ATME COLLEGE OF ENGINEERING

13th KM Stone, Bannur Road, Mysore - 570 028

DEPARTMENT OF ELECTRICAL & ELECTRONICS ENGINEERING

NOTES

COURSE TITLE: SIGNALS & DIGITAL SIGNAL PROCESSING

COURSE CODE: BEE502

SEMESTER: V

MODULE-5: DESIGN OF FIR DIGITAL FILTERS

INSTITUTIONAL VISION AND MISSION

VISION:

- Development of academically excellent, culturally vibrant, socially responsible, and globally competent human resources.

MISSION:

- To keep pace with advancements in knowledge and make the students competitive and capable at the global level.
- To create an environment for the students to acquire the right physical, intellectual, emotional, and moral foundations and shine as torchbearers of tomorrow's society.
- To strive to attain ever-higher benchmarks of educational excellence.

Department Vision and Mission

Vision:

To produce Electrical & Electronics Engineers through greatest quality of technical education, technical skill training and intellectual capacity building of individuals.

Mission:

- To provide knowledge to students that builds a strong foundation in the basic principles of electrical engineering, problem solving abilities, analytical skills, soft skills and communication skills for their overall development.
- To offer outcome based technical education.
- To encourage faculty in training & development and to offer consultancy through research & industry interaction.

MODULE-5: FIR FILTER DESIGN

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5.0 Objectives

1.Students will design and understand simple finite impulseresponse filters

2.Specific issues of stability of IIR filters is determined

5.1 Introduction

Two important classes of digital filters based on impulse response type are

Finite Impulse Response (FIR)

Infinite Impulse Response (IIR)

The filter can be expressed in two important forms as:

1) System function representation;

$$H(z) = \frac{\sum_{k=0}^{M} b_k z^{-k}}{1 + \sum_{k=1}^{N} a_k z^{-k}} \qquad (1)$$

2) Difference Equation representation;

$$\sum_{k=0}^{N} a_k y(n-k) = \sum_{k=0}^{M} b_k x(n-k) \qquad (2)$$

Each of this form allows various methods of implementation. The eq (2) can be viewed as a computational procedure (an algorithm) for determining the output sequence y(n) of the system from the input sequence x(n). Different realizations are possible with different arrangements of eq (2)

The major issues considered while designing a digital filters are :

- Realiability (causal or non causal)
- Stability (filter output will not saturate)
- Sharp Cutoff Characteristics
- Order of the filter need to be minimum (this leads to less delay)
- Generalized procedure (having single procedure for all kinds of filters)
- Linear phase characteristics

The factors considered with filter implementation are ,

a. It must be a simple design

b. There must be modularity in the implementation so that any order filter can be obtained with lower order modules.

c. Designs must be as general as possible. Having different design procedures for different types of filters(high pass, low pass,…) is cumbersome and complex.

d. Cost of implementation must be as low as possible

e. The choice of Software/Hardware realization

5.2 Features of IIR Filter

The important features of this class of filters can be listed as:

- Out put is a function of past o/p, present and past i/p's
- It is recursive in nature
- It has at least one Pole (in general poles and zeros)
- Sharp cutoff chas. is achievable with minimum order
- Difficult to have linear phase chas over full range of freq.
- Typical design procedure is analog design then conversion from analog to digital

The main features of FIR filter are,

- They are inherently stable
- Filters with linear phase characteristics can be designed
- Simple implementation – both recursive and nonrecursive structures possible
- Free of limit cycle oscillations when implemented on a finite-word length digital system

Disadvantages:

- Sharp cutoff at the cost of higher order
- Higher order leading to more delay, more memory and higher cost of implementation

5.3 Importance of Linear Phase

The group delay is defined as

$$\tau_g = -\frac{d\theta(\omega)}{d\omega}$$

which is negative differential of phase function.

Nonlinear phase results in different frequencies experiencing different delay and arriving at different time at the receiver. This creates problems with speech processing and data communication applications. Having linear phase ensures constant group delay for all frequencies.

The further discussions are focused on FIR filter.

Examples of simpleFIR filtering operations: ***1.UnityGain Filter***

y(n)=x(n)

2. Constant gain filter

y(n)=Kx(n)

3. Unit delay filter

y(n)=x(n-1)

4.Two - term Difference filter

y(n) = x(n)-x(n-1)

5. Two-term average filter

y(n) = 0.5(x(n)+x(n-1))

6. Three-term average filter (3-point moving average filter)

y(n) = 1/3[x(n)+x(n-1)+x(n-2)]

7. Central Difference filter

y(n)= 1/2[x(n) – x(n-2)]

When we say Order of the filter it is the number of previous inputs used to compute the current output and Filter coefficients are the numbers associated with each of the terms x(n), x(n-1),.. etc

The table below shows order and filter coefficients of above simple filter types:

Ex.	order	a0	a1	a2
1	0	1	-	-
2	0	K	-	-
3	1	0	1	-
4(HP)	1	1	-1	-
5(LP)	1	1/2	1/2	-
6(LP)	2	1/3	1/3	1/3
7(HP)	2	1/2	0	-1/2

5.4 Design of FIR filters

The section to follow will discuss on design of FIR filter. Since linear phase can be achieved with FIR filter we will discuss the conditions required to achieve this.

7.6.1 Symmetric and Antisymmetric FIR filters giving out Linear Phase characteristics:

Symmetry in filter impulse response will ensure linear phase

An FIR filter of length M with i/p x(n) & o/p y(n) is described by the difference equation:

$y(n) = b_0\, x(n) + b_1\, x(n-1) + \ldots\ldots + b_{M-1}\, x(n-(M-1)) = \sum_{k=0}^{M-1} b_k x(n-k)$ -(1)

Alternatively. it can be expressed in convolution form

$$y(n) = \sum_{k=0}^{M-1} h(k)x(n-k) \qquad - (2)$$

i.e $b_k = h(k)$, $k=0,1,\ldots..M-1$

Filter is also characterized by

$H(z) = \sum_{k=0}^{M-1} h(k) z^{-k}$ -(3) polynomial of degree M-1 in the variable z^{-1}. The roots of this polynomial constitute zeros of the filter.

An FIR filter has linear phase if its unit sample response satisfies the condition
h(n)= ± h(M-1-n) n=0,1,…….M-1 -(4)

Incorporating this symmetry & anti symmetry condition in eq 3 we can show linear phase chas of FIR filters

$$H(z) = h(0) + h(1)z^{-1} + h(2)z^{-2} + ... + h(M-2)z^{-(M-2)} + h(M-1)z^{-(M-1)}$$

If M is odd

$$H(z) = h(0) + h(1)z^{-1} + ... + h(\frac{M-1}{2})z^{-(\frac{M-1}{2})} + h(\frac{M+1}{2})z^{-(\frac{M+1}{2})} + h(\frac{M+3}{2})z^{-(\frac{M+3}{2})} + ...$$
$$+ h(M-2)z^{-(M-2)} + h(M-1)z^{-(M-1)}$$

$$= z^{-(\frac{M-1}{2})}\left[h(0)z^{(\frac{M-1}{2})} + h(1)z^{(\frac{M-3}{2})} + ... + h(\frac{M-1}{2}) + h(\frac{M+1}{2})z^{-1} + h(\frac{M+3}{2})z^{-2} + ...h(M-1)z^{-(\frac{M-1}{2})} \right]$$

Applying symmetry conditions for M odd

$$h(0) = \pm h(M-1)$$
$$h(1) = \pm h(M-2)$$
.
.
$$h(\frac{M-1}{2}) = \pm h(\frac{M-1}{2})$$
$$h(\frac{M+1}{2}) = \pm h(\frac{M-3}{2})$$
.
.
$$h(M-1) = \pm h(0)$$

$$H(z) = z^{-(\frac{M-1}{2})}\left[h(\frac{M-1}{2}) + \sum_{n=0}^{\frac{M-3}{2}} h(n)\{z^{(M-1-2n)/2} \pm z^{-(M-1-2n)/2}\} \right]$$

similarly for M even

$$H(z) = z^{-(\frac{M-1}{2})}\left[\sum_{n=0}^{\frac{M}{2}-1} h(n)\{z^{(M-1-2n)/2} \pm z^{-(M-1-2n)/2}\} \right]$$

Frequency response:

If the system impulse response has symmetry property (i.e.,h(n)=h(M-1-n)) and M is odd $H(e^{j\omega}) = e^{j\theta(\omega)} \mid H_r(e^{j\omega}) \mid$ where

$$H_r(e^{j\omega}) = \left[h(\frac{M-1}{2}) + 2\sum_{n=0}^{\frac{M-3}{2}} h(n)\cos\omega(\frac{M-1}{2} - n) \right]$$

$$\theta(\omega) = -(\frac{M-1}{2})\omega \quad \text{if } \mid H_r(e^{j\omega}) \mid \geq 0$$

$$= -(\frac{M-1}{2})\omega + \pi \quad \text{if } \mid H_r(e^{j\omega}) \mid \leq 0$$

In case of M even the phase response remains the same with magnitude response expressed as

$$H_r(e^{j\omega}) = \left[2\sum_{n=0}^{\frac{M}{2}-1} h(n)\cos\omega(\frac{M-1}{2} - n) \right]$$

If the impulse response satisfies anti symmetry property (i.e., h(n)=-h(M-1-n))then for
M odd we will have

$$h(\frac{M-1}{2}) = -h(\frac{M-1}{2}) \text{ i.e., } h(\frac{M-1}{2}) = 0$$

$$H_r(e^{j\omega}) = \left[2\sum_{n=0}^{\frac{M-3}{2}} h(n)\sin\omega(\frac{M-1}{2} - n) \right]$$

If M is even then,

$$H_r(e^{j\omega}) = \left[2\sum_{n=0}^{\frac{M}{2}-1} h(n)\sin\omega(\frac{M-1}{2}-n) \right]$$

In both cases the phase response is given by

$$\theta(\omega) = -(\frac{M-1}{2})\omega + \pi/2 \quad if\ |H_r(e^{j\omega})| \geq 0$$
$$= -(\frac{M-1}{2})\omega + 3\pi/2 \quad if\ |H_r(e^{j\omega})| \leq 0$$

Which clearly shows presence of Linear Phase characteristics.

Comments on filter coefficients:

- The number of filter coefficients that specify the frequency response is (M+1)/2 when is M odd and M/2 when M is even in case of symmetric conditions
- In case of impulse response antisymmetric we have h(M-1/2)=0 so that there are (M-1/2) filter coefficients when M is odd and M/2 coefficients when M is even

choice of Symmetric and antisymmetric unit sample response

When we have a choice between different symmetric properties, the particular one is picked up based on application for which the filter is used. The following points give an insight to this issue.

- If h(n)=-h(M-1-n) and M is odd, $H_r(w)$ implies that Hr(0)=0 & $H_r(\pi)=0$, consequently not suited for lowpass and highpass filter. This condition is suited in Band Pass filter design.
- Similarly if M is even $H_r(0)=0$ hence not used for low pass filter
- Symmetry condition h(n)=h(M-1-n) yields a linear-phase FIR filter with non zero response at w = 0 if desired.

Looking at these points, antisymmetric properties are not generally preferred.

Zeros of Linear Phase FIR Filters:

Consider the filter system function

$$H(z) = \sum_{n=o}^{M-1} h(n) z^{-n}$$

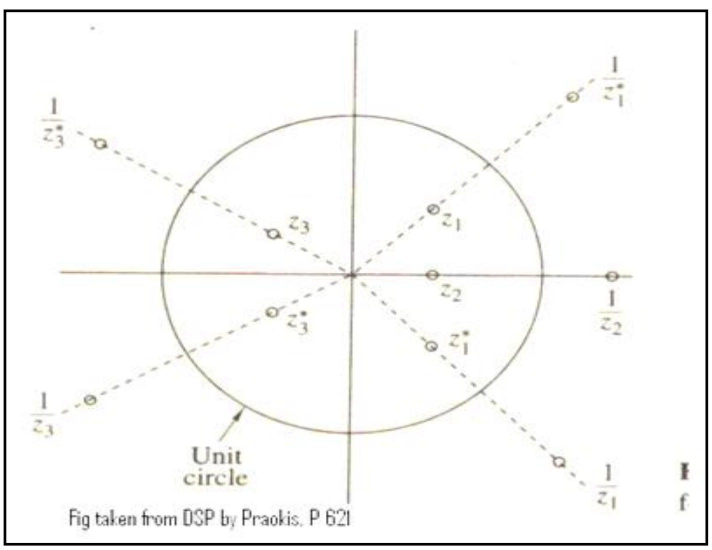
Expanding this equation

$$H(z) = h(0) + h(1)z^{-1} + h(2)z^{-2} + \ldots\ldots + h(M-2)z^{-(M-2)} + h(M-1)z^{-(M-1)}$$
$$\sin ce\ \ for\ \ Linear - phase\ we\ need$$
$$h(n) = h(M-1-n) \quad i.e.,$$
$$h(0) = h(M-1); h(1) = h(M-2);......h(M-1) = h(0);$$
$$then$$
$$H(z) = h(M-1) + h(M-2)z^{-1} + ... + h(1)z^{-(M-2)} + h(0)z^{-(M-1)}$$
$$H(z) = z^{-(M-1)}[h(M-1)z^{(M-1)} + h(M-2)z^{(M-2)} + ... + h(1)z + h(0)]$$
$$H(z) = z^{-(M-1)}[\sum_{n=0}^{M-1} h(n)(z^{-1})^{-n}] = z^{-(M-1)}H(z^{-1})$$

This shows that if z = z_1 is a zero then z=z_1^{-1} is also a zero

The different possibilities:

1. If z_1 = 1 then $z_1 = z_1^{-1}$ =1 is also a zero implying it is one zero
2. If the zero is real and |z|<1 then we have pair of zeros
3. If zero is complex and |z|=1then and we again have pair of complex zeros.
4. If zero is complex and |z|≠1 then and we have two pairs of complex zeros



The plot above shows distribution of zeros for a Linear – phase FIR filter. As it can be seen there is pattern in distribution of these zeros.

