j150\ \Omega
\end{aligned}
$$

The Norton equivalent circuit is as shown in Fig. 3.109(b).

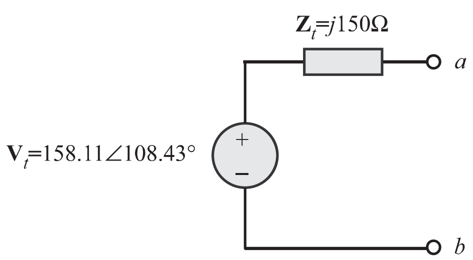


Figure 3.109(a)

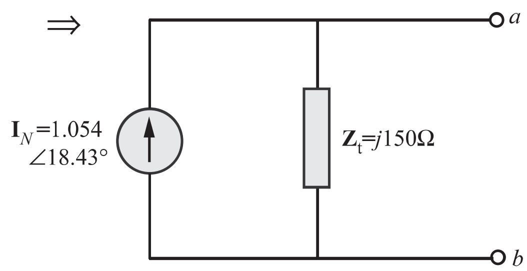


Figure 3.109(b)

3.6 Maximum power transfer theorem

We have earlier shown that for a resistive network, maximum power is transferred from a source to the load, when the load resistance is set equal to the Thevenin resistance with Thevenin equivalent source. Now we extend this result to the ac circuits.

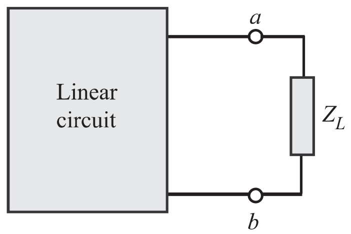


Figure 3.110 Linear circuit

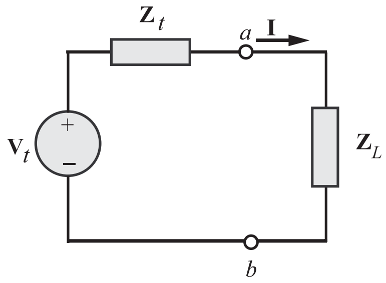


Figure 3.111 Thevenin equivalent circuit

In Fig. 3.110, the linear circuit is made up of impedances, independent and dependent sources. This linear circuit is replaced by its Thevenin equivalent circuit as shown in Fig. 3.111. The load impedance could be a model of an antenna, a TV, and so forth. In rectangular form, the Thevenin impedance $\mathbf{Z}_t$ and the load impedance $\mathbf{Z}_L$ are

$$\mathbf{Z}_t = R_t + jX_t$$

and

$$\mathbf{Z}_L = R_L + jX_L$$

The current through the load is

$$\mathbf{I} = \frac{\mathbf{V}_t}{\mathbf{Z}_t + \mathbf{Z}_L} = \frac{\mathbf{V}_t}{(R_t + jX_t) + (R_L + jX_L)}$$

The phasors $\mathbf{I}$ and $\mathbf{V}_t$ are the maximum values. The corresponding RMS values are obtained by dividing the maximum values by $\sqrt{2}$. Also, the RMS value of phasor current flowing in the load must be taken for computing the average power delivered to the load. The average power delivered to the load is given by

$$\begin{aligned} P &= \frac{1}{2}|\mathbf{I}|^2 R_L \\ &= \frac{|\mathbf{V}_t|^2 \frac{R_L}{2}}{(R_t + R_L)^2 (X_t + X_L)^2} \end{aligned} \tag{3.25}$$

Our idea is to adjust the load parameters R_L and X_L so that P is maximum. To do this, we get $\dfrac{\partial P}{\partial R_L}$ and $\dfrac{\partial P}{\partial X_L}$ equal to zero.

$$\frac{\partial P}{\partial X_L} = \frac{-|V_t|^2 R_L (X_t + X_L)}{\left[(R_t + R_L)^2 + (X_t + X_L)^2\right]^2}$$

$$\frac{\partial P}{\partial R_L} = \frac{|V_t|^2 \left[(R_t + R_L)^2 + (X_t + X_L)^2 - 2R_L (R_t + R_L)\right]}{2\left[(R_t + R_L)^2 + (X_t + X_L)^2\right]^2}$$

Setting $\quad \frac{\partial P}{\partial X_L} = 0$ gives

$$X_L = -X_t \tag{3.26}$$

and Setting $\quad \frac{\partial P}{\partial R_L} = 0$ gives

$$R_L = \sqrt{R_t^2 + (X_t + X_L)^2} \tag{3.27}$$

Combining equations (3.26) and (3.27), we can conclude that for maximum average power transfer, $\mathbf{Z}_L$ must be selected such that $X_L = -X_t$ and $R_L = R_t$. That is the maximum average power of a circuit with an impedance $\mathbf{Z}_t$ that is obtained when $\mathbf{Z}_L$ is set equal to complex conjugate of $\mathbf{Z}_t$.

Setting $R_L = R_t$ and $X_L = -X_t$ in equation (3.25), we get the maximum average power as

$$P = \frac{|V_t|^2}{8R_t}$$

In a situation where the load is purely real, the condition for maximum power transfer is obtained by putting $X_L = 0$ in equation (3.27). That is,

$$R_L = \sqrt{R_t^2 + X_t^2} = |\mathbf{Z}_t|$$

Hence for maximum average power transfer to a purely resistive load, the load resistance is equal to the magnitude of Thevenin impedance.

3.6.1 Maximum Power Transfer When Z is Restricted

Maximum average power can be delivered to $\mathbf{Z}_L$ only if $\mathbf{Z}_L = \mathbf{Z}_t^*$. There are few situations in which this is not possible. These situations are described below :

(i) R_L and X_L may be restricted to a limited range of values. With this restriction, choose X_L as close as possible to $-X_t$ and then adjust R_L as close as possible to $\sqrt{R_t^2 + (X_L + X_t)^2}$.

(ii) Magnitude of $\mathbf{Z}_L$ can be varied but its phase angle cannot be. Under this restriction, greatest amount of power is transferred to the load when $[\mathbf{Z}_L] = |\mathbf{Z}_t|$.

Z_t^* is the complex conjugate of Z_t.

EXAMPLE 3.42

Find the load impedance that transfers the maximum power to the load and determine the maximum power quantity obtained for the circuit shown in Fig. 3.112.

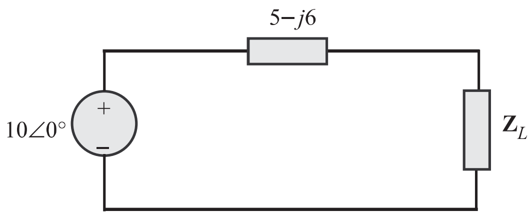
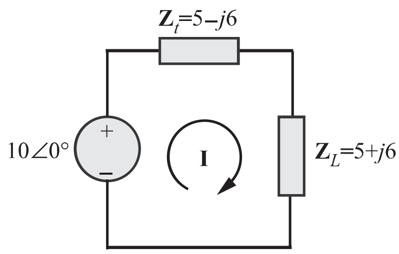


Figure 3.112

SOLUTION

We select, $\mathbf{Z}_L = \mathbf{Z}_t^*$ for maximum power transfer.



Hence
$$\mathbf{Z}_L = 5 + j6$$
$$\mathbf{I} = \frac{10\,\underline{/0^\circ}}{5+5} = 1\,\underline{/0^\circ}$$

Hence, the maximum average power transfered to the load is

$$P = \frac{1}{2}|\mathbf{I}|^2 R_L$$
$$= \frac{1}{2}(1)^2 \times 5 = \mathbf{2.5\ W}$$

EXAMPLE 3.43

Find the load impedance that transfers the maximum average power to the load and determine the maximum average power transferred to the load $\mathbf{Z}_L$ shown in Fig. 3.113.

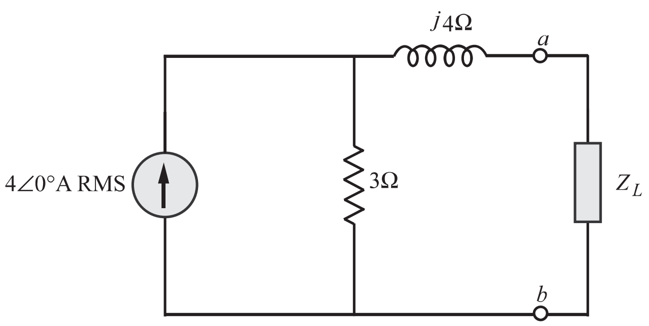


Figure 3.113

SOLUTION

The first step in the analysis is to find the Thevenin equivalent circuit by disconnecting the load $\mathbf{Z}_L$. This leads to a circuit diagram as shown in Fig. 3.114.