5.5 Methods of designing FIR filters

The standard methods of designing FIR filter can be listed as:

1. Fourier series based method
2. Window based method
3. Frequency sampling method

5.5.1 Design of Linear Phase FIR filter based on Fourier Series method

Motivation: Since the desired freq response $H_d(e^{j\omega})$ is a periodic function in ω with period 2π, it can be expressed as Fourier series expansion

$$H_d(e^{j\omega}) = \sum_{n=-\infty}^{\infty} h_d(n) e^{-j\omega n}$$

$$where \quad h_d(n) \quad are \quad fourier\ series\ coefficients$$

$$h_d(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} H_d(e^{j\omega}) e^{j\omega n} d\omega$$

This expansion results in impulse response coefficients which are infinite in duration and non causal. It can be made finite duration by truncating the infinite length. The linear phase can be obtained by introducing symmetric property in the filter impulse response, i.e., h(n) = h(-n). It can be made causal by introducing sufficient delay (depends on filter length)

5.5.2 Stepwise procedure

1. From the desired freq response using inverse FT relation obtain $h_d(n)$
2. Truncate the infinite length of the impulse response to finite length with (assuming M odd)

$$\begin{aligned} h(n) &= h_d(n) \quad for\ -(M-1)/2 \le n \le (M-1)/2 \\ &= 0 \quad otherwise \end{aligned}$$

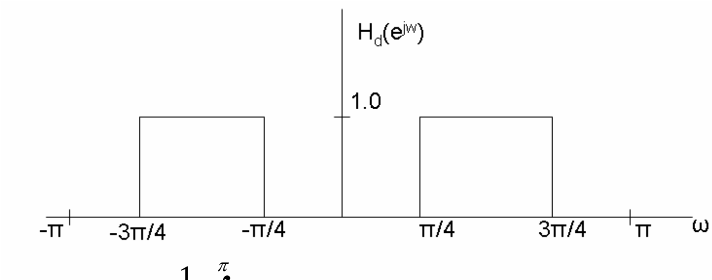
3. Introduce h(n) = h(-n) for linear phase characteristics
4. Write the expression for H(z); this is non-causal realization
5. To obtain causal realization $H'(z) = z^{-(M-1)/2} H(z)$

Exercise Problems

Problem 1 : Design an ideal bandpass filter with a frequency response:

$$H_d(e^{j\omega}) = 1 \qquad for\ \frac{\pi}{4} \le |\omega| \le \frac{3\pi}{4}$$
$$= 0 \qquad otherwise$$

Find the values of h(n) for M = 11 and plot the frequency response.



$$h_d(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} H_d(e^{j\omega})e^{j\omega n}d\omega$$
$$= \frac{1}{2\pi}\left[\int_{-3\pi/4}^{-\pi/4} e^{j\omega n}d\omega + \int_{\pi/4}^{3\pi/4} e^{j\omega n}d\omega\right]$$
$$= \frac{1}{\pi n}\left[\sin\frac{3\pi}{4}n - \sin\frac{\pi}{4}n\right] \quad -\infty \le n \le \infty$$

$$truncating\ to\ 11\ samples\ we\ have\ h(n) = h_d(n)\ \ for\ |n| \le 5$$
$$= 0 \quad otherwise$$

For n = 0 the value of h(n) is separately evaluated from the basic integration

h(0) = 0.5

Other values of h(n) are evaluated from h(n) expression

h(1)=h(-1)=0

h(2)=h(-2)=-0.3183

h(3)=h(-3)=0

h(4)=h(-4)=0

h(5)=h(-5)=0

The transfer function of the filter is

$$H(z) = h(0) + \sum_{n=1}^{(N-1)/2}[h(n)\{z^n + z^{-n}\}]$$
$$= 0.5 - 0.3183(z^2 + z^{-2})$$
the transfer function of the realizable filter is
$$H^{'}(z) = z^{-5}[0.5 - 0.3183(z^2 + z^{-2})]$$
$$= -0.3183z^{-3} + 0.5z^{-5} - 0.3183z^{-7}$$
the filter coeff are
$$h^{'}(0) = h'(10) = h'(1) = h'(9) = h'(2) = h'(8) = h'(4) = h'(6) = 0$$
$$h'(3) = h'(7) = -0.3183$$
$$h'(5) = 0.5$$

The magnitude response can be expressed as
$$|H(e^{j\omega})| = \sum_{n=1}^{(N-1)/2} a(n)\cos\omega n$$
comparing this exp *with*
$$|H(e^{j\omega})| = |z^{-5}[h(0) + 2\sum_{n=1}^{5} h(n)\cos\omega n]|$$

We have

a(0)=h(0)

a(1)=2h(1)=0

a(2)=2h(2)=-0.6366

a(3)=2h(3)=0

a(4)=2h(4)=0

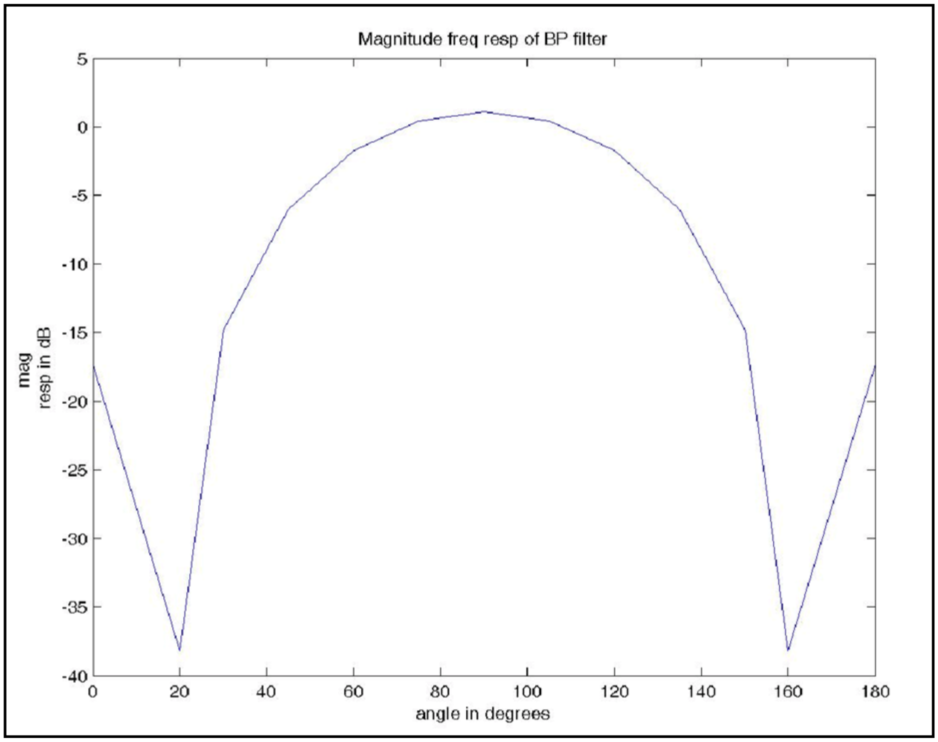
a(5)=2h(5)=0

The magnitude response function is

$|H(e^{j\omega})| = 0.5 – 0.6366 \cos 2\omega$ which can plotted for various values of ω

ω in degrees =[0 20 30 45 60 75 90 105 120 135 150 160 180];

|H(e $^{j\omega}$)| in dBs= [-17.3 -38.17 -14.8 -6.02 -1.74 0.4346 1.11 0.4346 -1.74 -6.02 -14.8 -38.17 -17.3];



Problem 2: Design an ideal lowpass filter with a freq response

$$H_d(e^{j\omega}) = 1 \qquad for -\frac{\pi}{2} \le \omega \le \frac{\pi}{2}$$

$$= 0 \qquad for \ \frac{\pi}{2} \le |\omega| \le \pi$$

Find the values of h(n) for N =11. Find H(z). Plot the magnitude response

From the freq response we can determine $h_d(n)$,

$$h_d(n) = \frac{1}{2\pi}\int_{-\pi/2}^{\pi/2} e^{j\omega n} d\omega = \frac{\sin\frac{\pi n}{2}}{\pi n} \quad -\infty \le n \le \infty \quad and \quad n \ne 0$$

Truncating $h_d(n)$ to 11 samples

h(0) = 1/2
h(1)=h(-1)=0.3183
h(2)=h(-2)=0
h(3)=h(-3)=-0.106
h(4)=h(-4)=0

h(5)=h(-5)=0.06366

The realizable filter can be obtained by shifting h(n) by 5 samples to right h'(n)=h(n-5)

h'(n)= [0.06366, 0, -0.106, 0, 0.3183, 0.5, 0.3183, 0, -0.106, 0, 0.06366];

$$H'(z) = 0.06366 - 0.106z^{-2} + 0.3183z^{-4} + 0.5z^{-5} + 0.3183z^{-6} - 0.106z^{-8} + 0.06366z^{-10}$$

Using the result of magnitude response for M odd and symmetry

$$H_r(e^{j\omega}) = [h(\frac{M-1}{2}) + \sum_{n=0}^{\frac{M-3}{2}} h(n)\cos\omega(\frac{M-1}{2} - n)]$$

$$|H_r(e^{j\omega})| = |[0.5 + 0.6366\cos w - 0.212\cos 3w + 0.127\cos 5w]|$$

Problem 3 :

Design an ideal band reject filter with a frequency response:

$$\begin{aligned} H_d(e^{j\omega}) &= 1 \qquad for|\omega| \le \frac{\pi}{3} \; and \; |\omega| \ge \frac{2\pi}{3} \\ &= 0 \qquad otherwise \end{aligned}$$

Find the values of h(n) for M = 11 and plot the frequency response

Ans:h(n)= [0 -0.1378 0 0.2757 0 0.667 0 0.2757 0 -0.1378 0];

5.6 Window based Linear Phase FIR filter design

The other important method of designing FIR filter is by making use of windows. The arbitrary truncation of impulse response obtained through inverse Fourier relation can lead to distortions in the final frequency response.The arbitrary truncation is equivalent to multiplying infinite length function with finite length rectangular window, i.e.,

h(n) = h_d(n) w(n) where w(n) = 1 for n = ±(M-1)/2

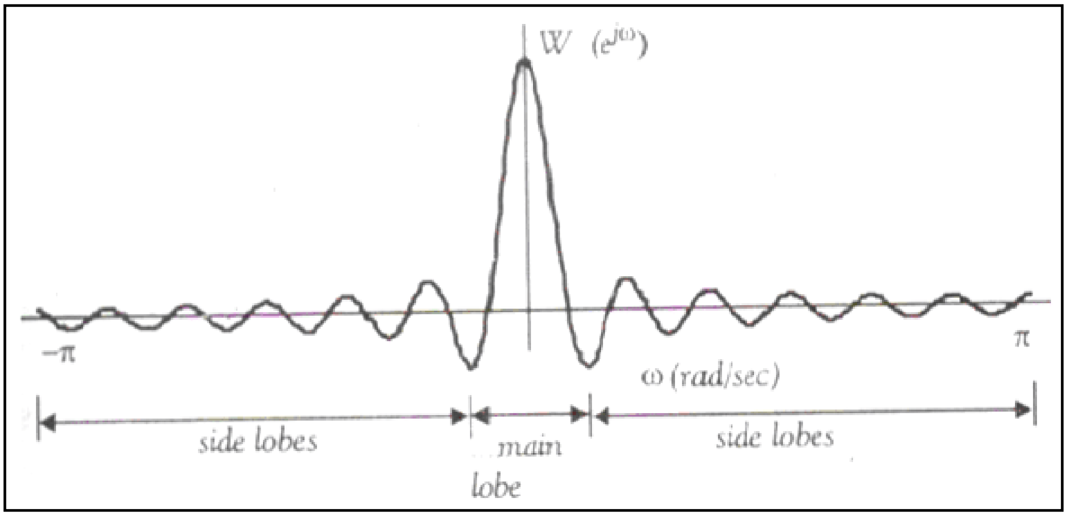
The above multiplication in time domain corresponds to convolution in freq domain, i.e.,

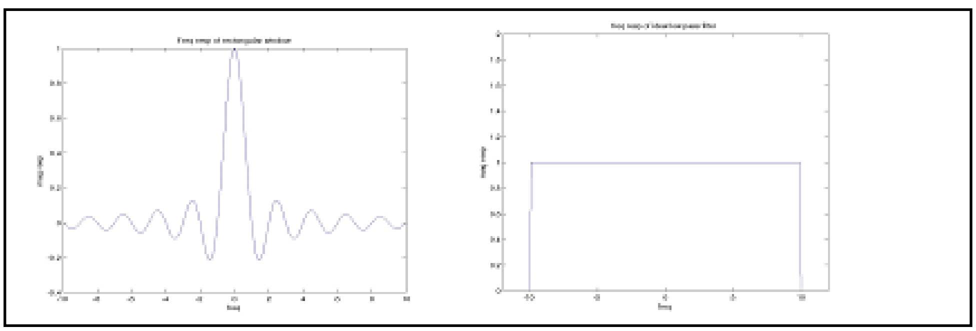
$H(e^{j\omega}) = H_d(e^{j\omega}) * W(e^{j\omega})$ where $W(e^{j\omega})$ is the FT of window function w(n).

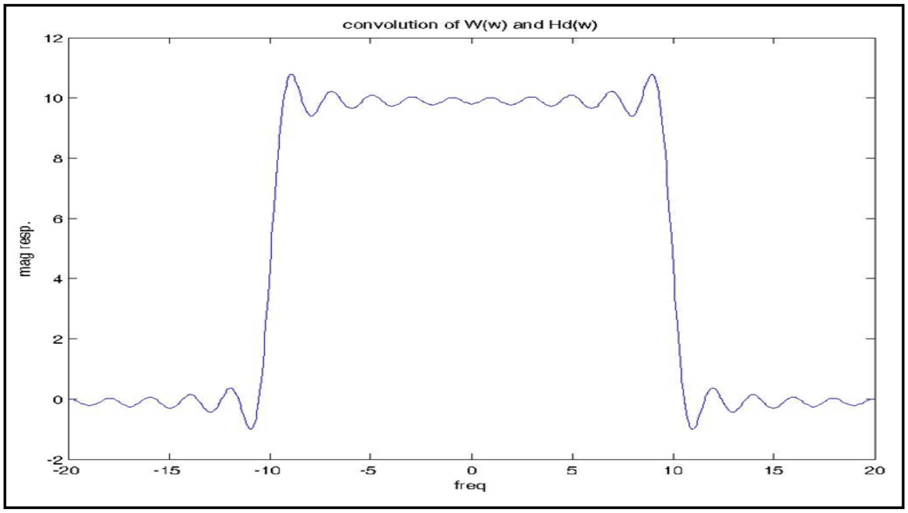
The FT of w(n) is given by

$$W(e^{j\omega}) = \frac{\sin(\omega M/2)}{\sin(\omega/2)}$$

The whole process of multiplying h(n) by a window function and its effect in freq domain are shown in below set of figures.







Suppose the filter to be designed is Low pass filter then the convolution of ideal filter freq response and window function freq response results in distortion in the resultant filter freq response. The ideal sharp cutoff chars are lost and presence of ringing effect is seen at the band edges which is referred to Gibbs Phenomena. This is due to main lobe width and side lobes of the window function freq response.The main lobe width introduces transition band and side lobes results in rippling characters in pass band and stop band. Smaller the main lobe width smaller will be the transition band. The ripples will be of low amplitude if the peak of the first side lobe is far below the main lobe peak.

How to reduce the distortions?

1. Increase length of the window

- as M increases the main lob width becomes narrower, hence the transition band width is decreased

-With increase in length the side lobe width is decreased but height of each side lobe increases in such a manner that the area under each sidelobe remains invariant to changes in M. Thus ripples and ringing effect in pass-band and stop-band are not changed.

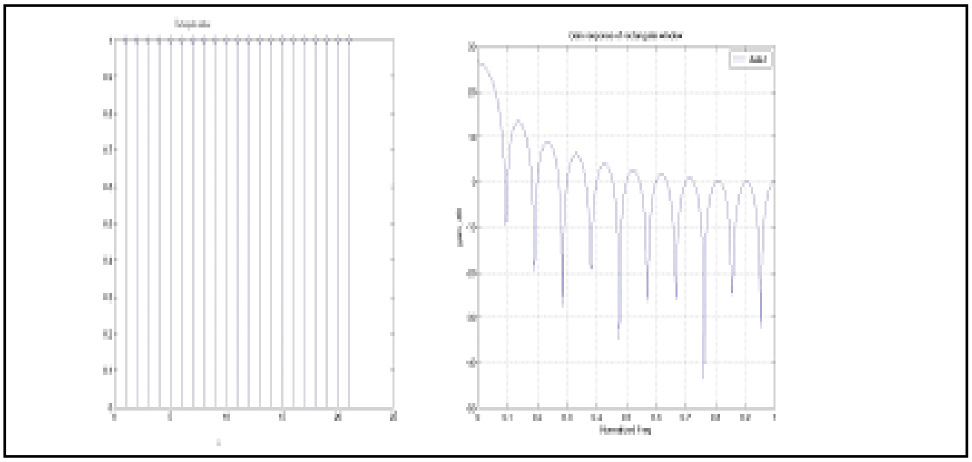
2. Choose windows which tapers off slowly rather than ending abruptly - Slow tapering reduces ringing and ripples but generally increases transition width since main lobe width of these kind of windows are larger.

What is ideal window characteristics?

Window having very small main lobe width with most of the energy contained with it (i.e.,ideal window freq response must be impulsive).Window design is a mathematical problem, more complex the window lesser are the distortions. Rectangular window is one of the simplest window in terms of computational complexity. Windows better than rectangular window are, Hamming, Hanning, Blackman, Bartlett, Traingular,Kaiser. The different window functions are discussed in the following sention.