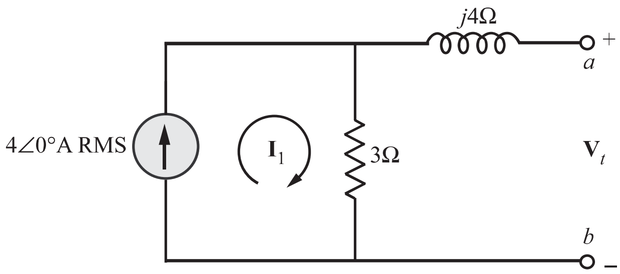


Figure 3.114

Hence
$$\mathbf{V}_t = \mathbf{V}_{oc} = 4\,\underline{/0^\circ} \times 3$$
$$= 12\,\underline{/0^\circ}\ \text{Volts(RMS)}$$

To find $\mathbf{Z}_t$, let us deactivate all the independent sources of Fig. 3.114. This leads to a circuit diagram as shown in Fig 3.114 (a):

$$\mathbf{Z}_t = 3 + j4\,\Omega$$

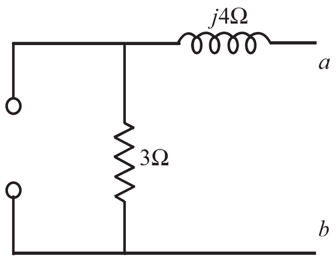


Figure 3.114 (a)

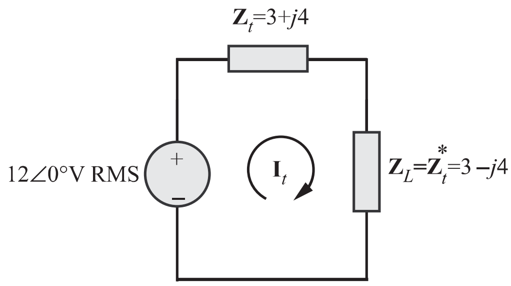


Figure 3.115

The Thevenin equivalent circuit with $\mathbf{Z}_L$ is as shown in Fig. 3.115.
For maximum average power transfer to the load, $\mathbf{Z}_L = \mathbf{Z}_t^* = 3 - j4$.

$$\mathbf{I}_t = \frac{12\,\underline{/0^\circ}}{3 + j4 + 3 - j4} = 2\,\underline{/0^\circ}\ \text{A(RMS)}$$

Hence, maximum average power delivered to the load is

$$P = |I_t|^2\,R_L = 4(3) = 12\ \text{W}$$

It may be noted that the scaling factor $\dfrac{1}{2}$ is not taken since the phase current is already expressed by its RMS value.

EXAMPLE 3.44

Refer the circuit given in Fig. 3.116. Find the value of R_L that will absorb the maximum average power.

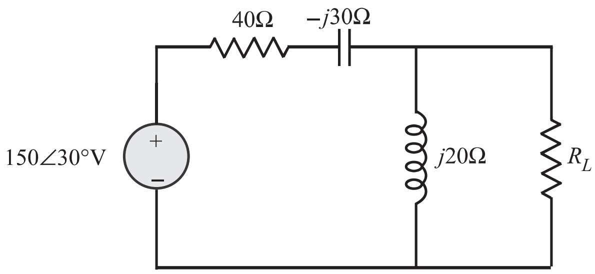


Figure 3.116

SOLUTION

Disconnecting the load resistor R_L from the original circuit diagram leads to a circuit diagram as shown in Fig. 3.117.

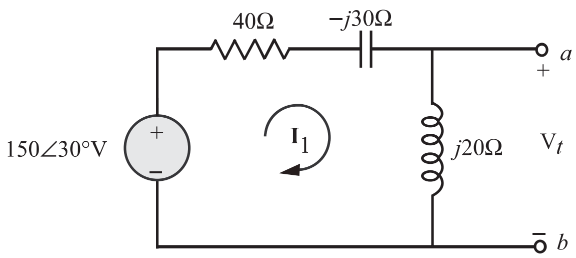


Figure 3.117

$$\begin{aligned}\mathbf{V}_t = \mathbf{V}_{oc} &= \mathbf{I}_1\,(j20)\\ &= \frac{150\,\underline{/30^\circ}\;\times j20}{(40 - j30 + j20)}\\ &= 72.76\,\underline{/134^\circ}\text{ Volts.}\end{aligned}$$

To find $\mathbf{Z}_t$, let us deactivate all the independent sources present in Fig. 3.117 as shown in Fig 3.117 (a).

$$\begin{aligned}\mathbf{Z}_t &= (40 - j30)\,||j20\\ &= \frac{j20\,(40 - j30)}{j20 + 40 - j30} = (9.412 + j22.35)\;\Omega\end{aligned}$$

The Value of R_L that will absorb the maximum average power is

$$R_L = |\mathbf{Z}_t| = \sqrt{(9.412)^2 + (22.35)^2}$$
$$= 24.25\ \Omega$$

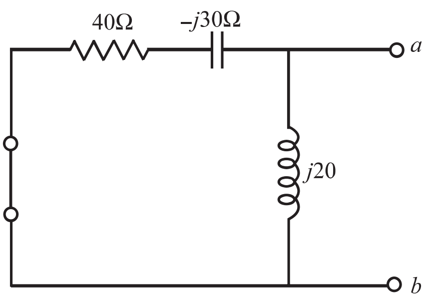


Figure 3.117 (a)

The Thevenin equivalent circuit with R_L inserted is as shown in Fig 3.117 (b).
Maximum average power absorbed by R_L is

$$P_{\max} = \frac{1}{2}\,|I_t|^2\,R_L$$

where $$\mathbf{I}_t = \frac{72.76\,\underline{/134^\circ}}{(9.412 + j22.35 + 24.25)}$$
$$= 1.8\,\underline{/100.2^\circ}\ \text{A}$$

$$\Rightarrow \quad P_{\max} = \frac{1}{2}(1.8)^2 \times 24.25$$
$$= \mathbf{39.29\ W}$$

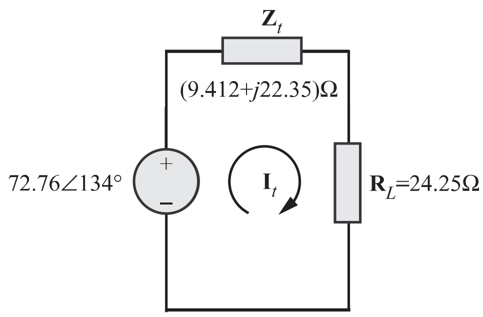


Figure 3.117 (b) Thevenin equivalent circuit

EXAMPLE 3.45

For the circuit of Fig. 3.118: (a) what is the value of Z_L that will absorb the maximum average power? (b) what is the value of maximum power?

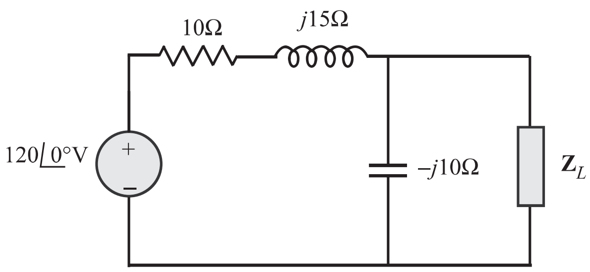


Figure 3.118

SOLUTION

Disconnecting $\mathbf{Z}_L$ from the original circuit we get the circuit as shown in Fig. 3.119. The first step is to find $\mathbf{V}_t$.

$$\mathbf{V}_t = \mathbf{V}_{oc} = \mathbf{I}_1\,(-j10)$$

$$= \left[\frac{120\,\underline{/0^\circ}}{10 + j15 - j10}\right](-j10)$$

$$= 107.33\,\underline{/-116.57^\circ}\;\;\text{V}$$

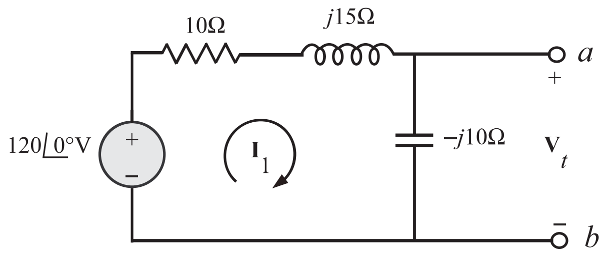
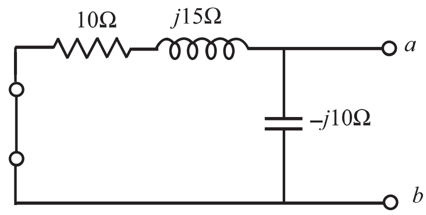


Figure 3.119

The next step is to find $\mathbf{Z}_t$. This requires deactivating the independent voltage source of Fig. 3.119.



$$\mathbf{Z}_t = (10 + j15)\,||\,(-j10)$$

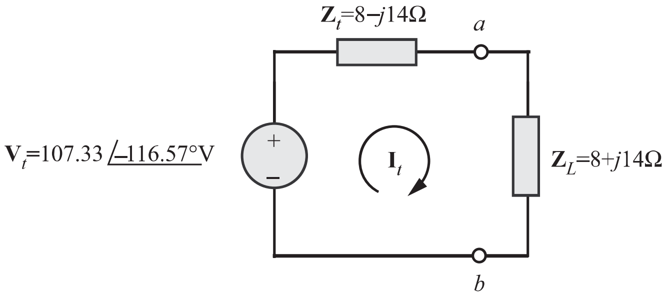
$$= \frac{-j10\,(10 + j15)}{-j10 + 10 + j15}$$

$$= 8 - j14\,\Omega$$

The value of $\mathbf{Z}_L$ for maximum average power absorbed is

$$\mathbf{Z}_t^* = 8 + j14\,\Omega$$

The Thevenin equivalent circuit along with $\mathbf{Z}_L = 8 + j14\,\Omega$ is as shown below:



$$\mathbf{I}_t = \frac{107.33\,\underline{/\,-116.57^\circ}}{8 - j14 + 8 + j14}$$

$$= \frac{107.33}{16}\,\underline{/-116.57^\circ}\;A$$

Hence,

$$P_{\max} = \frac{1}{2}\,|I_t|^2\,R_L$$

$$= \frac{1}{2}\left(\frac{107.33}{16}\right)^2 \times 8$$

$$= \mathbf{180\ Walts}$$

EXAMPLE 3.46

(a) For the circuit shown in Fig. 3.120, what is the value of $\mathbf{Z}_L$ that results in maximum average power that will be transferred to $\mathbf{Z}_L$? What is the maximum power ?

(b) Assume that the load resistance can be varied between 0 and 4000 Ω and the capacitive reactance of the load can be varied between 0 and $-2000\ \Omega$. What settings of R_L and X_C transfer the most average power to the load ? What is the maximum average power that can be transferred under these conditions?

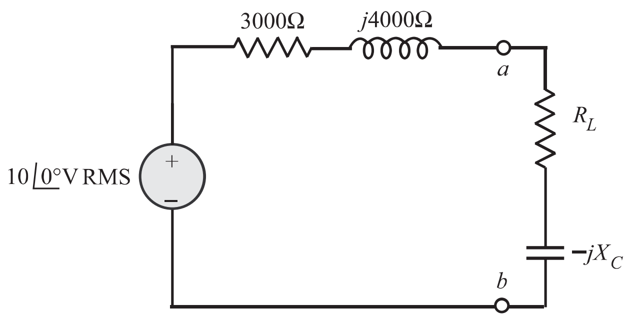


Figure 3.120

SOLUTION

(a) If there are no constraints on R_L and X_L, the load indepedance $\mathbf{Z}_L = \mathbf{Z}_t^* = (3000 - j4000)\ \Omega$.

Since the voltage source is given in terms of its RMS value, the average maximum power delivered to the load is

$$P_{\max} = |\mathbf{I}_t|^2\, R_L$$

where

$$\mathbf{I}_t = \frac{10\,\underline{/0^\circ}}{3000 + j4000 + 3000 - j4000} = \frac{10}{2 \times 3000}\text{A}$$

$$\Rightarrow \qquad P_{\max} = |\mathbf{I}_t|^2\, R_L = \frac{100}{4 \times (3000)^2} \times 3000 = \mathbf{8.33\ mW}$$

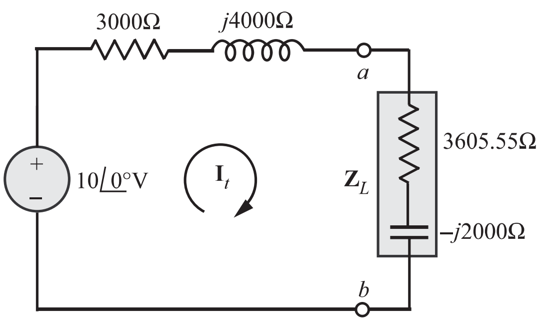
(b) Since R_L and X_C are restricted, we first set X_C as close to $-4000\ \Omega$ as possible; hence $X_C = -2000\ \Omega$. Next we set R_L as close to $\sqrt{R_t^2 + (X_C + X_L)^2}$ as possible.

Thus, $$R_L = \sqrt{3000^2 + (-2000 + 4000)^2} = 3605.55\ \Omega$$

Since R_L can be varied between 0 to $4000\ \Omega$, we can set R_L to $3605.55\ \Omega$. Hence $\mathbf{Z}_L$ is adjusted to a value

$$\mathbf{Z}_L = 3605.55 - j2000\ \Omega.$$

$$\mathbf{I}_t = \frac{10\,\underline{/0^\circ}}{3000 + j4000 + 3605.55 - j2000}$$
$$= 1.4489\,\underline{/-16.85^\circ}\ \text{mA}$$



The maximum average power delivered to the load is

$$P_{\max} = |\mathbf{I}_t|^2 R_L$$
$$= \left(1.4489 \times 10^{-3}\right)^2 \times 3605.55$$
$$= \mathbf{7.57\ mW}$$

Note that this is less than the power that can be delivered if there are no constraints on R_L and X_L.

EXAMPLE 3.47

A load impedance having a constant phase angle of -45° is connected across the load terminals a and b in the circuit shown in Fig. 3.121. The magnitude of $\mathbf{Z}_L$ is varied until the average power delivered, which is the maximum possible under the given restriction.

(a) Specify $\mathbf{Z}_L$ in rectangular form.

(b) Calculate the maximum average power delivered under this condition.

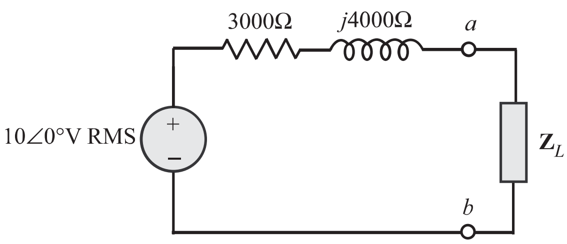


Figure 3.121

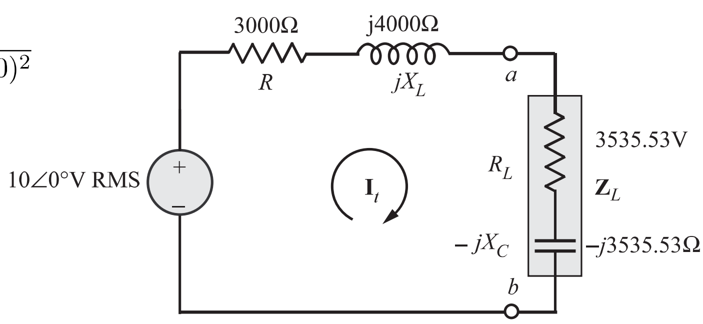
SOLUTION

Since the phase angle of $\mathbf{Z}_L$ is fixed at -45°, for maximum power transfer to $\mathbf{Z}_L$ it is mandatory that

$$|\mathbf{Z}_L| = |\mathbf{Z}_t|$$
$$= \sqrt{(3000)^2 + (4000)^2}$$
$$= 5000\ \Omega.$$

Hence, $\mathbf{Z}_L = |\mathbf{Z}_L|\,\underline{/-45^\circ}$

$$= \frac{5000}{\sqrt{2}} - j\frac{5000}{\sqrt{2}}$$



$$\mathbf{I}_t = \frac{10\,\underline{/0^\circ}}{(3000 + 3535.53) + j(4000 - 3535.53)}$$

$$= 1.526\,\underline{/-4.07^\circ}\ \text{mA}$$

$$P_{\max} = |\mathbf{I}_t|^2\, R_L$$

$$= \left(1.526 \times 10^{-3}\right)^2 \times 3535.53$$

$$= \mathbf{8.23\ mW}$$

This power is the maximum average power that can be delivered by this circuit to a load impedance whose angle is constant at -45°. Again this quantity is less than the maximum power that could have been delivered if there is no restriction on $\mathbf{Z}_L$. In example 3.46 part (a), we have shown that the maximum power that can be delivered without any restrictions on $\mathbf{Z}_L$ is 8.33 mW.

3.7 Reciprocity theorem

The reciprocity theorem states that in a linear bilateral single source circuit, the ratio of excitation to response is constant when the positions of excitation and response are interchanged.

Conditions to be met for the application of reciprocity theorem :

(i) The circuit must have a single source.

(ii) Initial conditions are assumed to be absent in the circuit.

(iii) Dependent sources are excluded even if they are linear.

(iv) When the positions of source and response are interchanged, their directions should be marked same as in the original circuit.

EXAMPLE 3.48

Find the current in 2 Ω resistor and hence verify reciprocity theorem.

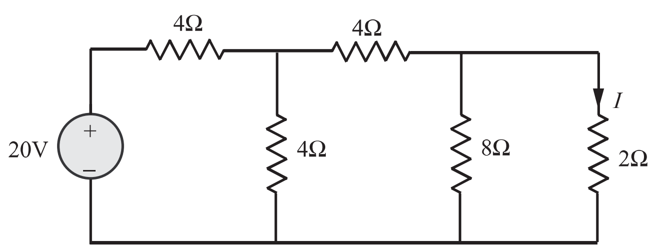


Figure 3.122

SOLUTION

The circuit is redrawn with markings as shown in Fig 3.123 (a).

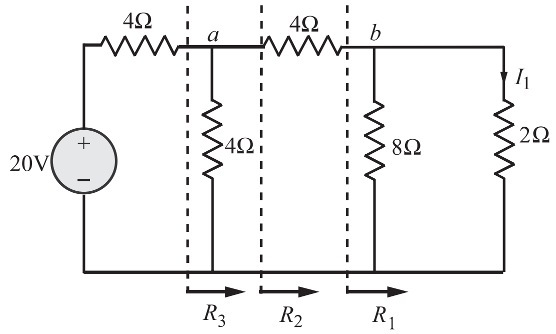


Figure 3.123 (a)

Then,
$$R_1 = (8^{-1} + 2^{-1})^{-1} = 1.6\Omega$$
$$R_2 = 1.6 + 4 = 5.6\Omega$$
$$R_3 = (5.6^{-1} + 4^{-1})^{-1} = 2.3333\Omega$$

Current supplied by the source $= \dfrac{20}{4 + 2.3333} = 3.16$ A

Current in branch $ab = I_{ab} = 3.16 \times \dfrac{4}{4 + 4 + 1.6} = 1.32$ A

Current in $2\Omega, I_1 = 1.32 \times \dfrac{8}{10} = 1.05$ A

Verification using reciprocity theorem

The circuit is redrawn by interchanging the position of excitation and response as shown in Fig 3.123 (b).