Rectangular window: The mathematical description is given by,

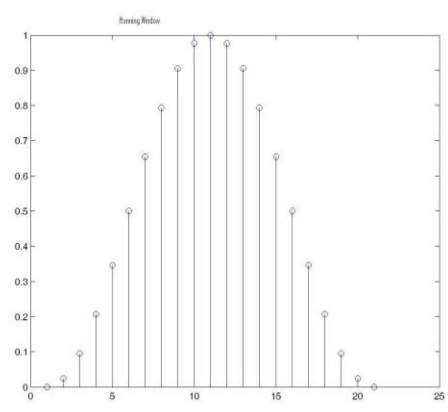
$$w_r(n) = 1 \ \ for\ 0 \le n \le M-1$$

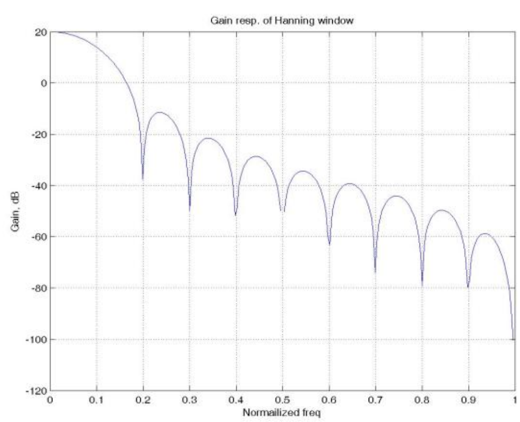


5.6.1 Hanning windows:

It is defined mathematically by,

$$w_{han}(n) = 0.5(1 - \cos\frac{2\pi n}{M-1}) \ \ for\ 0 \le n \le M-1$$



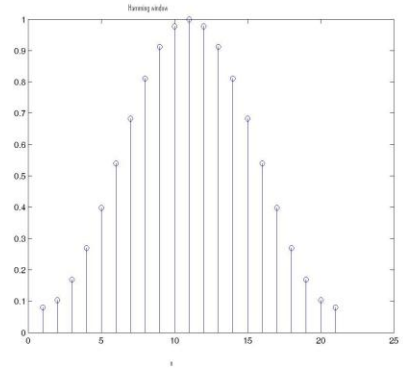


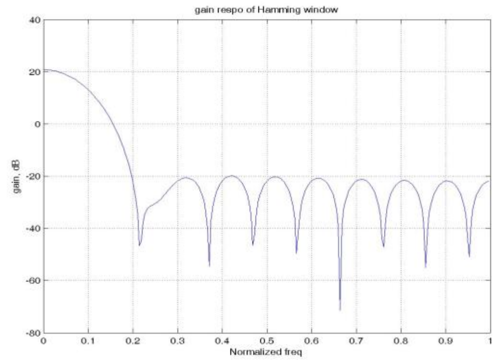
5.6.2 Hamming windows:

This window function is given by

,

$$w_{ham}(n) = 0.54 - 0.46\cos\frac{2\pi n}{M-1} \ \ for\ 0 \le n \le M-1$$

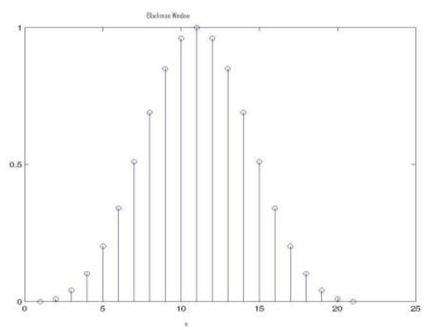


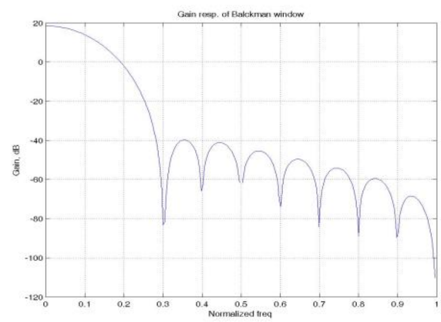


5.6.3 Blackman windows:

This windowfunction is given by,

$$w_{blk}(n) = 0.42 - 0.5\cos\frac{2\pi n}{M-1} + 0.08\cos\frac{4\pi n}{M-1} \quad for\ 0 \le n \le M-1$$

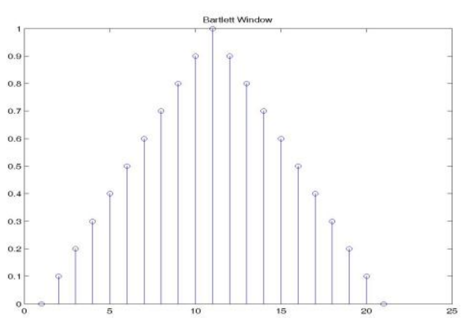


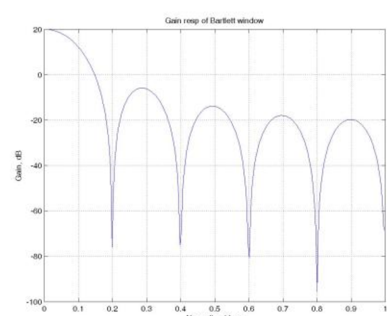


5.6.4 Bartlett (Triangular) windows:

The mathematical description is given by,

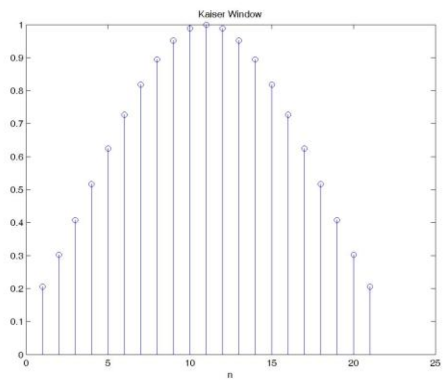
$$w_{bart}(n) = 1 - \frac{2\left|n - \frac{M-1}{2}\right|}{M-1} \qquad for\ 0 \le n \le M-1$$

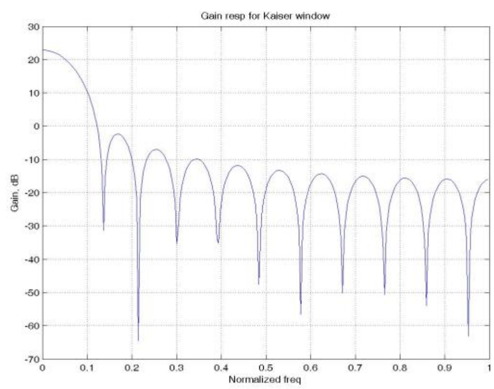




5.6.5 Kaiser windows: The mathematical description is given by,

$$w_k(n) = \frac{I_0\left[\alpha\sqrt{\left(\frac{M-1}{2}\right)^2 - \left(n - \frac{M-1}{2}\right)^2}\right]}{I_0\left[\alpha\left(\frac{M-1}{2}\right)\right]} \qquad for\ 0 \le n \le M-1$$





Type of window	Appr. Transition width of the main lobe	Peak sidelobe (dB)
Rectangular	4π/M	-13
Bartlett	8π/M	-27
Hanning	8π/M	-32
Hamming	8π/M	-43
Blackman	12π/M	-58

Looking at the above table we observe filters which are mathematically simple do not offer best characteristics. Among the window functions discussed Kaiser is the most complex one in terms of functional description whereas it is the one which offers maximum flexibility in the design.

5.6.6 Procedure for designing linear-phase FIR filters using windows:

1. Obtain $h_d(n)$ from the desired freq response using inverse FT relation
2. Truncate the infinite length of the impulse response to finite length with

(assuming M odd) choosing proper window

$$h(n) = h_d(n)w(n) \;\; where$$
$$w(n) \; is\; the\; window\; function\; defined\;\; for -(M-1)/2 \le n \le (M-1)/2$$

3. Introduce h(n) = h(-n) for linear phase characteristics

4. Write the expression for H(z); this is non-causal realization

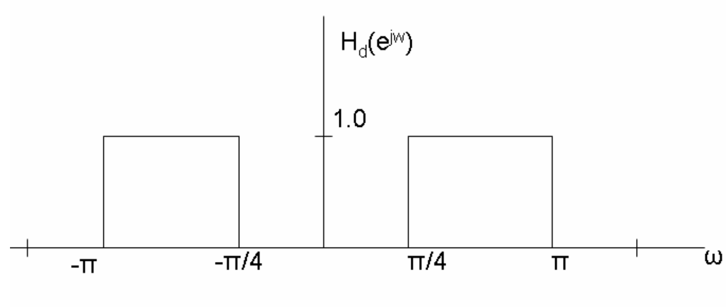
5. To obtain causal realization $H'(z) = z^{-(M-1)/2} H(z)$

Exercise Problems

Prob 1: Design an ideal highpass filter with a frequency response:

$$H_d(e^{j\omega}) = 1 \qquad for\; \frac{\pi}{4} \le |\omega| \le \pi$$
$$= 0 \qquad |\omega| < \frac{\pi}{4}$$

using a hanning window with M = 11 and plot the frequency response.



$$h_d(n) = \frac{1}{2\pi}[\int_{-\pi}^{-\pi/4} e^{j\omega n} d\omega + \int_{\pi/4}^{\pi} e^{j\omega n} d\omega]$$

$$h_d(n) = \frac{1}{\pi n}[\sin \pi n - \sin \frac{\pi n}{4}] \quad for \quad -\infty \le n \le \infty \quad and \quad n \ne 0$$

$$h_d(0) = \frac{1}{2\pi}[\int_{-\pi}^{-\pi/4} d\omega + \int_{\pi/4}^{\pi} d\omega] = \frac{3}{4} = 0.75$$

$h_d(1) = h_d(-1)=-0.225$
$h_d(2) = h_d(-2)= -0.159$
$h_d(3) = h_d(-3)= -0.075$
$h_d(4) = h_d(-4)= 0$
$h_d(5) = h_d(-5) = 0.045$

The hamming window function is given by

$$w_{hn}(n) = 0.5 + 0.5\cos\frac{2\pi n}{M-1} \qquad -(\frac{M-1}{2}) \le n \le (\frac{M-1}{2})$$
$$= 0 \qquad otherwise$$

$$for \quad N = 11$$
$$w_{hn}(n) = 0.5 + 0.5\cos\frac{\pi n}{5} \qquad -5 \le n \le 5$$

$w_{hn}(0) = 1$
$w_{hn}(1) = w_{hn}(-1)=0.9045$
$w_{hn}(2)= w_{hn}(-2)=0.655$
$w_{hn}(3)= w_{hn}(-3)= 0.345$
$w_{hn}(4)= w_{hn}(-4)=0.0945$
$w_{hn}(5)= w_{hn}(-5)=0$

$h(n)= w_{hn}(n)h_d(n)$

h(n)=[0 0 -0.026 -0.104 -0.204 0.75 -0.204 -0.104 -0.026 0 0]

$$h'(n) = h(n-5)$$
$$H'(z) = -0.026z^{-2} - 0.104z^{-3} - 0.204z^{-4} + 0.75z^{-5} - 0.204z^{-6} - 0.104z^{-7} - 0.026z^{-8}$$

Using the equation

$$H_r(e^{jw}) = [h(\frac{M-1}{2}) + 2\sum_{n=0}^{\frac{M-3}{2}} h(n)\cos\omega(\frac{M-1}{2} - n)$$
$$H_r(e^{jw}) = 0.75) + 2\sum_{n=0}^{4} h(n)\cos\omega(5-n)$$

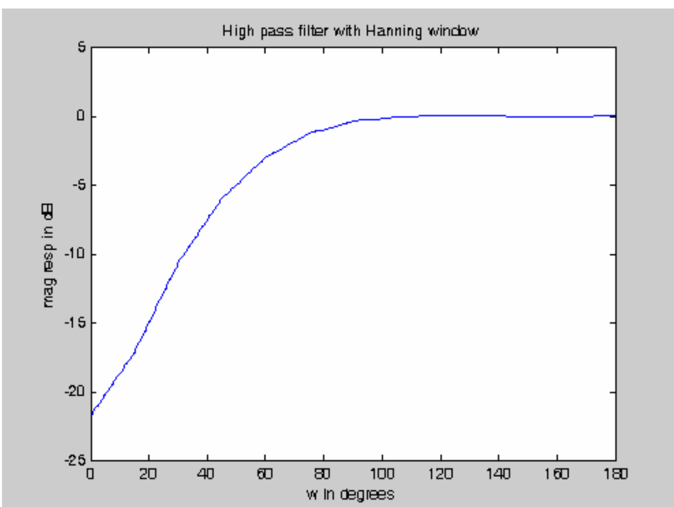
The magnitude response is given by,

$|Hr(e^{j\omega})| = |0.75 - 0.408\cos\omega - 0.208\cos 2\omega - 0.052\cos 3\omega|$

ω in degrees = [0 15 30 45 60 75 90 105 120 135 150 165 180]

$|H(e^{j\omega})|$ in dBs = [-21.72 -17.14 -10.67 -6.05 -3.07 -1.297 -0.3726

-0.0087 0.052 0.015 0 0 0.017]



Prob 2 : Design a filter with a frequency response:

$$H_d(e^{j\omega}) = e^{-j3\omega} \qquad for -\frac{\pi}{4} \le \omega \le \frac{\pi}{4}$$

$$= 0 \quad \frac{\pi}{4} < |\omega| \le \pi$$

using a Hanning window with M = 7

Soln:
The freq resp is having a term e $^{-j\omega(M-1)/2}$ which gives h(n) symmetrical about n = M-1/2 = 3 i.e we get a causal sequence.

$$h_d(n) = \frac{1}{2\pi} \int_{-\pi/4}^{\pi/4} e^{-j3\omega} e^{j\omega n} d\omega$$

$$= \frac{\sin \frac{\pi}{4}(n-3)}{\pi(n-3)}$$

$$this \;\; gives \; h_d(0) = h_d(6) = 0.075$$

$$h_d(1) = h_d(5) = 0.159$$

$$h_d(2) = h_d(4) = 0.22$$

$$h_d(3) = 0.25$$

The Hanning window function values are given by

$w_{hn}(0) = w_{hn}(6) = 0$

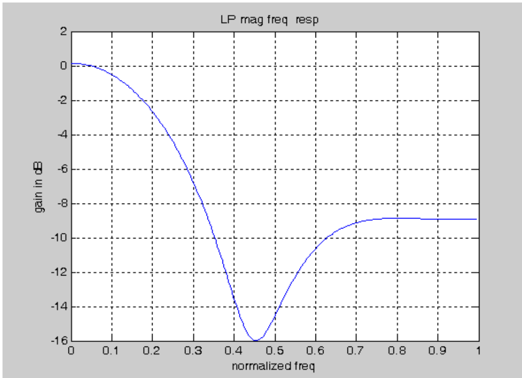
$w_{hn}(1) = w_{hn}(5) = 0.25$

$w_{hn}(2) = w_{hn}(4) = 0.75$

$w_{hn}(3) = 1$

$h(n) = h_d(n)\, w_{hn}(n)$

h(n)=[0 0.03975 0.165 0.25 0.165 0.3975 0]



5.7 Design of Linear Phase FIR filters using Frequency Sampling method

6.9.1 Motivation: We know that DFT of a finite duration DT sequence is obtained by sampling FT of the sequence then DFT samples can be used in reconstructing original time domain samples if frequency domain sampling was done correctly. The samples of FT of h(n) i.e., H(k) are sufficient to recover h(n).

Since the designed filter has to be realizable then h(n) has to be real, hence even symmetry properties for mag response |H(k)| and odd symmetry properties for phase response can be applied. Also, symmetry for h(n) is applied to obtain linear phase chas.

Fro DFT relationship we have

$$h(n) = \frac{1}{N}\sum_{k=0}^{N-1} H(k) e^{j2\pi kn/N} \qquad for \quad n = 0,1,......N-1$$

$$H(k) = \sum_{n=0}^{N-1} h(n) e^{-j2\pi kn/N} \qquad for \quad k = 0,1,.........N-1$$

Also we know H(k) = $H(z)|_{z=e^{j2\pi kn/N}}$

The system function H(z) is given by

$$H(z) = \sum_{n=0}^{N-1} h(n) z^{-n}$$

Substituting for h(n) from IDFT relationship

$$H(z) = \frac{1-z^{-N}}{N}\sum_{k=0}^{N-1} \frac{H(k)}{1-e^{j2\pi kn/N} z^{-1}}$$

Since H(k) is obtained by sampling $H(e^{j\omega})$ hence the method is called Frequency Sampling Technique.