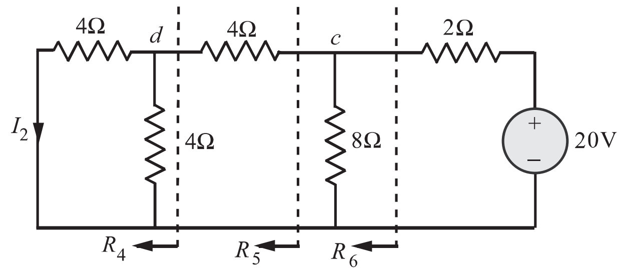


Figure 3.123 (b)

Solving the equivalent resistances,

$$R_4 = 2\Omega, \quad R_5 = 6\Omega, \quad R_6 = 3.4286\Omega$$

Now the current supplied by the source

$$= \frac{20}{3.4286 + 2} = 3.6842\text{A}$$

Therefore,

$$I_{cd} = 3.6842 \times \frac{8}{8+6} = 2.1053\text{A}$$
$$I_2 = \frac{2.1053}{2} = 1.05\text{A}$$

As $I_1 = I_2 = 1.05$ A, reciprocity theorem is verified.

EXAMPLE 3.49

In the circuit shown in Fig. 3.124, find the current through 1.375 Ω resistor and hence verify reciprocity theorem.

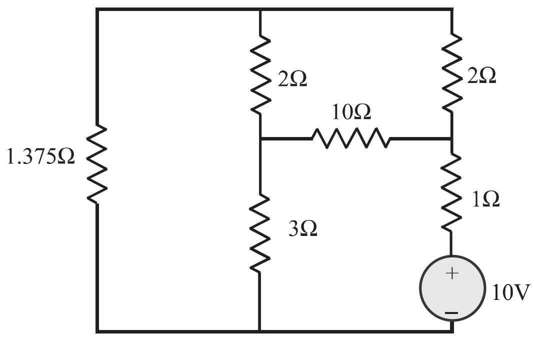


Figure 3.124

SOLUTION

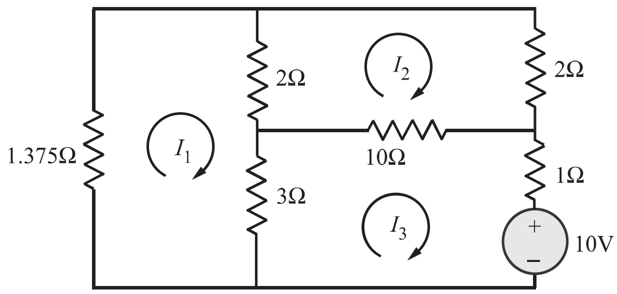


Figure 3.125

KVL clockwise for mesh 1 :

$$6.375I_1 - 2I_2 - 3I_3 = 0$$

KVL clockwise for mesh 2 :

$$-2I_1 + 14I_2 - 10I_3 = 0$$

KVL clockwise for mesh 3 :

$$-3I_1 - 10I_2 + 14I_3 = -10$$

Putting the above three mesh equations in matrix form, we get

$$\begin{bmatrix} 6.375 & -2 & -3 \\ -2 & 14 & -10 \\ -3 & -10 & 14 \end{bmatrix} \begin{bmatrix} I_1 \\ I_2 \\ I_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ -10 \end{bmatrix}$$

Using Cramer's rule, we get

$$I_1 = -2\text{A}$$

Negative sign indicates that the assumed direction of current flow should have been the other way.

Verification using reciprocity theorem :

The circuit is redrawn by interchanging the positions of excitation and response. The new circuit is shown in Fig. 3.126.

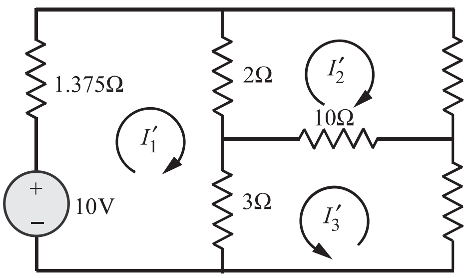


Figure 3.126

The mesh equations in matrix form for the circuit shown in Fig. 3.126 is

$$\begin{bmatrix} 6.375 & -2 & 3 \\ -2 & 14 & 10 \\ 3 & 10 & 14 \end{bmatrix} \begin{bmatrix} I'_1 \\ I'_2 \\ I'_3 \end{bmatrix} = \begin{bmatrix} 10 \\ 0 \\ 0 \end{bmatrix}$$

Using Cramer's rule, we get

$$I'_3 = -2 \text{ A}$$

Since $I_1 = I'_3 = -2$ A, the reciprocity theorem is verified.

EXAMPLE 3.50

Find the current $\mathbf{I}_x$ in the $j2\ \Omega$ impedance and hence verify reciprocity theorem.

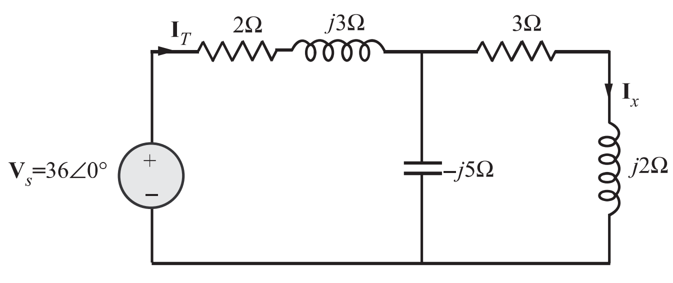


Figure 3.127

SOLUTION

With reference to the Fig. 3.127, the current through $j2\,\Omega$ impepance is found using series−parallel reduction techniques.

Total impedance of the circuit is

$$\mathbf{Z}_T = (2 + j3) + (-j5)||(3 + j2)$$
$$= 2 + j3 + \frac{(-j5)(3 + j2)}{-j5 + 3 + j2}$$
$$= 6.537 \underline{/19.36^\circ} \;\; \Omega$$

The total current in the network is

$$\mathbf{I}_T = \frac{36 \underline{/0^\circ}}{6.537 \underline{/19.36^\circ}}$$
$$= 5.507 \underline{/-19.36^\circ} \;\; \text{A}$$

Using the principle of current division, we find that

$$\mathbf{I}_x = \frac{\mathbf{I}_T\,(-j5)}{-j5 + 3 + j2}$$
$$= 6.49 \underline{/-64.36^\circ} \;\; \text{A}$$

Verification of reciprocity theorem :

The circuit is redrawn by changing the positions of excitation and response. This circuit is shown in Fig. 3.128.

Total impedance of the circuit shown in Fig. 3.128 is

$$\mathbf{Z}'_T = (3 + j2) + (2 + j3)\,||\,(-j5)$$
$$= (3 + j2) + \frac{(2 + j3)\,(-j5)}{2 + j3 - j5}$$
$$= 9.804 \underline{/19.36^\circ} \;\; \Omega$$

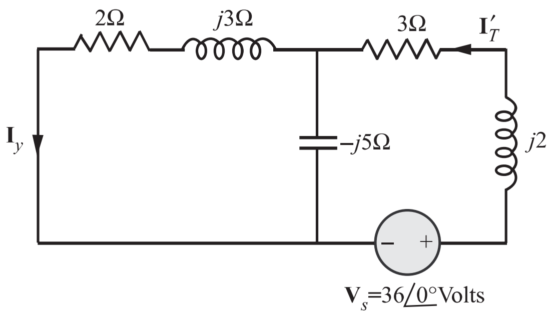


Figure 3.128

The total current in the circuit is

$$\mathbf{I}'_T = \frac{36 \underline{/0^\circ}}{Z'_T} = 3.672 \underline{/-19.36^\circ} \;\; \text{A}$$

Using the principle of current division,

$$\mathbf{I}_y = \frac{\mathbf{I}'_T\,(-j5)}{-j5 + 2 + j3} = 6.49 \underline{/-64.36^\circ} \;\; \text{A}$$

It is found that $\mathbf{I}_x = \mathbf{I}_y$, thus verifying the reciprocity theorem.

EXAMPLE 3.51

Refer the circuit shown in Fig. 3.129. Find current through the ammeter, and hence verify reciprocity theorem.

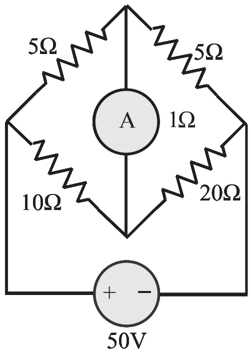


Figure 3.129

SOLUTION

To find the current through the ammeter :

By inspection the loop equations for the circuit in Fig. 3.130 can be written in the matrix form as

$$\begin{bmatrix} 16 & -1 & -10 \\ -1 & 26 & -20 \\ -10 & -20 & 30 \end{bmatrix} \begin{bmatrix} I_1 \\ I_2 \\ I_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 50 \end{bmatrix}$$

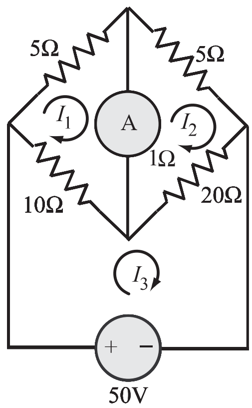


Figure 3.130

Using Cramer's rule, we get

$$I_1 = 4.6 \text{ A}$$
$$I_2 = 5.4 \text{ A}$$

Hence current through the ammeter $= I_2 - I_1 = 5.4 - 4.6 = 0.8$A.

Verification of reciprocity theorem:

The circuit is redrawn by interchanging the positions of excitation and response as shown in Fig. 3.131.

By inspection the loop equations for the circuit can be written in matrix form as

$$\begin{bmatrix} 15 & 0 & -10 \\ 0 & 25 & -20 \\ -10 & -20 & 31 \end{bmatrix} \begin{bmatrix} I_1' \\ I_2' \\ I_3' \end{bmatrix} = \begin{bmatrix} -50 \\ 50 \\ 0 \end{bmatrix}$$

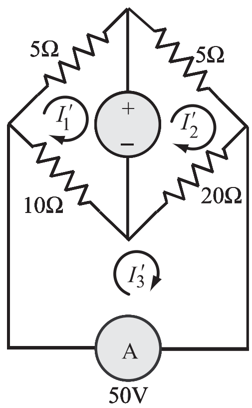


Figure 3.131

Using Cramer's rule we get

$$I_3' = 0.8 \text{ A}$$

Hence, current through the Ammeter = 0.8 A.

It is found from both the cases that the response is same. Hence the reciprocity theorem is verified.

EXAMPLE 3.52

Find current through 5 ohm resistor shown in Fig. 3.132 and hence verify reciprocity theorem.

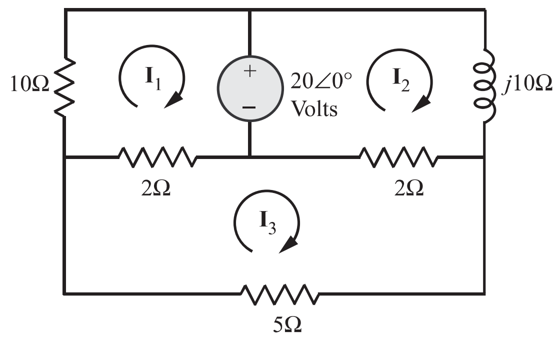


Figure 3.132

SOLUTION

By inspection, we can write

$$\begin{bmatrix} 12 & 0 & -2 \\ 0 & 2+j10 & -2 \\ -2 & -2 & 9 \end{bmatrix}\begin{bmatrix} \mathbf{I}_1 \\ \mathbf{I}_2 \\ \mathbf{I}_3 \end{bmatrix} = \begin{bmatrix} -20 \\ 20 \\ 0 \end{bmatrix}$$

Using Cramer's rule, we get

$$\mathbf{I}_3 = 0.5376\,\underline{/-126.25^\circ}\ \text{A}$$

Hence, current through 5 ohm resistor $= 0.5376\,\underline{/-126.25^\circ}\ \text{A}$

Verification of reciprocity theorem:

The original circuit is redrawn by interchanging the excitation and response as shown in Fig. 3.133.

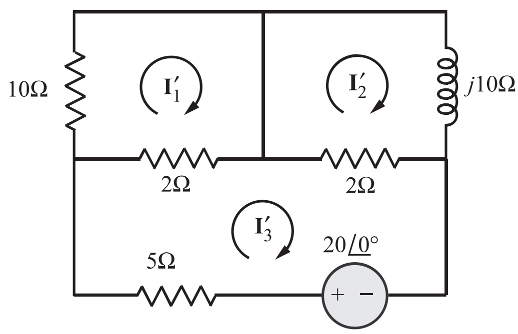


Figure 3.133

Putting the three equations in matrix form, we get

$$\begin{bmatrix} 12 & 0 & -2 \\ 0 & 2+j10 & -2 \\ -2 & -2 & 9 \end{bmatrix} \begin{bmatrix} \mathbf{I}_1' \\ \mathbf{I}_2' \\ \mathbf{I}_3' \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 20 \end{bmatrix}$$

Using Cramer's rule, we get

$$\mathbf{I}_1' = 0.3876 \underline{/-2.35} \text{ A}$$

$$\mathbf{I}_2' = 0.456 \underline{/-78.9^\circ} \text{ A}$$

Hence,

$$\mathbf{I}_2' - \mathbf{I}_1' = -0.3179 - j0.4335$$

$$= 0.5376 \underline{/-126.25^\circ} \text{ A}$$

The response in both cases remains the same. Thus verifying reciprocity theorem.

3.8 Millman's theorem

It is possible to combine number of voltage sources or current sources into a single equivalent voltage or current source using Millman's theorem. Hence, this theorem is quite useful in calculating the total current supplied to the load in a generating station by a number of generators connected in parallel across a busbar.

Millman's theorem states that if n number of generators having generated emfs $\mathbf{E}_1, \mathbf{E}_2, \cdots \mathbf{E}_n$ and internal impedances $\mathbf{Z}_1, \mathbf{Z}_2, \cdots \mathbf{Z}_n$ are connected in parallel, then the emfs and impedances can be combined to give a single equivalent emf of $\mathbf{E}$ with an internal impedance of equivalent value $\mathbf{Z}$.

where
$$\mathbf{E} = \frac{\mathbf{E}_1\mathbf{Y}_1 + \mathbf{E}_2\mathbf{Y}_2 + \ldots + \mathbf{E}_n\mathbf{Y}_n}{\mathbf{Y}_1 + \mathbf{Y}_2 + \ldots + \mathbf{Y}_n}$$

and
$$\mathbf{Z} = \frac{1}{\mathbf{Y}_1 + \mathbf{Y}_2 + \ldots + \mathbf{Y}_n}$$

where $\mathbf{Y}_1, \mathbf{Y}_2 \cdots \mathbf{Y}_n$ are the admittances corresponding to the internal impedances $\mathbf{Z}_1, \mathbf{Z}_2 \cdots \mathbf{Z}_n$ and are given by

$$\mathbf{Y}_1 = \frac{1}{\mathbf{Z}_1}$$

$$\mathbf{Y}_2 = \frac{1}{\mathbf{Z}_2}$$

$$\vdots$$

$$\mathbf{Y}_n = \frac{1}{\mathbf{Z}_n}$$

Fig. 3.134 shows a number of generators having emfs $\mathbf{E}_1, \mathbf{E}_2 \cdots \mathbf{E}_n$ connected in parallel across the terminals x and y. Also, $\mathbf{Z}_1, \mathbf{Z}_2 \cdots \mathbf{Z}_n$ are the respective internal impedances of the generators.

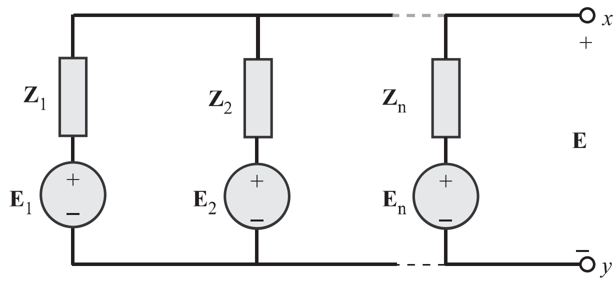


Figure 3.134

The Thevenin equivalent circuit of Fig. 3.134 using Millman's theorem is shown in Fig. 3.135. The nodal equation at x gives

$$\frac{\mathbf{E}_1 - \mathbf{E}}{\mathbf{Z}_1} + \frac{\mathbf{E}_2 - \mathbf{E}}{\mathbf{Z}_2} + \cdots + \frac{\mathbf{E}_n - \mathbf{E}}{\mathbf{Z}_n} = 0$$

$$\Rightarrow \quad \left[\frac{\mathbf{E}_1}{\mathbf{Z}_1} + \frac{\mathbf{E}_2}{\mathbf{Z}_2} + \cdots + \frac{\mathbf{E}_n}{\mathbf{Z}_n}\right] = \mathbf{E}\left[\frac{1}{\mathbf{Z}_1} + \frac{1}{\mathbf{Z}_2} + \cdots + \frac{1}{\mathbf{Z}_n}\right]$$

$$\Rightarrow \quad \mathbf{E}_1\mathbf{Y}_1 + \mathbf{E}_2\mathbf{Y}_2 + \cdots + \mathbf{E}_n\mathbf{Y}_n = \mathbf{E}\left[\frac{1}{\mathbf{Z}}\right]$$

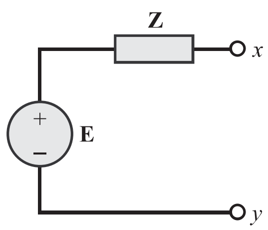


Figure 3.135

where **Z** = Equivalent internal impedance.

$$\text{or} \quad [\mathbf{E}_1\mathbf{Y}_1 + \mathbf{E}_2\mathbf{Y}_2 + \cdots + \mathbf{E}_n\mathbf{Y}_n] = \mathbf{E}\mathbf{Y}$$

$$\Rightarrow \quad \mathbf{E} = \frac{\mathbf{E}_1\mathbf{Y}_1 + \mathbf{E}_2\mathbf{Y}_2 + \cdots + \mathbf{E}_n\mathbf{Y}_n}{\mathbf{Y}}$$

$$\text{where} \quad \mathbf{Y} = \mathbf{Y}_1 + \mathbf{Y}_2 + \cdots + \mathbf{Y}_n$$

$$\text{and} \quad \mathbf{Z} = \frac{1}{\mathbf{Y}} = \frac{1}{\mathbf{Y}_1 + \mathbf{Y}_2 + \cdots + \mathbf{Y}_n}$$

EXAMPLE 3.53

Refer the circuit shown in Fig. 3.136. Find the current through $10\ \Omega$ resistor using Millman's theorem.