Since the impulse response samples or coefficients of the filter has to be real for filter to be realizable with simple arithmetic operations, properties of DFT of real sequence can be used. The following properties of DFT for real sequences are useful:

$H^*(k) = H(N-k)$

$|H(k)|=|H(N-k)|$ - magnitude response is even

$\theta(k) = -\theta(N-k)$ – Phase response is odd

$h(n) = \frac{1}{N}\sum_{k=0}^{N-1} H(k)e^{j2\pi kn/N}$ can be rewritten as (for N odd)

$$h(n) = \frac{1}{N}\left[H(0) + \sum_{k=1}^{N-1} H(k)e^{j2\pi kn/N}\right]$$
$$h(n) = \frac{1}{N}\left[H(0) + \sum_{k=1}^{N-1/2} H(k)e^{j2\pi kn/N} + \sum_{k=N-1/2}^{N-1} H(k)e^{j2\pi kn/N}\right]$$

Using substitution k = N – r or r = N- k in the second substitution with r going from now (N- 1)/2 to 1 as k goes from 1 to (N-1)/2

$$h(n) = \frac{1}{N}\left[H(0) + \sum_{k=1}^{(N-1)/2} H(k)e^{j2\pi kn/N} + \sum_{k=1}^{(N-1)/2} H(N-k)e^{-j2\pi kn/N}\right]$$
$$h(n) = \frac{1}{N}\left[H(0) + \sum_{k=1}^{(N-1)/2} H(k)e^{j2\pi kn/N} + \sum_{k=1}^{(N-1)/2} H^*(k)e^{-j2\pi kn/N}\right]$$
$$h(n) = \frac{1}{N}\left[H(0) + \sum_{k=1}^{(N-1)/2} H(k)e^{j2\pi kn/N} + \sum_{k=1}^{(N-1)/2} (H(k)e^{j2\pi kn/N})^*\right]$$
$$h(n) = \frac{1}{N}\left[H(0) + \sum_{k=1}^{(N-1)/2} (H(k)e^{j2\pi kn/N} + (H(k)e^{j2\pi kn/N})^*\right]$$
$$h(n) = \frac{1}{N}\left[H(0) + 2\sum_{k=1}^{(N-1)/2} \mathrm{Re}(H(k)e^{j2\pi kn/N}\right]$$

Similarly for N even we have

$$h(n) = \frac{1}{N}\left[H(0) + 2\sum_{k=1}^{(N-1)/2} \mathrm{Re}(H(k)e^{j2\pi kn/N}\right]$$

Using the symmetry property h(n)= h (N-1-n) we can obtain Linear phase FIR filters using the frequency sampling technique.

Exercise problems

Prob 1 : Design a LP FIR filter using Freq sampling technique having cutoff freq of π/2 rad/sample. The filter should have linear phase and length of 17.

The desired response can be expressed as

$$\begin{aligned} H_d(e^{j\omega}) &= e^{-j\omega(\frac{M-1}{2})} \quad for \quad |\omega| \le \omega c \\ &= 0 \qquad otherwise \\ with \quad M &= 17 \quad and \quad \omega c = \pi/2 \end{aligned}$$

$$\begin{aligned} H_d(e^{j\omega}) &= e^{-j\omega 8} \quad for \quad 0 \le \omega \le \pi/2 \\ &= 0 \qquad for \quad \pi/2 \le \omega \le \pi \end{aligned}$$

Selecting $\omega_k = \dfrac{2\pi k}{M} = \dfrac{2\pi k}{17} \quad for \quad k = 0,1,......16$

$$H(k) = H_d(e^{j\omega})\Big|_{\omega=\frac{2\pi k}{17}}$$

$$\begin{aligned} H(k) &= e^{-j\frac{2\pi k}{17}8} \quad for \quad 0 \le \frac{2\pi k}{17} \le \frac{\pi}{2} \\ &= 0 \qquad for \quad \pi/2 \le \frac{2\pi k}{17} \le \pi \\ H(k) &= e^{-j\frac{16\pi k}{17}} \quad for \quad 0 \le k \le \frac{17}{4} \\ &= 0 \qquad for \quad \frac{17}{4} \le k \le \frac{17}{2} \end{aligned}$$

The range for “k” can be adjusted to be an integer such as

$$\begin{aligned} &0 \le k \le 4 \\ &and \quad 5 \le k \le 8 \end{aligned}$$

The freq response is given by

$$H(k) = e^{-j\frac{2\pi k}{17}8} \qquad for \quad 0 \le k \le 4$$
$$= 0 \qquad for \qquad 5 \le k \le 8$$

Using these value of H(k) we obtain h(n) from the equation

$$h(n) = \frac{1}{M}(H(0) + 2\sum_{k=1}^{(M-1)/2} \text{Re}(H(k)e^{j2\pi kn/M}))$$

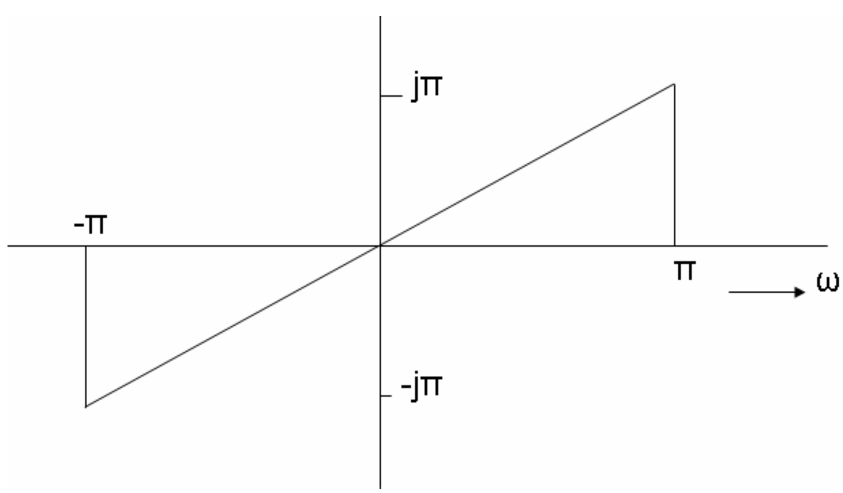
i.e., $h(n) = \frac{1}{17}(1 + 2\sum_{k=1}^{4} \text{Re}(e^{-j16\pi k/17}e^{j2\pi kn/17}))$

$$h(n) = \frac{1}{17}(H(0) + 2\sum_{k=1}^{4}\cos(\frac{2\pi k(8-n)}{17}) \quad for \quad n = 0,1,........16$$

- Even though k varies from 0 to 16 since we considered ω varying between 0 and π/2 only k values from 0 to 8 are considered
- While finding h(n) we observe symmetry in h(n) such that n varying 0 to 7 and 9 to 16 have same set of h(n)

5.8 Design of FIR Differentiator

Differentiators are widely used in Digital and Analog systems whenever a derivative of the signal is needed. Ideal differentiator has pure linear magnitude response in the freq range –π to +π. The typical frequency response characteristics is as shown in the below figure.



$$H_r(e^{j\omega}) = 2\sum_{n=0}^{4} h(n)\sin\omega(5-n)$$

$$H(e^{j\omega}) = j\,|\,H_r(e^{j\omega})\,| = j\{0.254\sin 5\omega + 0.424\sin 3\omega + 1.272\sin\omega\}$$

b) Blackman Window

window function is defined as

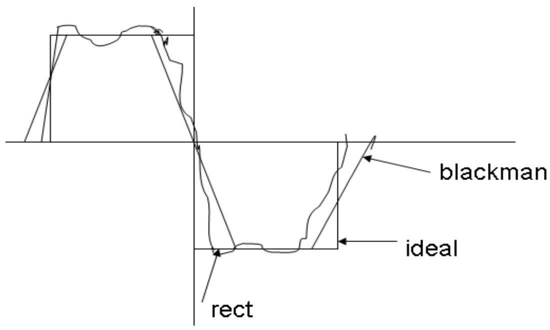
$$w_b(n) = 0.42 + 0.5\cos\frac{\pi n}{5} + 0.08\cos\frac{2\pi n}{5} \qquad -5 \le n \le 5$$

$$= 0 \qquad otherwise$$

$W_b(n)$ = [0, 0.04, 0.2, 0.509,0.849,1,0.849, 0.509, 0.2, 0.04,0] for -5≥n≥5

h'(n) = h(n-5) = [0, 0, -0.0424, 0, -0.5405, 0, 0.5405, 0, 0.0424, 0, 0]

$$H(e^{j\omega}) = -j[0.0848\sin 3\omega + 1.0810\sin\omega]$$



Outcomes

Analyse and Design FIR Filters using various techniques(Window functions and frequency sampling techniques)

Further Readings

1. nptel.ac.in/video.php?subjectId=117102060
2. www.journals.elsevier.com/digital-signal-processing
3. www.dspguide.com/whatdsp.htm

5.9 Realization of Digital Filters

Introduction

The two important forms of expressing system leading to different realizations of FIR & IIR filters are

a) Difference equation form

$$y(n) = -\sum_{k=1}^{N} a_k y(n-k) + \sum_{k=1}^{M} b_k x(n-k)$$

b) Ration of polynomials

$$H(Z) = \frac{\sum_{k=0}^{M} b_k Z^{-k}}{1 + \sum_{k=1}^{N} a_k Z^{-k}}$$

The following factors influence choice of a specific realization,

- Computational complexity
- Memory requirements
- Finite-word-length
- Pipeline / parallel processing

Computation Complexity

This is do with number of arithmetic operations i.e. multiplication, addition & divisions. If the realization can have less of these then it will be less complex computationally.
In the recent processors the fetch time from memory & number of times a comparison between two numbers is performed per output sample is also considered and found to be important from the point of view of computational complexity.

Memory requirements

This is basically number of memory locations required to store the system parameters, past inputs, past outputs, and any intermediate computed values. Any realization requiring less of these is preferred.

Finite-word-length effects

These effects refer to the quantization effects that are inherent in any digital implementation of the system, either in hardware or in software. No computing system has infinite precision. With finite precision there is bound to be errors. These effects are basically to do with truncation & rounding-off of samples. The extent of this effect varies with type of arithmetic used(fixed or floating). The serious issue is that the effects have influence on system characteristics. A structure which is less sensitive to this effect need to be chosen.

Pipeline / Parallel Processing

This is to do with suitability of the structure for pipelining & parallel processing. The parallel processing can be in software or hardware. Longer pipelining make the system more efficient.

5.9.2 Structure for FIR Systems

FIR system is described by,

$$y(n) = \sum_{k=0}^{M-1} b_k x(n-k)$$

Or equivalently, the system function

$$H(Z) = \sum_{k=0}^{M-1} b_k Z^{-k}$$

Where we can identify $h(n) = \begin{cases} b_n & 0 \le n \le n-1 \\ 0 & otherwise \end{cases}$

Different FIR Structures used in practice are,

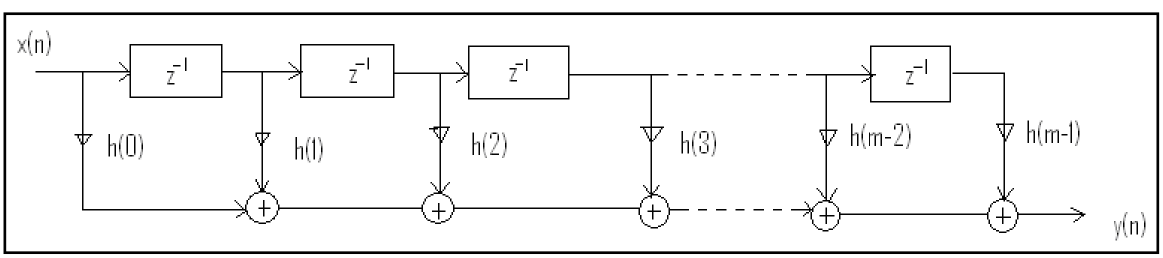
1. Direct form
2. Cascade form
3. Frequency-sampling realization
4. Lattice realization

Direct – Form Structure

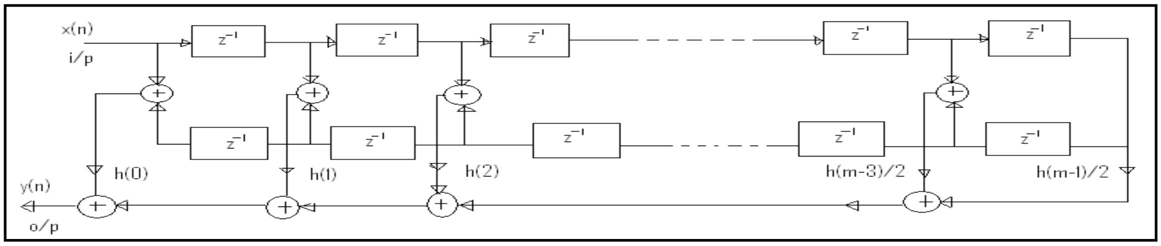
Convolution formula is used to express FIR system given by,

$$y(n) = \sum_{k=0}^{M-1} h(k)\ x(n-k)$$

- It is Non recursive in structure



- As can be seen from the above implementation it requires M-1 memory locations for storing the M-1 previous inputs
- It requires computationally M multiplications and M-1 additions per output point
- It is more popularly referred to as tapped delay line or transversal system
- Efficient structure with linear phase characteristics are possible where $h(n) = \pm h(M-1-n)$

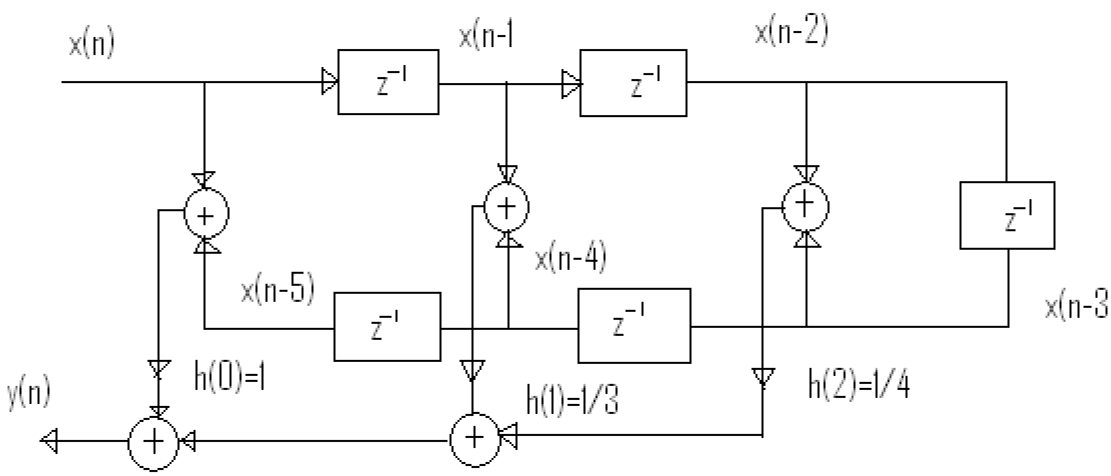


Prob:
Realize the following system function using minimum number of multiplication

(1) $H(Z) = 1 + \frac{1}{3}Z^{-1} + \frac{1}{4}Z^{-2} + \frac{1}{4}Z^{-3} + \frac{1}{3}Z^{-4} + Z^{-5}$

We recognize $h(n) = \left[1, \frac{1}{3}, \frac{1}{4}, \frac{1}{4}, \frac{1}{3}, 1\right]$

M is even = 6, and we observe h(n) = h(M-1-n)   h(n) = h(5-n)
i.e h(0) = h(5)   h(1) = h(4)   h(2) = h(3)
Direct form structure for Linear phase FIR can be realized

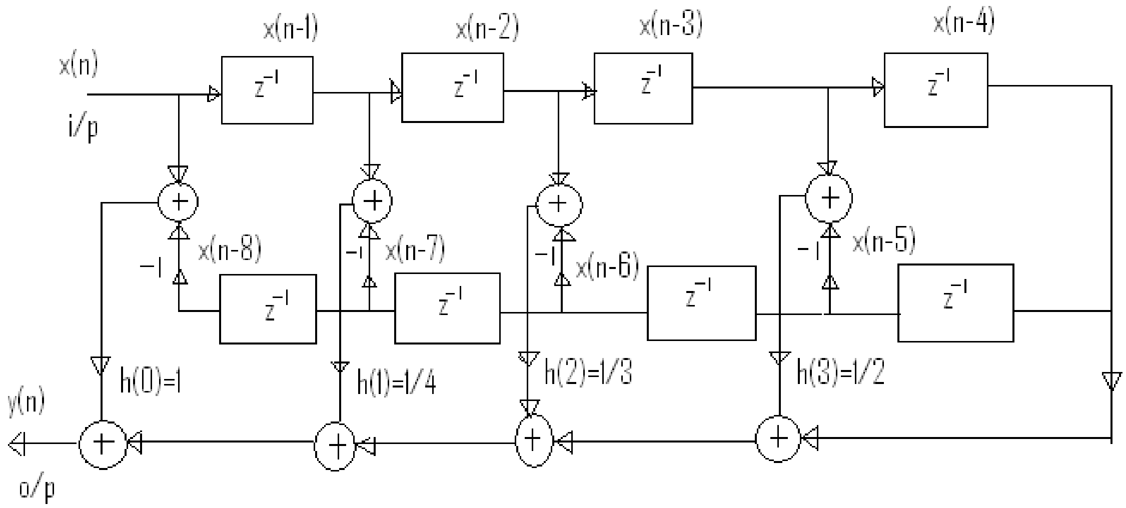


Exercise: Realize the following using system function using minimum number of multiplication.