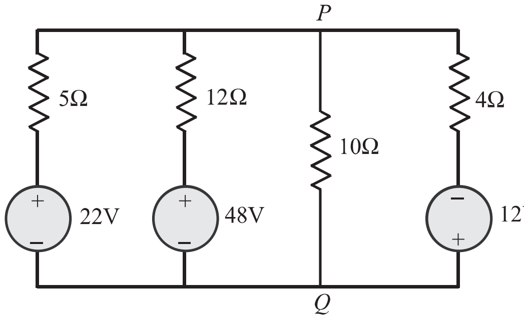


Figure 3.136

SOLUTION

Using Millman's theorem, the circuit shown in Fig. 3.136 is replaced by its Thevenin equivalent circuit across the terminals PQ as shown in Fig. 3.137.

$$\mathbf{E} = \frac{\mathbf{E}_1\mathbf{Y}_1 + \mathbf{E}_2\mathbf{Y}_2 - \mathbf{E}_3\mathbf{Y}_3}{\mathbf{Y}_1 + \mathbf{Y}_2 + \mathbf{Y}_3}$$

$$= \frac{22\left(\frac{1}{5}\right) + 48\left(\frac{1}{12}\right) - 12\left(\frac{1}{4}\right)}{\frac{1}{5} + \frac{1}{12} + \frac{1}{4}}$$

$$= 10.13 \text{ Volts}$$

$$R = \frac{1}{\mathbf{Y}_1 + \mathbf{Y}_2 + \mathbf{Y}_3}$$

$$= \frac{1}{0.2 + 0.083 + 0.25}$$

$$= 1.88\ \Omega$$

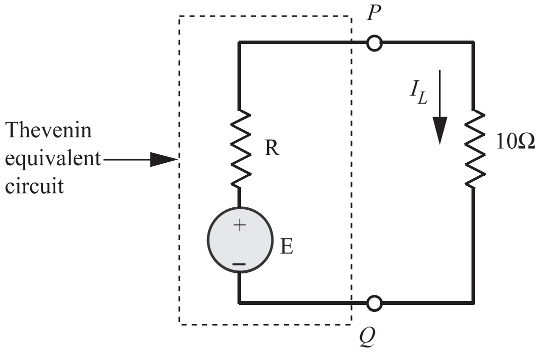


Figure 3.137

Hence, $$I_L = \frac{E}{R + 10} = 0.853 \text{ A}$$

EXAMPLE 3.54

Find the current through $(10 - j3)\Omega$ using Millman's theorem. Refer Fig. 3.138.

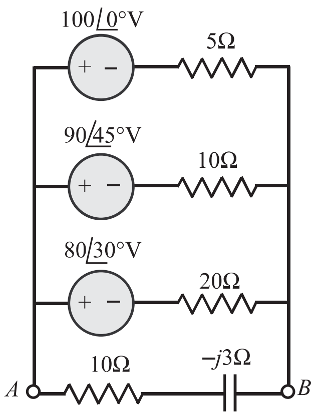


Figure 3.138

SOLUTION

The circuit shown in Fig. 3.138 is replaced by its Thevenin equivalent circuit as seen from the terminals, A and B using Millman's theorem. Fig. 3.139 shows the Thevenin equivalent circuit along with $\mathbf{Z}_L = 10 - j3\ \Omega$.

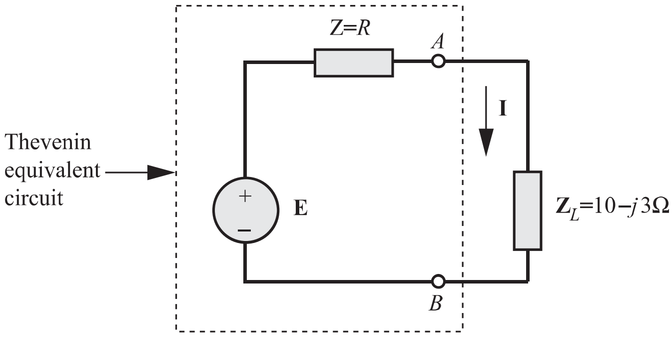


Figure 3.139

$$\mathbf{E} = \frac{\mathbf{E}_1\mathbf{Y}_1 + \mathbf{E}_2\mathbf{Y}_2 - \mathbf{E}_3\mathbf{Y}_3}{\mathbf{Y}_1 + \mathbf{Y}_2 + \mathbf{Y}_3}$$

$$= \frac{100\,\underline{/0^\circ}\left(\frac{1}{5}\right) + 90\,\underline{/45^\circ}\left(\frac{1}{10}\right) + 80\,\underline{/30^\circ}\left(\frac{1}{20}\right)}{\frac{1}{5} + \frac{1}{10} + \frac{1}{20}}$$

$$= 88.49\,\underline{/15.66^\circ}\ \mathrm{V}$$

$$\mathbf{Z} = R = \frac{1}{\mathbf{Y}_1 + \mathbf{Y}_2 + \mathbf{Y}_3} = \frac{1}{\frac{1}{5} + \frac{1}{10} + \frac{1}{20}} = 2.86\ \Omega$$

$$\mathbf{I} = \frac{\mathbf{E}}{\mathbf{Z} + \mathbf{Z}_L} = \frac{88.49\,\underline{/15.66}}{2.86 + 10 - j3} = 6.7\,\underline{/28.79^\circ}\ \mathbf{A}$$

Alternately,

$$\mathbf{E} = \frac{\mathbf{E}_1\mathbf{Y}_1 + \mathbf{E}_2\mathbf{Y}_2 + \mathbf{E}_3\mathbf{Y}_3 + \mathbf{E}_4\mathbf{Y}_4}{\mathbf{Y}_1 + \mathbf{Y}_2 + \mathbf{Y}_3 + \mathbf{Y}_4}$$

$$= \frac{100 \times 5^{-1} + 90\,\underline{/45^\circ} \times 10^{-1} + 80\,\underline{/30^\circ} \times 20^{-1}}{5^{-1} + 10^{-1} + 20^{-1} + (10 - j3)^{-1}}$$

$$= 70\,\underline{/12^\circ}\,\mathrm{V}$$

Therefore,

$$I = \frac{70\,\underline{/12^\circ}}{10 - j3}$$

$$= 6.7\,\underline{/28.8^\circ}\,\mathrm{A}$$

EXAMPLE 3.55

Refer the circuit shown in Fig. 3.140. Use Millman's theorem to find the current through $(5+j5)\ \Omega$ impedance.

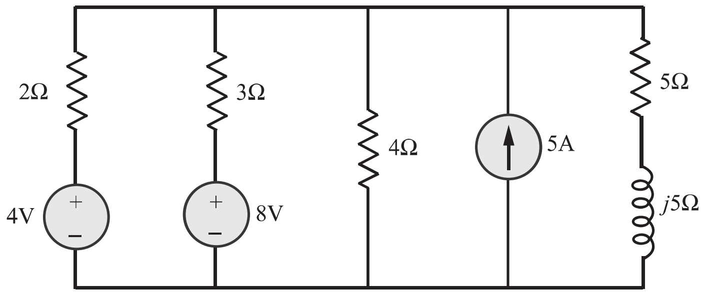


Figure 3.140

SOLUTION

The original circuit is redrawn after performing source transformation of 5 A in parallel with 4 Ω resistor into an equivalent voltage source and is shown in Fig. 3.141.

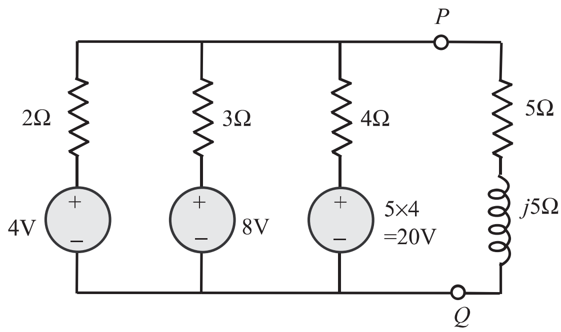


Figure 3.141

Treating the branch $5 + j5\Omega$ as a branch with $\mathbf{E}_s = 0V$,

$$\begin{aligned}\mathbf{E}_{PQ} &= \frac{\mathbf{E}_1\mathbf{Y}_1 + \mathbf{E}_2\mathbf{Y}_2 + \mathbf{E}_3\mathbf{Y}_3 + \mathbf{E}_4\mathbf{Y}_4}{\mathbf{Y}_1 + \mathbf{Y}_2 + \mathbf{Y}_3 + \mathbf{Y}_4} \\ &= \frac{4 \times 2^{-1} + 8 \times 3^{-1} + 20 \times 4^{-1}}{2^{-1} + 3^{-1} + 4^{-1} + (5 - j5)^{-1}} \\ &= 8.14\,\underline{/4.83^\circ}\,\mathrm{V}\end{aligned}$$

Therefore current in $(5 + j5)\Omega$ is

$$\mathbf{I} = \frac{8.14\,\underline{/4.83^\circ}}{5 + j5} = 1.15\,\underline{/-40.2^\circ}\,\mathrm{A}$$

Alternately

$\mathbf{E}_{PQ}$ with $(5 + j5)$ open

$$\begin{aligned}\mathbf{E}_{PQ} &= \frac{\mathbf{E}_1\mathbf{Y}_1 + \mathbf{E}_2\mathbf{Y}_2 + \mathbf{E}_3\mathbf{Y}_3}{\mathbf{Y}_1 + \mathbf{Y}_2 + \mathbf{Y}_3} \\ &= \frac{4 \times 2^{-1} + 8 \times 3^{-1} + 20 \times 4^{-1}}{2^{-1} + 3^{-1} + 4^{-1}} \\ &= 8.9231\mathrm{V}\end{aligned}$$

Equivalent resistance $R = (2^{-1} + 3^{-1} + 4^{-1})^{-1} = 0.9231\Omega$
Therefore current in $(5 + j5)\Omega$ is

$$I = \frac{8.9231}{0.9231 + 5 + j5} = 1.15\,\underline{/-40.2^\circ}\ \text{A}$$

EXAMPLE 3.56

Find the current through 2 Ω resistor using Millman's theorem. Refer the circuit shown in Fig. 3.142.