$H(Z) = 1 + \frac{1}{4}Z^{-1} + \frac{1}{3}Z^{-2} + \frac{1}{2}Z^{-3} - \frac{1}{2}Z^{-5} - \frac{1}{3}Z^{-6} - \frac{1}{4}Z^{-7} - Z^{-8}$

m=9 $h(n) = \left[1, \frac{1}{4}, \frac{1}{3}, \frac{1}{2}, -\frac{1}{2}, -\frac{1}{3}, -\frac{1}{4}, -1\right]$

odd symmetry
h(n) = -h(M-1-n);   h(n) = -h(8-n);   h(m-1/2) = h(4) = 0
h(0) = -h(8);   h(1) = -h(7);   h(2) = -h(6);   h(3) = -h(5)



Cascade – Form Structure

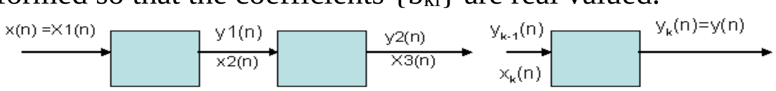
The system function H(Z) is factored into product of second – order FIR system

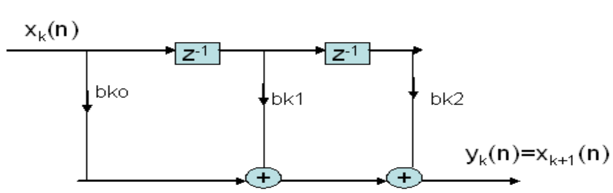
$$H(Z) = \prod_{k=1}^{K} H_k(Z)$$

Where $H_k(Z) = b_{k0} + b_{k1}Z^{-1} + b_{k2}Z^{-2}$ k = 1, 2, ….. K

and K = integer part of (M+1) / 2

The filter parameter b_0 may be equally distributed among the K filter section, such that $b_0 = b_{10}\, b_{20} \ldots. b_{k0}$ or it may be assigned to a single filter section. The zeros of H(z) are grouped in pairs to produce the second – order FIR system. Pairs of complex-conjugate roots are formed so that the coefficients $\{b_{ki}\}$ are real valued.

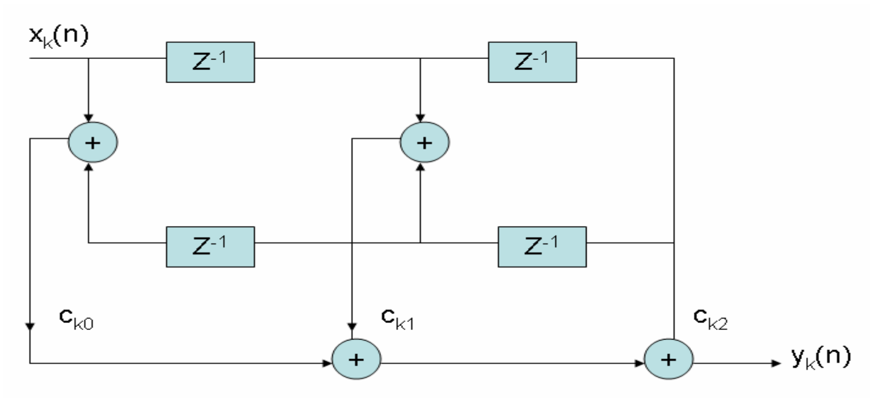




In case of linear –phase FIR filter, the symmetry in h(n) implies that the zeros of H(z) also exhibit a form of symmetry. If zk and zk* are pair of complex – conjugate zeros then

1/zk and 1/zk* are also a pair complex –conjugate zeros. Thus simplified fourth order sections are formed. This is shown below,

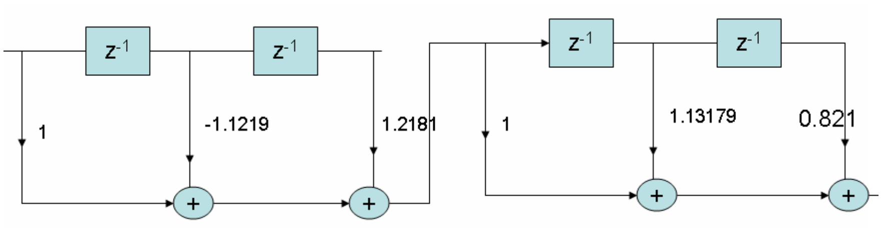
$$H_k(z) = C_{k0}(1 - z_k z^{-1})(1 - z_k * z^{-1})(1 - z^{-1}/z_k)(1 - z^{-1}/z_{k*})$$
$$= C_{k0} + C_{k1}z^{-1} + C_{k2}z^{-2} + C_{k1}z^{-3} + z^{-4}$$



Problem: Realize the difference equation
$y(n) = x(n) + 0.25x(n-1) + 0.5x(n-2) + 0.75x(n-3) + x(n-4)$
in cascade form.

Soln:

$$Y(z) = X(z)\{1 + 0.25z^{-1} + 0.5z^{-2} + 0.75z^{-3} + z^{-4})$$
$$H(z) = 1 + 0.25z^{-1} + 0.5z^{-2} + 0.75z^{-3} + z^{-4}$$
$$H(z) = (1 - 1.1219z^{-1} + 1.2181z^{-2})(1 + 1.3719z^{-1} + 0.821z^{-2})$$
$$H(z) = H_1(z)H_2(z)$$



Frequency sampling realization

We can express system function H(z) in terms of DFT samples H(k) which is given by

$$H(z) = (1 - z^{-N})\frac{1}{N}\sum_{k=0}^{N-1}\frac{H(k)}{1 - W_N^{-k}z^{-1}}$$

This form can be realized with cascade of FIR and IIR structures. The term ($1-z^{-N}$) is realized as FIR and the term $\frac{1}{N}\sum_{k=0}^{N-1}\frac{H(k)}{1 - W_N^{-k}z^{-1}}$ as IIR structure.

The realization of the above freq sampling form shows necessity of complex arithmetic.

Incorporating symmetry in h(n) and symmetry properties of DFT of real sequences the realization can be modified to have only real coefficients.

Lattice structures

Lattice structures offer many interesting features:

1. Upgrading filter orders is simple. Only additional stages need to be added instead of redesigning the whole filter and recalculating the filter coefficients.
2. These filters are computationally very efficient than other filter structures in a filter bank applications (eg. Wavelet Transform)
3. Lattice filters are less sensitive to finite word length effects.

Consider

$$H(z) = \frac{Y(z)}{X(z)} = 1 + \sum_{i=1}^{m} a_m(i) z^{-i}$$

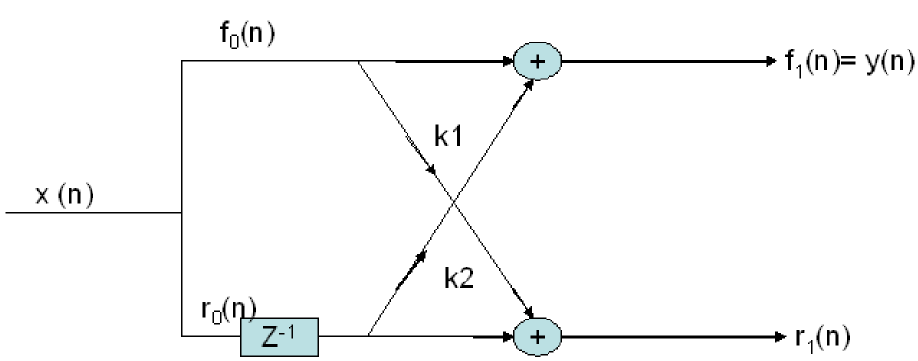
m is the order of the FIR filter and $a_m(0)=1$

when m = 1 $Y(z)/X(z) = 1 + a_1(1)z^{-1}$

$y(n)= x(n)+ a_1(1)x(n-1)$

f1(n) is known as upper channel output and r1(n)as lower channel output.

$f_0(n)= r_0(n)=x(n)$



The outputs are

$$f_1(n) = f_0(n) + k_1 r_0(n-1) \qquad 1a$$
$$r_1(n) = k_1 f_0(n) + r_0(n-1) \qquad 1b$$
$$if \quad k_1 = a_1(1), then \quad f_1(n) = y(n)$$

If m=2

$$\frac{Y(z)}{X(z)} = 1 + a_2(1)z^{-1} + a_2(2)z^{-2}$$
$$y(n) = x(n) + a_2(1)x(n-1) + a_2(2)x(n-2)$$
$$y(n) = f_1(n) + k_2 r_1(n-1) \qquad (2)$$

Substituting 1a and 1b in (2)

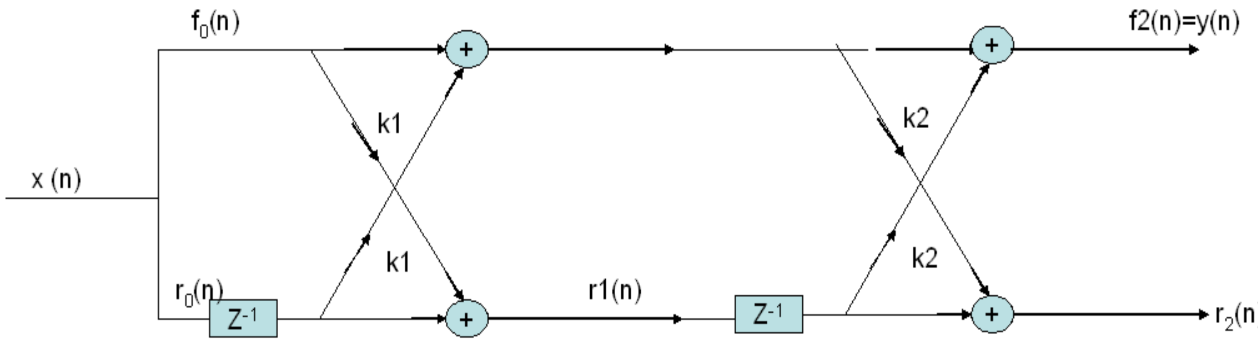
$$y(n) = f_0(n) + k_1 r_0(n-1) + k_2[k_1 f_0(n-1) + r_0(n-2)]$$
$$= f_0(n) + k_1 r_0(n-1) + k_2 k_1 f_0(n-1) + k_2 r_0(n-2)]$$
$$\sin ce \quad f_0(n) = r_0(n) = x(n)$$
$$y(n) = x(n) + k_1 x(n-1) + k_2 k_1 x(n-1) + k_2 x(n-2)]$$
$$= x(n) + (k_1 + k_1 k_2)x(n-1) + k_2 x(n-2)$$

We recognize
$$a_2(1) = k_1 + k_1 k_2$$
$$a_2(1) = k_2$$

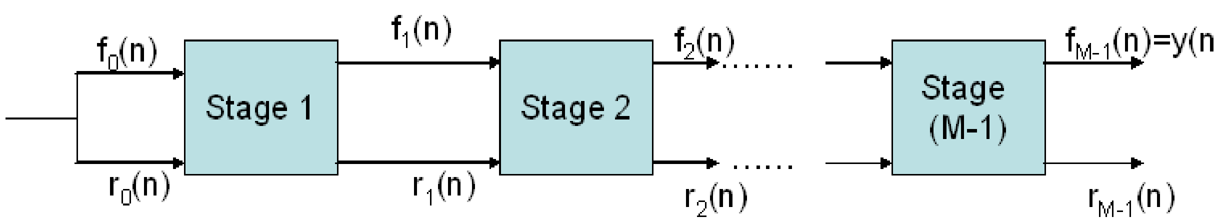
Solving the above equation we get

$$k_1 = \frac{a_2(1)}{1 + a_2(2)} \quad and \quad k_2 = a_2(2) \qquad (4)$$

Equation (3) means that, the lattice structure for a second-order filter is simply a cascade of two first-order filters with k1 and k2 as defined in eq (4)



Similar to above, an Mth order FIR filter can be implemented by lattice structures with M – stages



Direct Form – I to lattice structure

For m = M, M-1, ………..2, 1 do

$$k_m = a_m(m)$$

$$a_{m-1}(i) = \frac{a_m(i) - a_m(m)a_m(m-i)}{1-k_m^2} \qquad 1 \le i \le m-1$$

- The above expression fails if k_m=1. This is an indication that there isa zero on the unit circle. If k_m=1, factor out this root from A(z) and the recursive formula can be applied for reduced order system.

$for\ m = 2\ and\ m = 1$
$k_2 = a_2(2) \quad \& \quad k_1 = a_1(1)$

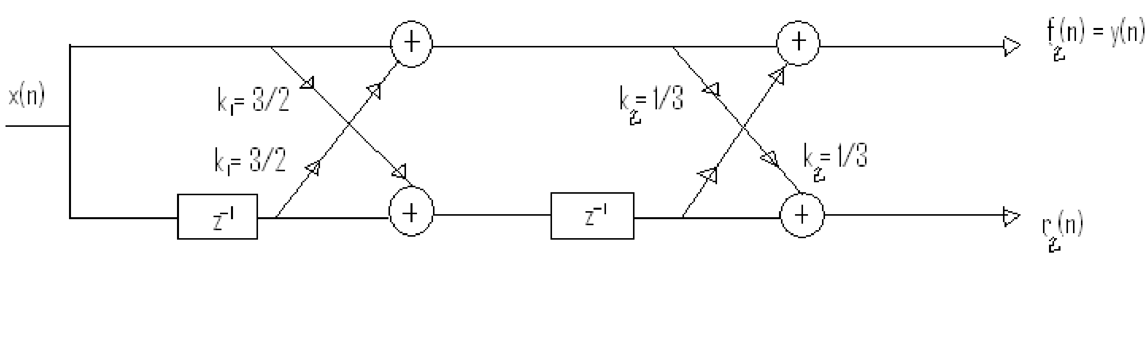
$for\ m = 2\ \&\ i = 1$
$$a_1(1) = \frac{a_2(1) - a_2(2)a_2(1)}{1 - k_2^2} = \frac{a_2(1)[1 - a_2(2)]}{1 - a_2^2(2)} = \frac{a_2(1)}{1 + a_2(2)}$$
$$Thus \quad k_1 = \frac{a_2(1)}{1 + a_2(2)}$$

8.4.2 Lattice to direct form –I

For m = 1,2,…….M-1

$a_m(0) = 1$
$a_m(m) = k_m$
$a_m(i) = a_{m-1}(i) + a_m(m)a_{m-1}(m-i) \quad 1 \le i \le m-1$

Problem:
Given FIR filter $H(Z) = 1 + 2Z^{-1} + \frac{1}{3}Z^{-2}$ obtain lattice structure for the same
Given $a_1(1) = 2$, $a_2(2) = 1/3$
Using the recursive equation for
m = M, M-1, ……, 2, 1
here M=2 therefore m = 2, 1
if m=2 $k_2 = a_2(2) = 1/3$
if m=1 $k_1 = a_1(1)$
also, when m=2 and i=1
$$a_1(1) = \frac{a_2(1)}{1 + a_2(2)} = \frac{2}{1 + 1/3} = \frac{3}{2}$$
Hence $k_1 = a_1(1) = 3/2$