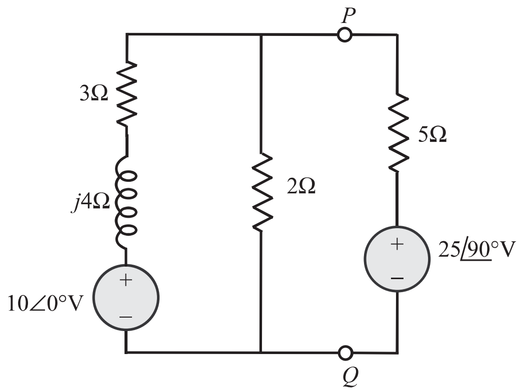


Figure 3.142

SOLUTION

The Thevenin equivalent circuit using Millman's theorem for the given problem is as shown in Fig. 3.142(a).

where
$$\mathbf{E} = \frac{\mathbf{E}_1\mathbf{Y}_1 + \mathbf{E}_2\mathbf{Y}_2}{\mathbf{Y}_1 + \mathbf{Y}_2}$$

$$= \frac{10\,\underline{/10^\circ}\left[\dfrac{1}{3+j4}\right] + 25\,\underline{/90^\circ}\left[\dfrac{1}{5}\right]}{\dfrac{1}{3+j4} + \dfrac{1}{5}}$$

$$= 10.06\,\underline{/97.12^\circ}\ \text{V}$$

$$\mathbf{Z} = \frac{1}{\mathbf{Y}_1 + \mathbf{Y}_2} = \frac{1}{\dfrac{1}{3+j4} + \dfrac{1}{5}}$$

$$= 2.8\,\underline{/26.56^\circ}\ \Omega$$

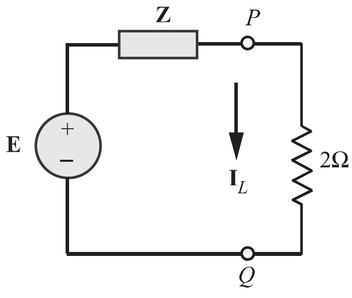


Figure 3.142(a)

Hence,
$$\mathbf{I}_L = \frac{\mathbf{E}}{\mathbf{Z}+2} = \frac{10.06\,\underline{/97.12^\circ}}{2.8\,\underline{/26.56^\circ} + 2}$$

$$= 2.15\,\underline{/81.63^\circ}\ \text{A}$$

Reinforcement problems

R.P 3.1

Find the current in 2 Ω resistor connected between A and B by using superposition theorem.

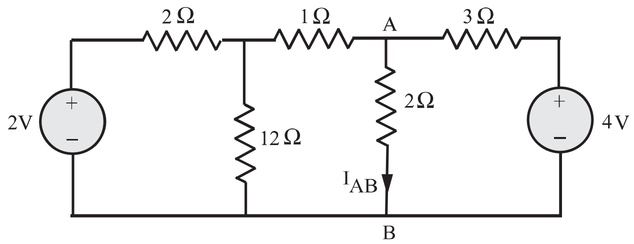


Figure R.P. 3.1

SOLUTION

Fig. R.P. 3.1(a), shows the circuit with 2V-source acting alone (4V-source is shorted).

Resistance as viewed from 2V-source is $2 + R_1$ Ω,

$$\text{where} \quad R_1 = \left(\frac{3 \times 2}{5} + 1\right) \Big\| \, 12$$

$$= \frac{(1.2 + 1) \times 12}{14.2} = 1.8592\ \Omega$$

$$\text{Hence,} \quad I_a = \frac{2}{2 + 1.8592} = 0.5182\ \text{A}$$

$$\text{Then,} \quad I_b = I_a \times \frac{12}{12 + 1 + 1.2} = 0.438\ \text{A}$$

$$\therefore \quad I_1 = 0.438 \times \frac{3}{5} = 0.2628\ \text{A}$$

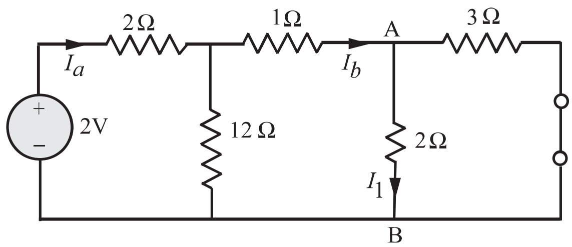


Figure R.P. 3.1(a)

With 4V-source acting alone, the circuit is as shown in Fig. R.P. 3.1(b).

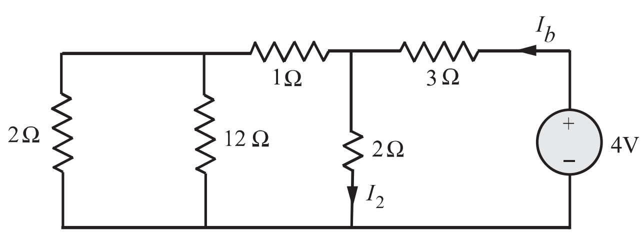


Figure R.P.3.1(b)

The resistance as seen by 4V-source is $3 + R_2$ where

$$R_2 = \left(\frac{2 \times 12}{14} + 1\right) \Big\| 2$$

$$= \frac{2.7143 \times 2}{4.7143} = 1.1551\ \Omega$$

Hence, $$\mathbf{I}_b = \frac{4}{3 + 1.1551} = 0.9635\ \text{A}$$

Thus, $$\mathbf{I}_2 = \frac{\mathbf{I}_b \times 2.7143}{4.7143} = 0.555\ \text{A}$$

Finally, applying the principle of superposition,

we get, $$\mathbf{I}_{AB} = \mathbf{I}_1 + \mathbf{I}_2$$
$$= 0.2628 + 0.555$$
$$= 0.818\ \text{A}$$

R.P 3.2

For the network shown in Fig. R.P. 3.2, apply superposition theorem and find the current **I**.

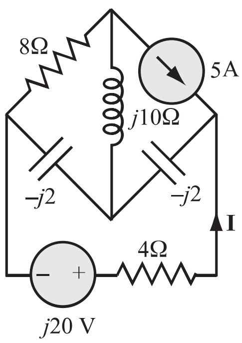


Figure R.P. 3.2

SOLUTION

Open the 5A-current source and retain the voltage source. The resulting network is as shown in Fig. R.P. 3.2(a).

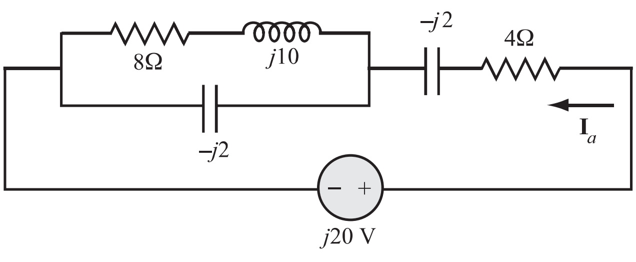


Figure R.P. 3.2(a)

The impedance as seen from the voltage source is

$$\mathbf{Z} = (4 - j2) + \frac{(8 + j10)\,(-j2)}{8 + j8} = 6.01\,\underline{/-45^\circ}\quad \Omega$$

Hence, $$\mathbf{I}_a = \frac{j20}{\mathbf{Z}} = 3.328\,\underline{/135^\circ}\ \text{A}$$

Next, short the voltage source and retain the current source. The resulting network is as shown in Fig. R.P. 3.2 (b).

Here, $\mathbf{I}_3 = 5\text{A}$. Applying *KVL* for mesh 1 and mesh 2, we get

$$8\mathbf{I}_1 + (\mathbf{I}_1 - 5)\,j10 + (\mathbf{I}_1 - \mathbf{I}_2)\,(-j2) = 0$$

and $$(\mathbf{I}_2 - \mathbf{I}_1)\,(-j2) + (\mathbf{I}_2 - 5)\,(-j2) + 4\mathbf{I}_2 = 0$$

Simplifying, we get

$$(8 + j8)\mathbf{I}_1 + j2\mathbf{I}_2 = j50$$

and $$j2\mathbf{I}_1 + (4 - j4)\mathbf{I}_2 = -j10$$

Solving, we get

$$\mathbf{I}_b = \mathbf{I}_2 = \frac{\begin{vmatrix} 8 + j8 & j50 \\ j2 & -j10 \end{vmatrix}}{\begin{vmatrix} 8 + j8 & j2 \\ j2 & 4 - j4 \end{vmatrix}}$$

$$= 2.897\,\underline{/-23.96^\circ}\ \text{A}$$

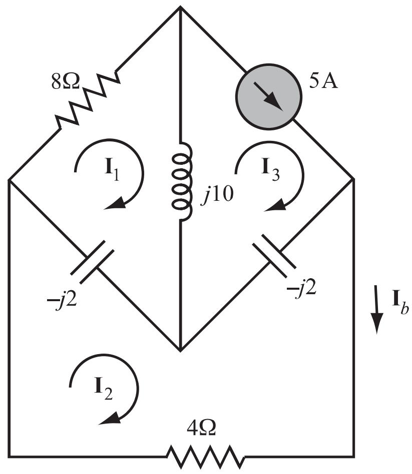


Figure R.P. 3.2(b)

Since, $\mathbf{I}_a$ and $\mathbf{I}_b$ are flowing in opposite directions, we have

$$\mathbf{I} = \mathbf{I}_a - \mathbf{I}_b = 6.1121\,\underline{/144.78^\circ}\ \text{A}$$

R.P 3.3

Apply superposition theorem and find the voltage across 1 Ω resistor. Refer the circuit shown in Fig. R.P. 3.3. Take $v_1(t) = 5\cos(t + 10^\circ)$ and $i_2(t) = 3\sin 2t$ A.