Recommended questions with solution

1. Consider an FIR lattice filter with co-efficients $k_1 = \frac{1}{2}$, $k_2 = \frac{1}{3}$, $k_3 = \frac{1}{4}$. Determine the FIR filter co-efficient for the direct form structure
($H(Z) = a_3(0) + a_3(1)Z^{-1} + a_3(2)Z^{-2} + a_3(3)Z^{-3}$)

$a_3(0) = 1$ $\quad a_3(3) = k_3 = 1/4$ $\quad a_2(2) = k_2 = \frac{1}{3}$

$a_1(1) = k_1 = \frac{1}{2}$

for m=2, i=1

$$a_2(1) = a_1(1) + a_2(2)a_1(1)$$

$$= a_1(1)[1 + a_2(2)] = \frac{1}{2}\left[1 + \frac{1}{3}\right]$$

$$= \frac{4}{6} = \frac{2}{3}$$

for m=3, i=1

$$a_3(1) = a_2(1) + a_3(3)a_2(2)$$

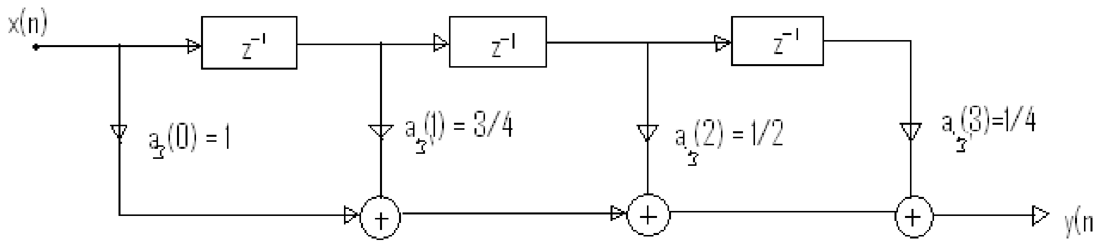
$$= \frac{2}{3} + \frac{1}{4}.\frac{1}{3}$$
$$= \frac{2}{3} + \frac{1}{12} = \frac{8+1}{12}$$

$$= \frac{9}{12} = \frac{3}{4}$$

for m=3 & i=2

$$a_3(2) = a_2(2) + a_3(3)a_2(1)$$
$$= \frac{1}{3} + \frac{1}{4}.\frac{2}{3}$$
$$= \frac{1}{3} + \frac{1}{6} = \frac{2+1}{6}$$
$$= \frac{3}{6} = \frac{1}{2}$$

$a_3(0) = 1,\qquad a_3(1) = \frac{3}{4},\qquad a_3(2) = \frac{1}{2}\ ,\qquad a_3(3) = \frac{1}{4}$



Structures for IIR Filters

The IIR filters are represented by system function;

$$H(Z) = \frac{\sum_{k=0}^{M} b_k z^{-k}}{1 + \sum_{k=1}^{N} a_k z^{-k}}$$

and corresponding difference equation given by,

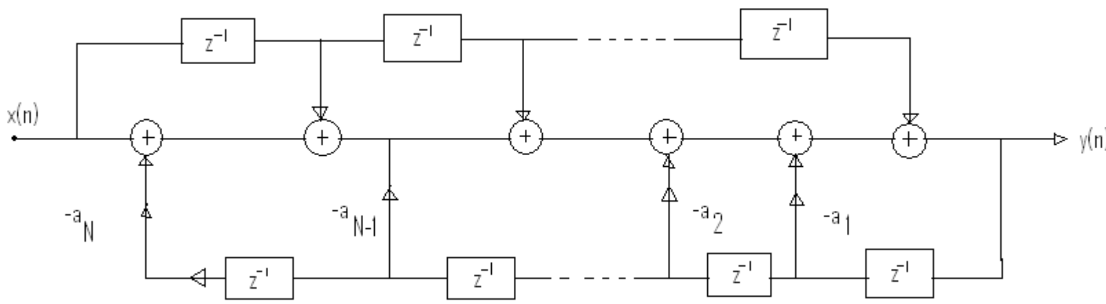
$$y(n) = -\sum_{k=1}^{N} a_k y(n-k) + \sum_{k=0}^{N} b_k x(n-k)$$

Different realizations for IIR filters are,

1. Direct form-I
2. Direct form-II
3. Cascade form
4. Parallel form
5. Lattice form

Direct form-I

This is a straight forward implementation of difference equation which is very simple. Typical Direct form – I realization is shown below . The upper branch is forward path and lower branch is feedback path. The number of delays depends on presence of most previous input and output samples in the difference equation.



Direct form-II

The given transfer function H(z) can be expressed as,

$$H(z) = \frac{Y(z)}{X(z)} = \frac{V(z)}{X(z)} . \frac{Y(z)}{V(z)}$$

where V(z) is an intermediate term. We identify,

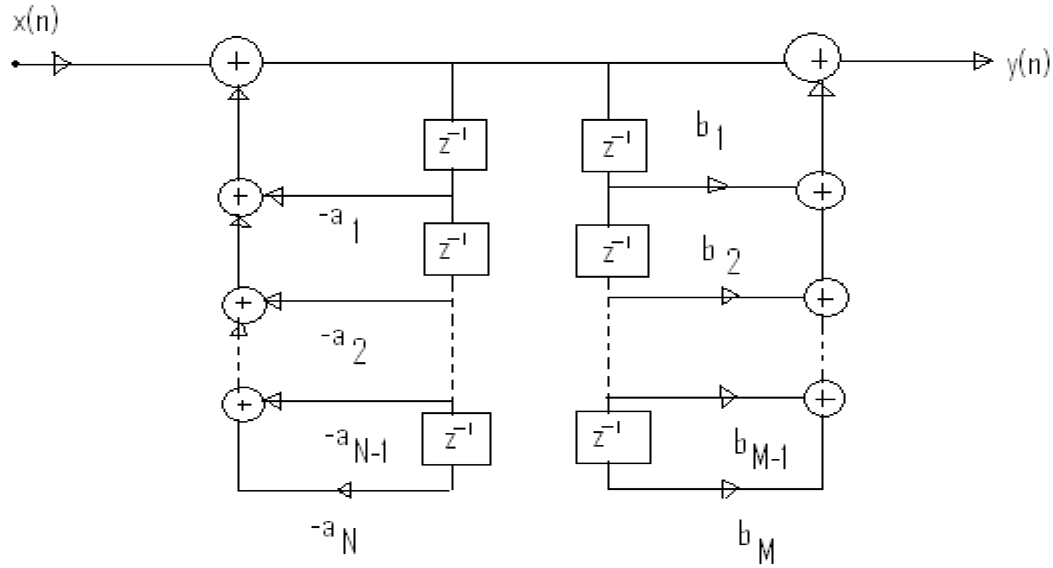
$$\frac{V(z)}{X(z)} = \frac{1}{1 + \sum_{k=1}^{N} a_k z^{-k}}$$ -------------------all poles

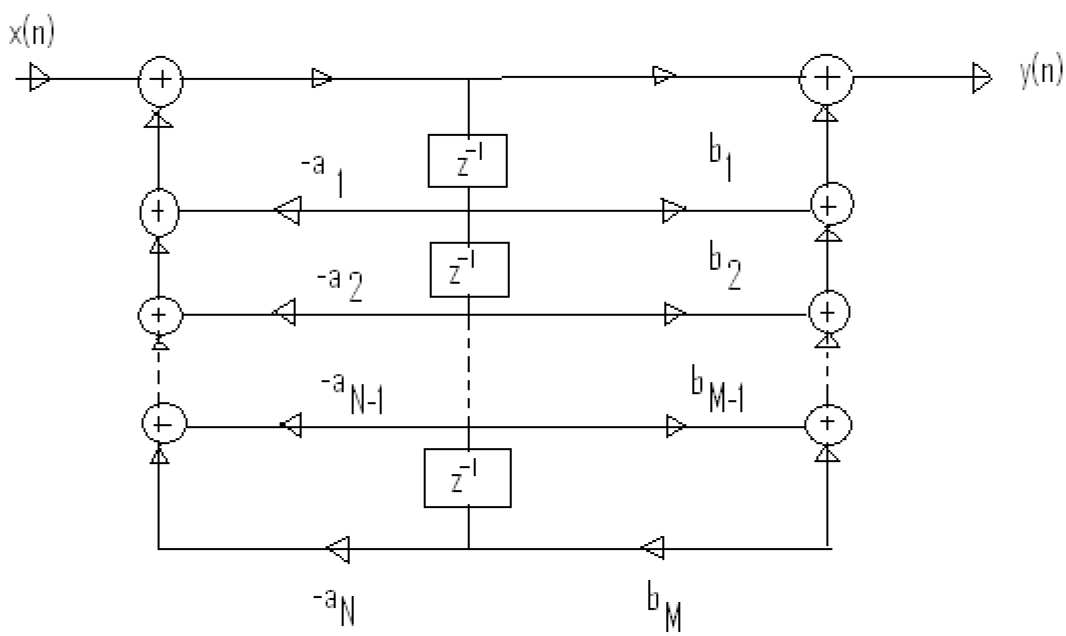
$$\frac{Y(z)}{V(z)} = \left(1 + \sum_{k=1}^{M} b_k z^{-k}\right)$$ -------------------all zeros

The corresponding difference equations are,

$$v(n) = x(n) - \sum_{k=1}^{N} a_k v(n-k)$$

$$y(n) = v(n) + \sum_{k=1}^{M} b_k v(n-1)$$





This realization requires M+N+! multiplications, M+N addition and the maximum of {M, N} memory location

Cascade Form

The transfer function of a system can be expressed as,

$$H(z) = H_1(z)H_2(z)....H_k(z)$$

Where $H_k(Z)$ could be first order or second order section realized in Direct form – II form i.e.,

$$H_k(Z) = \frac{b_{k0} + b_{k1}Z^{-1} + b_{k2}Z^{-2}}{1 + a_{k1}Z^{-1} + a_{k2}Z^{-2}}$$

where K is the integer part of (N+1)/2

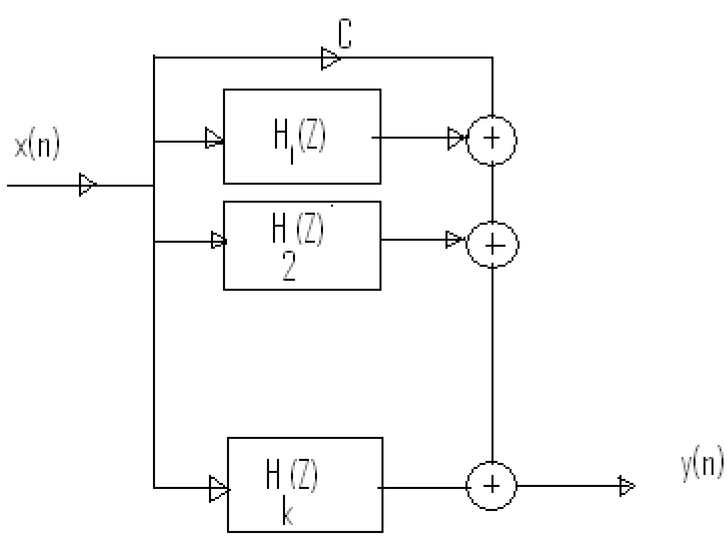
Similar to FIR cascade realization, the parameter b_0 can be distributed equally among the k filter section B_0 that $b_0 = b_{10}b_{20}.....b_{k0}$. The second order sections are required to realize section which has complex-conjugate poles with real co-efficients. Pairing the two complex-conjugate poles with a pair of complex-conjugate zeros or real-valued zeros to form a subsystem of the type shown above is done arbitrarily. There is no specific rule used in the combination. Although all cascade realizations are equivalent for infinite precision arithmetic, the various realizations may differ significantly when implemented with finite precision arithmetic.

Parallel form structure

In the expression of transfer function, if $N \geq M$ we can express system function

$$H(Z) = C + \sum_{k=1}^{N} \frac{A_k}{1 - p_k Z^{-1}} \qquad = C + \sum_{k=1}^{N} H_k(Z)$$

Where $\{p_k\}$ are the poles, $\{A_k\}$ are the coefficients in the partial fraction expansion, and the constant C is defined as $C = b_N / a_N$, The system realization of above form is shown below.



Where $H_k(Z) = \dfrac{b_{k0} + b_{k1}Z^{-1}}{1 + a_{k1}Z^{-1} + a_{k2}Z^{-2}}$

Once again choice of using first- order or second-order sections depends on poles of the denominator polynomial. If there are complex set of poles which are conjugative in nature then a second order section is a must to have real coefficients.

Problem 2

Determine the

(i)Direct form-I (ii) Direct form-II (iii) Cascade &

(iv)Parallel form realization of the system function

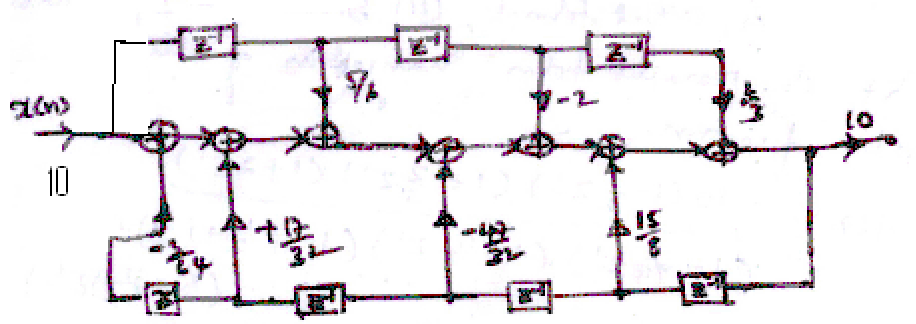
$$H(Z)=\frac{10\left(1-\frac{1}{2}Z^{-1}\right)\left(1-\frac{2}{3}Z^{-1}\right)\left(1+2Z^{-1}\right)}{\left(1-\frac{3}{4}Z^{-1}\right)\left(1-\frac{1}{8}Z^{-1}\right)\left(1-\left(\frac{1}{2}+j\frac{1}{2}\right)Z^{-1}\right)\left(1-\left(\frac{1}{2}-j\frac{1}{2}\right)Z^{-1}\right)}$$

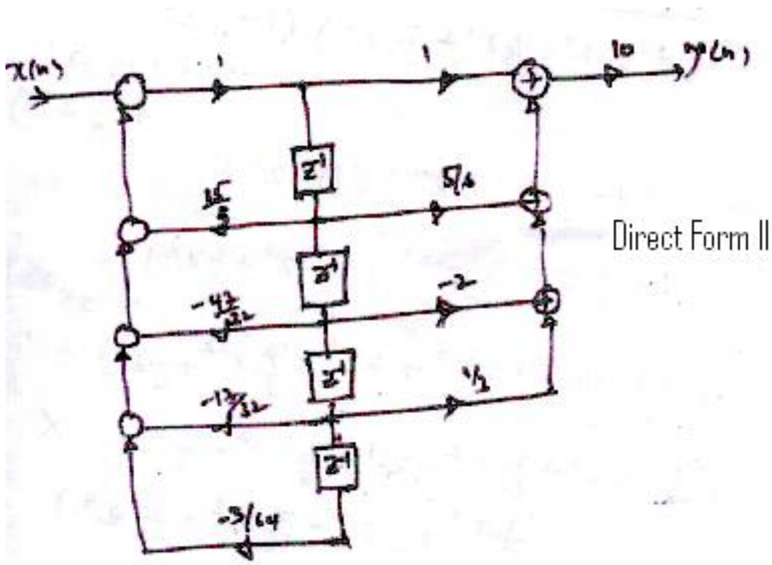
$$=\frac{10\left(1-\frac{7}{6}Z^{-1}+\frac{1}{3}Z^{-2}\right)\left(1+2Z^{-1}\right)}{\left(1+\frac{7}{8}Z^{-1}+\frac{3}{32}Z^{-2}\right)\left(1-Z^{-1}+\frac{1}{2}Z^{-2}\right)}$$

$$H(Z)=\frac{10\left(1+\frac{5}{6}Z^{-1}-2Z^{-2}+\frac{2}{3}Z^{-3}\right)}{\left(1-\frac{15}{8}Z^{-1}+\frac{47}{32}Z^{-2}-\frac{17}{32}Z^{-3}+\frac{3}{64}Z^{-4}\right)}$$

$$H(z)=\frac{(-14.75-12.90z^{-1})}{(1+\frac{7}{8}z^{-1}+\frac{3}{32}z^{-2})}+\frac{(24.50+26.82z^{-1})}{(1-z^{-1}+\frac{1}{2}z^{-2})}$$