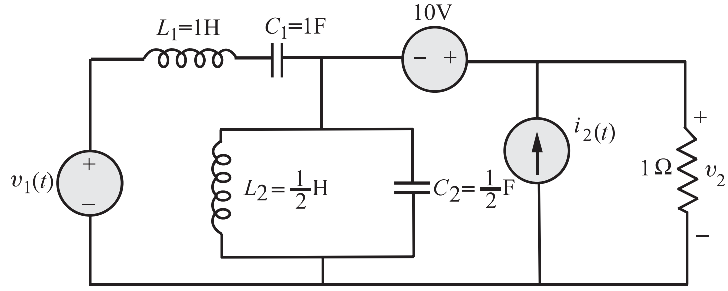


Figure R.P. 3.3

SOLUTION

To begin with let us assume $v_1(t)$ alone is acting. Accordingly, short 10V - source and open $i_2(t)$. The resulting phasor network is shown in Fig. R.P. 3.3(a).

$$\begin{aligned}
\omega &= 1\text{rad/sec} \\
5\cos(t+10^\circ) &\rightarrow 5\,\underline{/10^\circ}\ \text{V} \\
L_1 = 1\text{H} &\rightarrow j\,\omega L_1 = j1\,\Omega \\
C_1 = 1\text{F} &\rightarrow \frac{1}{j\,\omega C_1} = -j1\,\Omega \\
L_2 = \frac{1}{2}\text{H} &\rightarrow j\,\omega L_2 = j\frac{1}{2}\,\Omega \\
C_2 = \frac{1}{2}\text{F} &\rightarrow \frac{1}{j\,\omega C_2} = -j2\,\Omega
\end{aligned}$$

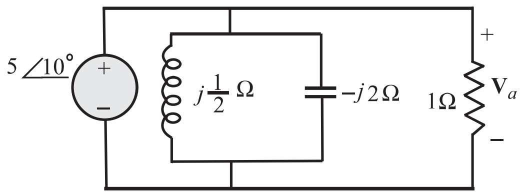


Figure R.P. 3.3(a)

$$\therefore \qquad \mathbf{V}_a = 5\,\underline{/10^\circ}\ \text{V}$$
$$\Rightarrow \qquad v_a(t) = 5\cos[t+10^\circ]$$

Let us next assume that $i_2(t)$ alone is acting. The resulting network is shown in Fig. R.P. 3.3(b).

$$\begin{aligned}
\omega &= 2\ \text{rad/sec} \\
3\sin 2t &\rightarrow 3\,\underline{/0^\circ}\ \text{A} \\
C_1 = 1\text{F} &\rightarrow \frac{1}{j\,\omega C_1} = -j\frac{1}{2}\,\Omega \\
L_1 = 1\text{H} &\rightarrow j\,\omega L_1 = j2\,\Omega \\
C_2 = \frac{1}{2}\text{F} &\rightarrow \frac{1}{j\,\omega C_2} = -j1\,\Omega \\
L_2 = \frac{1}{2}\text{H} &\rightarrow j\,\omega L_2 = j1\,\Omega
\end{aligned}$$

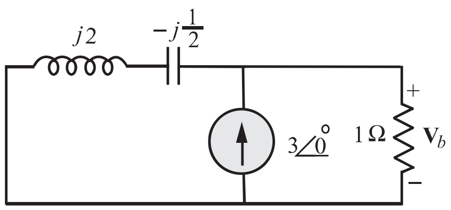


Figure R.P. 3.3(b)

$$\mathbf{V}_b = 3\,\underline{/0^\circ} \times \frac{j1.5}{1+j1.5} = 2.5\,\underline{/33.7^\circ}\ \text{A}$$
$$\Rightarrow \qquad v_b(t) = 2.5\sin[2t+33.7^\circ]\,\text{A}$$

Finally with 10V-source acting alone, the network is as shown in Fig. R.P. 3.3(c). Since $\omega = 0$, inductors are shorted and capacitors are opened.

Hence, $\mathbf{V}_c = 10$ V

Applying principle of superposition, we get.

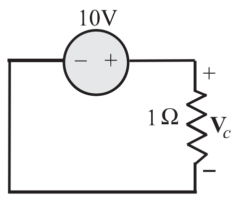


Figure R.P. 3.3(c)

$$\begin{aligned}
v_2(t) &= v_a(t) = v_b(t) + \mathbf{V}_c \\
&= 5\cos(t+10^\circ) + 2.5\sin(2t+33.7^\circ) + 10\text{Volts}
\end{aligned}$$

R.P 3.4

Calculate the current through the galvanometer for the Kelvin double bridge shown in Fig. R.P. 3.4. Use Thevenin's theorem. Take the resistance of the galvanometer as 30 Ω.

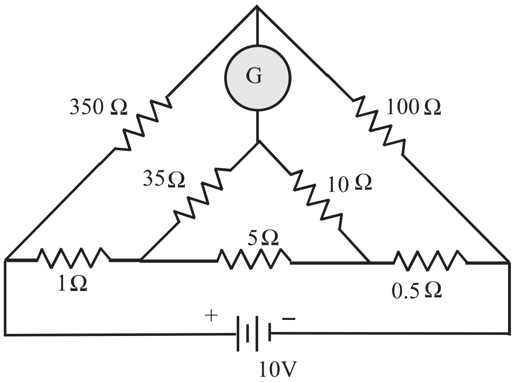


Figure R.P. 3.4

SOLUTION

With G being open, the resulting network is as shown in Fig. R.P. 3.4(a).

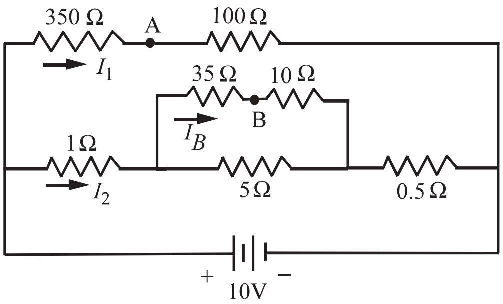


Figure 3.4(a)

$$V_A = I_1 \times 100 = \frac{10}{450} \times 100 = \frac{20}{9} \text{ V}$$

$$I_2 = \frac{10}{1.5 + \dfrac{45 \times 5}{50}} = 1.66, \quad I_B = \frac{I_2 \times 5}{45 + 5} = 0.1 I_2$$

Hence,
$$V_B = I_2 \times 0.5 + I_B \times 10$$
$$= 2.5 \text{ V}$$

Thus,
$$V_{AB} = V_t = V_A - V_B = \frac{20}{9} - 2.5 = \frac{-5}{18} \text{ Volts}$$

To find R_t, short circuit the voltage source. The resulting network is as shown in Fig. R.P. 3.4(b).

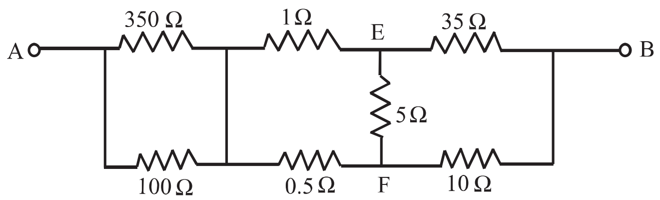


Figure R.P. 3.4 (b)

Transforming the Δ between B, E and F into an equivalent Y, we get

$$R_B = \frac{35 \times 10}{50} = 7\ \Omega, \quad R_E = \frac{35 \times 5}{50} = 3.5\ \Omega, \quad R_F = \frac{5 \times 10}{50} = 1\ \Omega$$

The reduced network after transformation is as shown in Fig. R.P. 3.4(c).

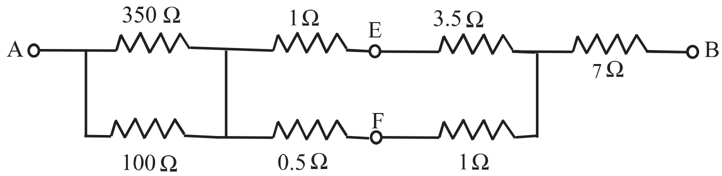


Figure R.P. 3.4(c)

Hence, $$R_{AB} = R_t = \frac{350 \times 100}{450} + \frac{4.5 \times 1.5}{6} + 7$$
$$= 85.903\ \Omega$$

The Thevenin's equivalent circuit as seen from A and B with 30 Ω connected between A and B is as shown in Fig. R.P. 3.4(d).

$$I_G = \frac{-\frac{5}{18}}{85.903 + 30} = -2.4\text{mA}$$

Negative sign implies that the current flows from B to A.

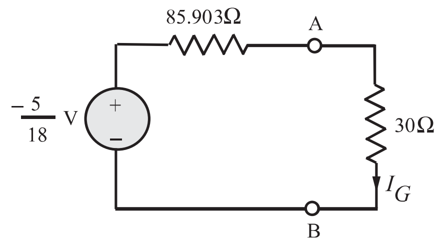


Figure R.P. 3.4(d)

Outcomes:
1. To analyze circuit systems using application of Superposition theorem.
2. To analyze circuit systems using application of Thevenin’s theorem.
3. To analyze circuit systems using application of Norton’s theorem.
4. To analyze circuit systems using application of Millman’s theorem.
5. To analyze circuit systems using application of Reciprocity theorem.
6. To understand the use of circuit analysis and methods using Maximum Power Transfer Theorem.

Resources:
1. http://en.wikipedia.org/wiki/Network_analysis_(electrical_circuits)
2. http://freevideolectures.com/Course/2336/Circuit-Theory
3. http://nptel.ac.in/courses/108102042/