Direct Form I

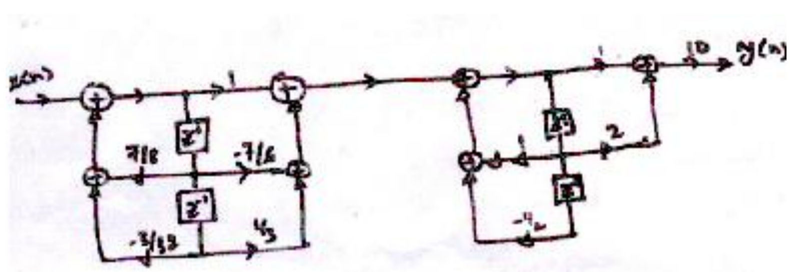




Cascade Form
H(z) = H_1(z) H_2(z)
Where

$$H_1(z) = \frac{1 - \frac{7}{6}z^{-1} + \frac{1}{3}z^{-2}}{1 - \frac{7}{8}z^{-1} + \frac{3}{32}z^{-2}}$$

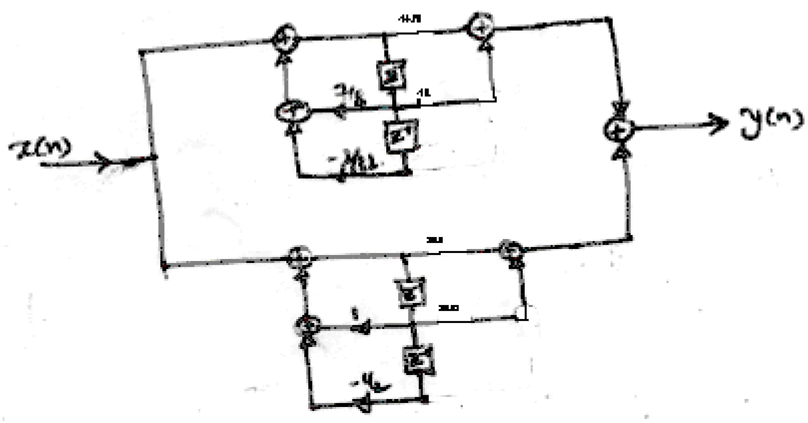
$$H_1(z) = \frac{10(1 + 2z^{-1})}{1 - z^{-1} + \frac{1}{2}z^{-2}}$$



Parallel Form

H(z) = $H_1(z) + H_2(z)$

$$H(z)=\frac{(-14.75-12.90z^{-1})}{(1+\frac{7}{8}z^{-1}+\frac{3}{32}z^{-2})}+\frac{(24.50+26.82z^{-1})}{(1-z^{-1}+\frac{1}{2}z^{-2})}$$



Problem: 3

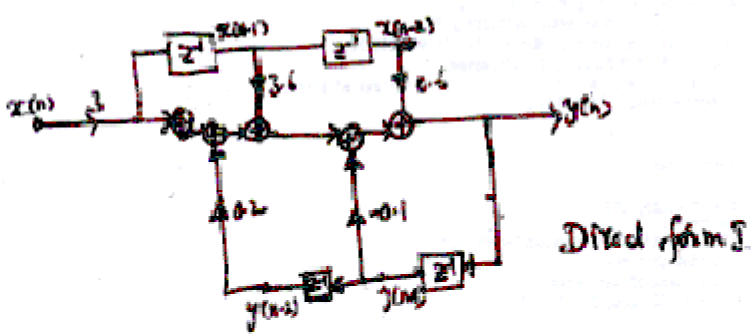
Obtain the direct form – I, direct form-II

Cascade and parallel form realization for the following system,

y(n)= -0.1 y(n-1)+0.2y(n-2)+3x(n)+3.6 x(n-1)+0.6 x(n-2)

Solution:

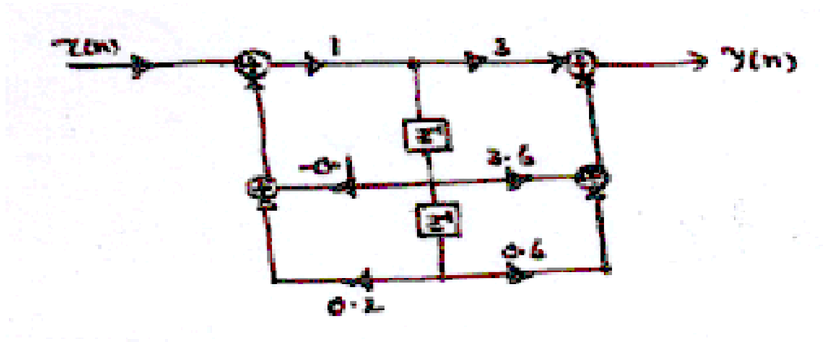
The Direct form realization is done directly from the given i/p – o/p equation, show in below diagram



Direct form –II realization

Taking ZT on both sides and finding H(z)

$$H(z)=\frac{Y(z)}{X(z)}=\frac{3+3.6z^{-1}+0.6z^{-2}}{1+0.1z^{-1}-0.2z^{-2}}$$

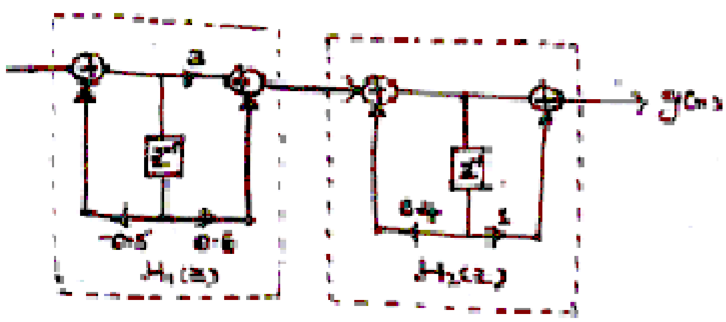


Cascade form realization
The transformer function can be expressed as:

$$H(z)=\frac{(3+0.6z^{-1})(1+z^{-1})}{(1+0.5z^{-1})(1-0.4z^{-1})}$$

which can be re written as

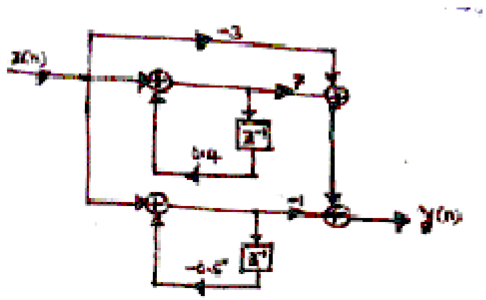
where $H_1(z)=\dfrac{3+0.6z^{-1}}{1+0.5z^{-1}}$ and $H_2(z)=\dfrac{1+z^{-1}}{1-0.4z^{-1}}$



Parallel Form realization

The transfer function can be expressed as
$H(z) = C + H_1(z) + H_2(z)$ where $H_1(z)$ & $H_2(z)$ is given by,

$$H(z)=-3+\frac{7}{1-0.4z^{-1}}-\frac{1}{1+0.5z^{-1}}$$



Lattice Structure for IIR System

Consider an All-pole system with system function.

$$H(Z) = \frac{1}{1+\sum_{k=1}^{N} a_N(k)Z^{-k}} = \frac{1}{A_N(Z)}$$

The corresponding difference equation for this IIR system is,

$$y(n) = -\sum_{k=1}^{N} a_N(k)y(n-k) + x(n)$$

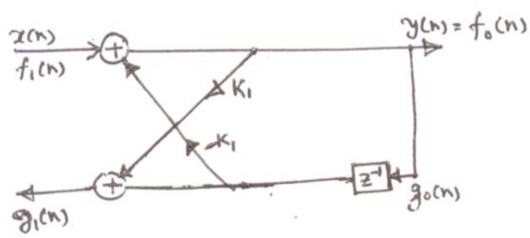
OR

$$x(n) = y(n) + \sum_{k=1}^{N} a_N(k)y(n-k)$$

For N=1

$$x(n) = y(n) + a_1(1)y(n-1)$$

Which can realized as,



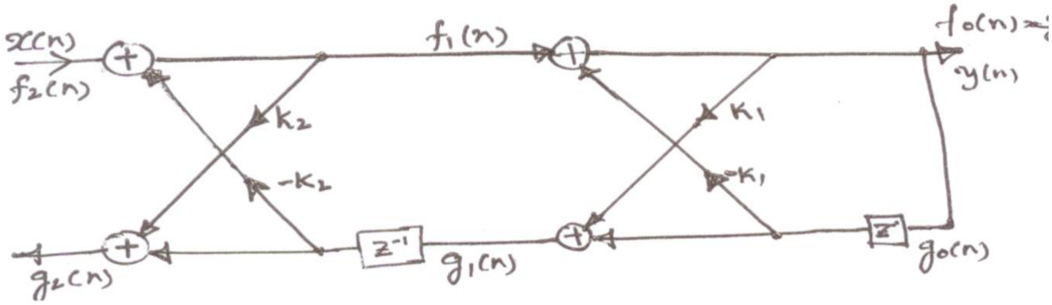
We observe

$$x(n) = f_1(n)$$
$$y(n) = f_0(n) = f_1(n) - k_1 g_0(n-1)$$
$$= x(n) - k_1 y(n-1) \qquad\qquad k_1 = a_1(1)$$
$$g_1(n) = k_1 f_0(n) + g_0(n-1) = k_1 y(n) + y(n-1)$$

For N=2, then

$$y(n) = x(n) - a_2(1)y(n-1) - a_2(2)y(n-2)$$

This output can be obtained from a two-stage lattice filter as shown in below fig



$$
\begin{aligned}
f_2(n) &= x(n) \\
f_1(n) &= f_2(n) - k_2 g_1(n-1) \\
g_2(n) &= k_2 f_1(n) + g_1(n-1) \\
f_0(n) &= f_1(n) - k_1 g_0(n-1) \\
g_1(n) &= k_1 f_0(n) + g_0(n-1)
\end{aligned}
$$

$$
\begin{aligned}
y(n) &= f_0(n) = g_0(n) = f_1(n) - k_1 g_0(n-1) \\
&= f_2(n) - k_2 g_1(n-1) - k_1 g_0(n-1) \\
&= f_2(n) - k_2\left[k_1 f_0(n-1) + g_0(n-2)\right] - k_1 g_0(n-1) \\
&= x(n) - k_2\left[k_1 y(n-1) + y(n-2)\right] - k_1 y(n-1) \\
&= x(n) - k_1(1+k_2) y(n-1) - k_2 y(n-2)
\end{aligned}
$$

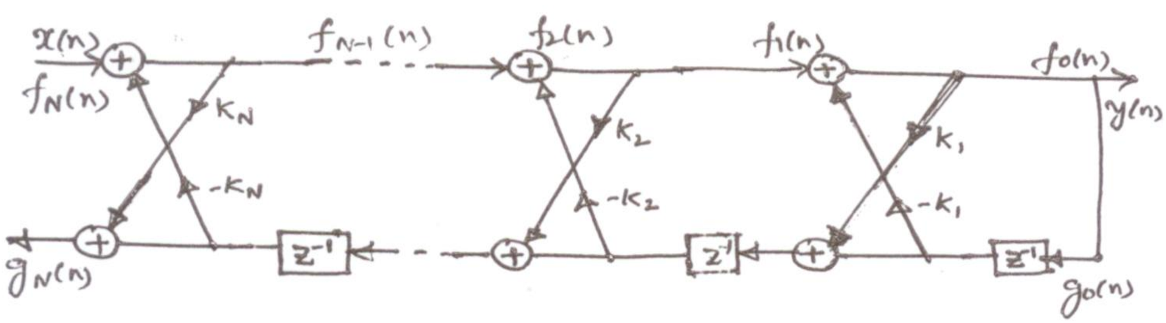
Similarly

$$g_2(n) = k_2 y(n) + k_1(1+k_2) y(n-1) + y(n-2)$$

We observe

$$a_2(0) = 1; a_2(1) = k_1(1+k_2); a_2(2) = k_2$$

N-stage IIR filter realized in lattice structure is,



$$
\begin{aligned}
f_N(n) &= x(n) \\
f_{m-1}(n) &= f_m(n) - k_m g_{m-1}(n-1) \qquad \text{m=N, N-1,---1} \\
g_m(n) &= k_m f_{m-1}(n) + g_{m-1}(n-1) \qquad \text{m=N, N-1,---1}
\end{aligned}
$$

$$y(n) = f_0(n) = g_0(n)$$

Conversion from lattice structure to direct form:

$$a_m(m) = k_m; \qquad a_m(0) = 1$$
$$a_m(k) = a_{m-1}(k) + a_m(m)a_{m-1}(m-k)$$

Conversion from direct form to lattice structure

$$a_{m-1}(0) = 1 \qquad\qquad k_m = a_m(m)$$

$$a_{m-1}(k) = \frac{a_m(k) - a_m(m)\, a_m(m-k)}{1 - a_m^2(m)}$$

Lattice – Ladder Structure:

A general IIR filter containing both poles and zeros can be realized using an all pole lattice as the basic building block.

If,

$$H(Z) = \frac{B_M(Z)}{A_N(Z)} = \frac{\sum_{k=0}^{M} b_M(k)Z^{-k}}{1 + \sum_{k=1}^{N} a_N(k)Z^{-k}}$$

Where $N \geq M$

A lattice structure can be constructed by first realizing an all-pole lattice co-efficients $k_m, \;\; 1 \leq m \leq N$ for the denominator $A_N(Z)$, and then adding a ladder part for M=N. The output of the ladder part can be expressed as a weighted linear combination of $\{g_m(n)\}$.

Now the output is given by

$$y(n) = \sum_{m=0}^{M} C_m g_m(n)$$

Where $\{C_m\}$ are called the ladder co-efficient and can be obtained using the recursive relation,

$$C_m = b_m - \sum_{i=m+1}^{M} C_i a_i(i-m); \qquad\qquad m=M, M-1, \ldots.0$$

4. Convert the following pole-zero IIR filter into a lattice ladder structure,

$$H(Z)=\frac{1+2Z^{-1}+2Z^{-2}+Z^{-3}}{1+\frac{13}{24}Z^{-1}+\frac{5}{8}Z^{-2}+\frac{1}{3}Z^{-3}}$$

Solution:

Given $b_M(Z)=1+2Z^{-1}+2Z^{-2}+Z^{-3}$

And $A_N(Z)=1+\frac{13}{24}Z^{-1}+\frac{5}{8}Z^{-2}+\frac{1}{3}Z^{-3}$

$a_3(0)=1; \quad a_3(1)=\frac{13}{24}; \quad a_3(2)=\frac{5}{8}; \quad a_3(3)=\frac{1}{3}$

$$k_3=a_3(3)=\tfrac{1}{3}$$

Using the equation

$$a_{m-1}(k)=\frac{a_m(k)-a_m(m)a_m(m-k)}{1-a^2m(m)}$$

for m=3, k=1

$$a_2(1)=\frac{a_3(1)-a_3(3)a_3(2)}{1-a_3^2(3)}=\frac{\frac{13}{24}-\frac{1}{3}\cdot\frac{5}{8}}{1-\left(\frac{1}{3}\right)^2}=\frac{3}{8}$$

for m=3, & k=2

$$a_2(2)=k_2=\frac{a_3(2)-a_3(3)a_3(1)}{1-a_3^2(3)}$$

$$\frac{\frac{5}{8}-\frac{1}{3}\cdot\frac{13}{24}}{1-\frac{1}{9}}=\frac{\frac{45-13}{72}}{\frac{8}{9}}=\frac{1}{2}$$

for m=2, & k=1

$$a_1(1)=k_1=\frac{a_2(1)-a_2(2)a_2(1)}{1-a_2^2(2)}$$

$$\frac{\frac{3}{8}-\frac{1}{2}\cdot\frac{3}{8}}{1-(\frac{1}{2})^2}=\frac{\frac{3}{8}-\frac{3}{16}}{1-\frac{1}{4}}=\frac{1}{4}$$

for lattice structure $k_1=\frac{1}{4}, \quad k_2=\frac{1}{2}, \quad k_3=\frac{1}{3}$

For ladder structure

$$C_m=b_m-\sum_{i=m+1}^{M}C_1.a_1(1-m) \qquad \text{m=M, M-1,1,0}$$

M=3 $\quad C_3=b_3=1; \quad C_2=b_2-C_3a_3(1)$

$$=2-1.(\tfrac{13}{24})=1.4583$$

$$C_1=b_1-\sum_{i=2}^{3}c_1a_1(i-m) \qquad \text{m=1}$$

$$=b_1-\left[c_2a_2(1)+c_3a_{3(2)}\right]$$

$$=2-\left[(1.4583)(\tfrac{3}{8})+\tfrac{5}{8}\right]=0.8281$$

$$c_0=b_0-\sum_{i=1}^{3}c_1a_1(i-m)$$

$$= b_0 - [c_1 a_1(1) + c_2 a_2(2) + c_3 a_3(3)]$$
$$= 1 - [08281(\tfrac{1}{4}) + 1.4583(\tfrac{1}{2}) + \tfrac{1}{3}] = -02695$$

To convert a lattice- ladder form into a direct form, we find an equation to obtain $a_N(k)$ from k_m (m=1,2,………N) then equation for c_m is recursively used to compute b_m (m=0,1,2,………M).

6. Consider a FIR filter with system function:
H(z) = 1+2.82 Z^{-1} +3.4048z-^{2} +1.74z^{-3}. Sketch the direct form and lattice realizations of the filter.

Sol. : $A_3(z) = H(z) = 1 + 2.82z^{-1} + 3.4048z^{-2} + 1.74z^{-3}$

$$B_3(z) = 1.74 + 3.4048z^{-1} + 2.82z^{-2} + z^{-3}$$

Hence $k_3 = 1.74$

$$A_2(z) = \frac{A_3(z) - k_3 B_3(z)}{1 - k_4^2}$$

$$= \frac{1 + 2.82z^{-1} + 3.4048z^{-2} + 1.74z^{-3} - 3.0276 - 5.9243z^{-1} - 4.9068z^{-2} - 1.74z^{-3}}{(-2.0276)}$$

$$= \frac{-2.0276 - 3.1043z^{-1} - 1.502z^{-2}}{(-2.0276)} = 1 + 1.531z^{-1} + 0.7407z^{-2}$$

$$B_2(z) = 0.7407 + 1.531z^{-1} + z^{-2}$$

$$k_2 = 0.7407$$

$$A_1(z) = \frac{A_2(z) - k_2 B_2(z)}{1 - k_2^2}$$

$$= \frac{1 + 1.531z^{-1} + 0.7407z^{-2} - 0.5486 - 1.134z^{-1} - 0.7407z^{-2}}{0.4514}$$

$$= 1 + 0.8795z^{-1}$$

$$k_1 = 0.8795$$

Direct form realization :

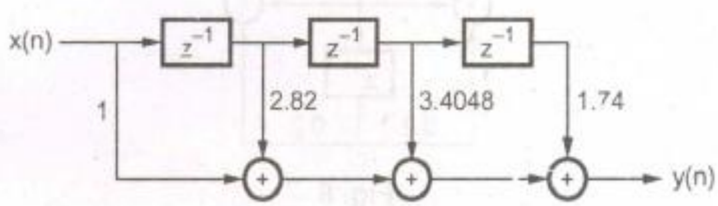


Fig. 5

the two important forms of expressing system leading to different realizations of FIR & IIR filters are

c) Difference equation form

$$y(n) = -\sum_{k=1}^{N} a_k y(n-k) + \sum_{k=1}^{M} b_k x(n-k)$$

d) Ration of polynomials

$$H(Z) = \frac{\sum_{k=0}^{M} b_k Z^{-k}}{1 + \sum_{k=1}^{N} a_k Z^{-k}}$$

Exercise: Realize the following using system function using minimum number of multiplication.

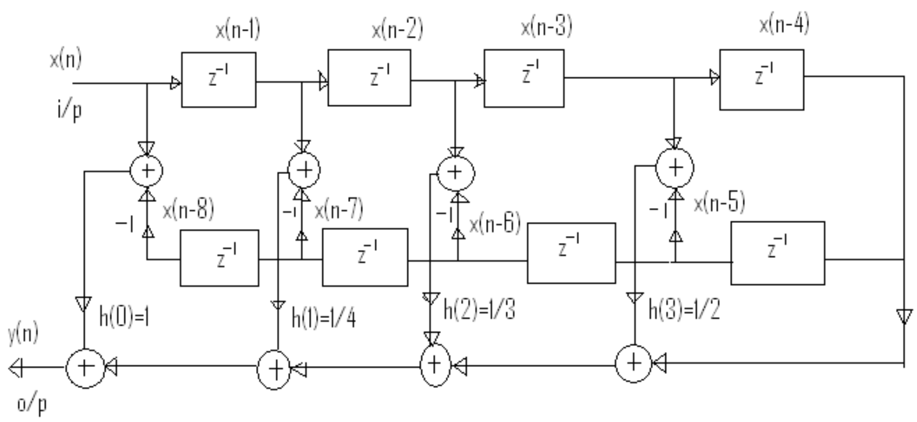
$$H(Z) = 1 + \frac{1}{4}Z^{-1} + \frac{1}{3}Z^{-2} + \frac{1}{2}Z^{-3} - \frac{1}{2}Z^{-5} - \frac{1}{3}Z^{-6} - \frac{1}{4}Z^{-7} - Z^{-8}$$

m=9 $\qquad h(n) = \left[1, \frac{1}{4}, \frac{1}{3}, \frac{1}{2}, -\frac{1}{2}, -\frac{1}{3}, -\frac{1}{4}, -1\right]$

odd symmetry

h(n) = -h(M-1-n); h(n) = -h(8-n); h(m-1/2) = h(5) = 0

h(0) = -h(8); h(1) = -h(7); h(2) = -h(6); h(3) = -h(5)



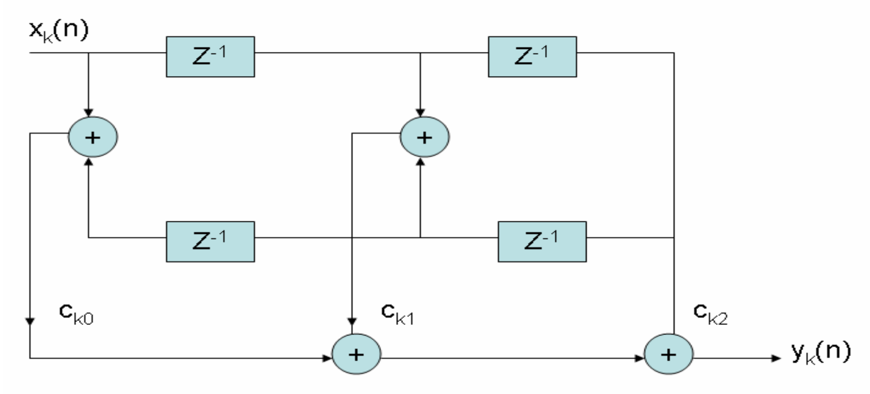
Cascade – Form Structure

The system function H(Z) is factored into product of second – order FIR system

$$H(Z) = \prod_{k=1}^{K} H_k(Z)$$

Where $H_k(Z) = b_{k0} + b_{k1}Z^{-1} + b_{k2}Z^{-2}$ k = 1, 2, ….. K

and K = integer part of (M+1) / 2



Problem: Realize the difference equation

$$y(n) = x(n) + 0.25x(n-1) + 0.5x(n-2) + 0.75x(n-3) + x(n-4)$$

in cascade form.

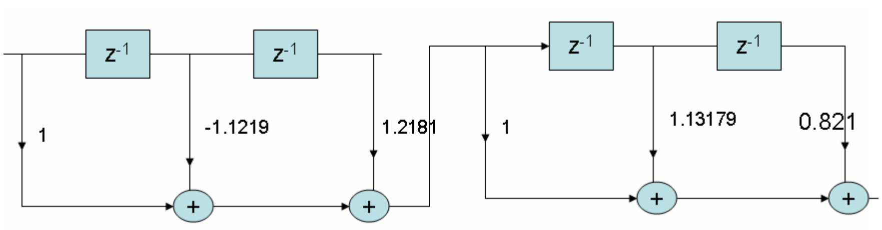
Soln:

$$Y(z) = X(z)\{1 + 0.25z^{-1} + 0.5z^{-2} + 0.75z^{-3} + z^{-4})$$

$$H(z) = 1 + 0.25z^{-1} + 0.5z^{-2} + 0.75z^{-3} + z^{-4}$$

$$H(z) = (1 - 1.1219z^{-1} + 1.2181z^{-2})(1 + 1.3719z^{-1} + 0.821z^{-2})$$

$$H(z) = H_1(z)H_2(z)$$



Lattice structures

Lattice structures offer many interesting features:

4. Upgrading filter orders is simple. Only additional stages need to be added instead of redesigning the whole filter and recalculating the filter coefficients.

5. These filters are computationally very efficient than other filter structures in a filter bank applications (eg. Wavelet Transform)
6. Lattice filters are less sensitive to finite word length effects.

Consider

$$H(z) = \frac{Y(z)}{X(z)} = 1 + \sum_{i=1}^{m} a_m(i) z^{-i}$$

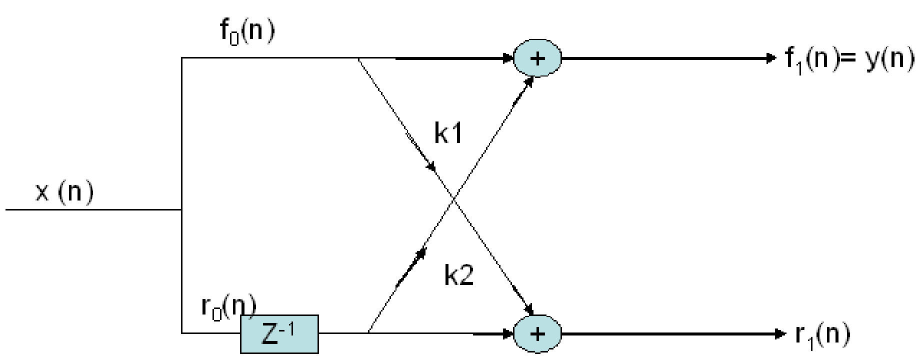
m is the order of the FIR filter and $a_m(0)=1$

when m = 1   Y(z)/ X(z) = 1+ $a_1(1)\, z^{-1}$

y(n)= x(n)+ $a_1(1)$x(n-1)

f1(n) is known as upper channel output and r1(n)as lower channel output.

$f_0(n) = r_0(n) = x(n)$



The outputs are

$$f_1(n) = f_0(n) + k_1 r_0(n-1) \qquad 1a$$
$$r_1(n) = k_1 f_0(n) + r_0(n-1) \qquad 1b$$
$$if \quad k_1 = a_1(1), then \quad f_1(n) = y(n)$$

If m=2

$$\frac{Y(z)}{X(z)} = 1 + a_2(1) z^{-1} + a_2(2) z^{-2}$$
$$y(n) = x(n) + a_2(1) x(n-1) + a_2(2) x(n-2)$$
$$y(n) = f_1(n) + k_2 r_1(n-1) \qquad (2)$$

Substituting 1a and 1b in (2)

$$
\begin{aligned}
y(n) &= f_0(n) + k_1 r_0(n-1) + k_2[k_1 f_0(n-1) + r_0(n-2)] \\
&= f_0(n) + k_1 r_0(n-1) + k_2 k_1 f_0(n-1) + k_2 r_0(n-2)] \\
\sin ce \quad & f_0(n) = r_0(n) = x(n) \\
y(n) &= x(n) + k_1 x(n-1) + k_2 k_1 x(n-1) + k_2 x(n-2)] \\
&= x(n) + (k_1 + k_1 k_2) x(n-1) + k_2 x(n-2)
\end{aligned}
$$

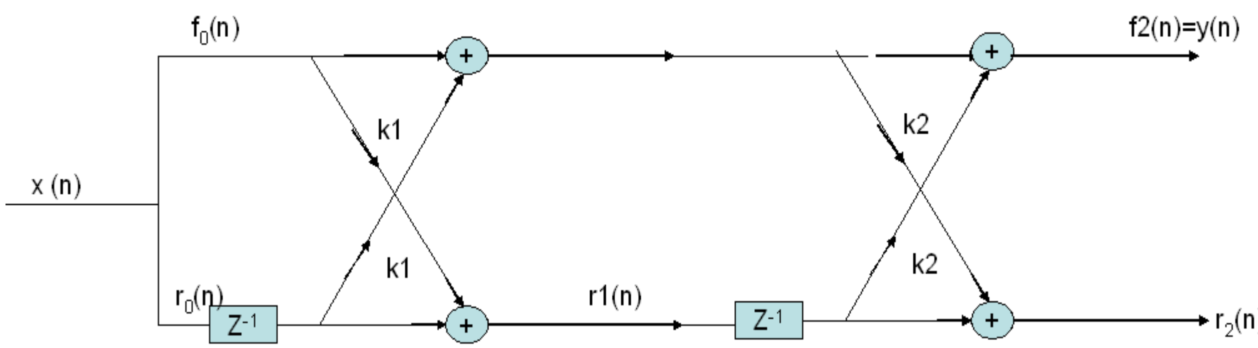
We recognize

$$
\begin{aligned}
a_2(1) &= k_1 + k_1 k_2 \\
a_2(1) &= k_2
\end{aligned}
$$

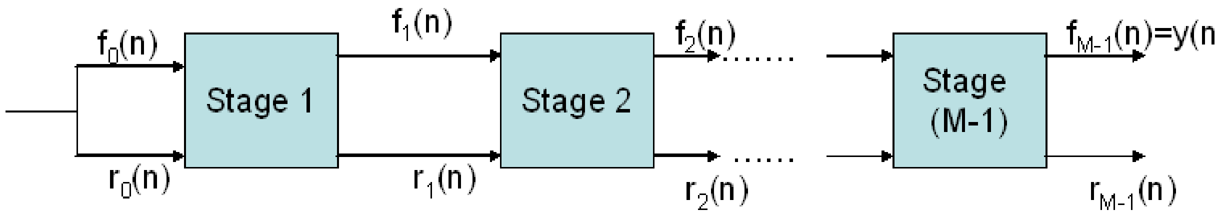
Solving the above equation we get

$$k_1 = \frac{a_2(1)}{1 + a_2(2)} \quad and \quad k_2 = a_2(2) \qquad (4)$$

Equation (3) means that, the lattice structure for a second-order filter is simply a cascade of two first-order filters with k1 and k2 as defined in eq (5)



Similar to above, an Mth order FIR filter can be implemented by lattice structures with M – stages



Direct Form –I to lattice structure

For m = M, M-1, ………..2, 1 do

$$k_m = a_m(m)$$

$$a_{m-1}(i) = \frac{a_m(i) - a_m(m)a_m(m-i)}{1-k_m^2} \qquad 1 \le i \le m-1$$

- The above expression fails if k_m=1. This is an indication that there isa zero on the unit circle. If k_m=1, factor out this root from A(z) and the recursive formula can be applied for reduced order system.

$$for\ m = 2\ and\ m = 1$$

$$k_2 = a_2(2) \quad \& \quad k_1 = a_1(1)$$

$$for\ m = 2\ \&\ i = 1$$

$$a_1(1) = \frac{a_2(1) - a_2(2)a_2(1)}{1-k_2^2} = \frac{a_2(1)[1-a_2(2)]}{1-a_2^2(2)} = \frac{a_2(1)}{1+a_2(2)}$$

$$Thus \quad k_1 = \frac{a_2(1)}{1+a_2(2)}$$

Lattice to direct form –I

For m = 1,2,…….M-1

$$a_m(0) = 1$$

$$a_m(m) = k_m$$

$$a_m(i) = a_{m-1}(i) + a_m(m)a_{m-1}(m-i) \quad 1 \le i \le m-1$$

Further Readings

1. nptel.ac.in/video.php?subjectId=117102060
2. www.journals.elsevier.com/digital-signalprocessing
3. www.dspguide.com/whatdsp.htm