ATME COLLEGE OF ENGINEERING

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DEPARTMENT OF ELECTRICAL & ELECTRONICS ENGINEERING

NOTES

COURSE TITLE: SIGNALS & DIGITAL SIGNAL PROCESSING

COURSE CODE: BEE502

SEMESTER: V

MODULE-4: DESIGN OF IIR DIGITAL FILTERS

INSTITUTIONAL VISION AND MISSION

VISION:

- Development of academically excellent, culturally vibrant, socially responsible, and globally competent human resources.

MISSION:

- To keep pace with advancements in knowledge and make the students competitive and capable at the global level.
- To create an environment for the students to acquire the right physical, intellectual, emotional, and moral foundations and shine as torchbearers of tomorrow's society.
- To strive to attain ever-higher benchmarks of educational excellence.

Department Vision and Mission

Vision:

To produce Electrical & Electronics Engineers through greatest quality of technical education, technical skill training and intellectual capacity building of individuals.

Mission:

- To provide knowledge to students that builds a strong foundation in the basic principles of electrical engineering, problem solving abilities, analytical skills, soft skills and communication skills for their overall development.
- To offer outcome based technical education.
- To encourage faculty in training & development and to offer consultancy through research & industry interaction.

MODULE- 4: Design of Digital IIR Filters

Structure

4.0 OBJECTIVES

1. Students will design the IIR Digital filter using Impulse invarience method
2. Students will design the IIR Digital filter using Bilinear transformation

4.1 Introduction

A digital filter is a linear shift-invariant discrete-time system that is realized using finite Precision arithmetic. The design of digital filters involves three basic steps:

- The specification of the desired properties of the system.
- The approximation of these specifications using a causal discrete-time system.
- The realization of these specifications using _nite precision arithmetic.

These three steps are independent; here we focus our attention on the second step. The desired digital filter is to be used to filter a digital signal that is derived from an analog signal by means of periodic sampling. The speci_cations for both analog and digital filters are often given in the frequency domain, as for example in the design of low pass, high pass, band pass and band elimination filters. Given the sampling rate, it is straight forward to convert from frequency specifications on an analog filter to frequency speci_cations on the corresponding digital filter, the analog frequencies being in terms of Hertz and digital frequencies being in terms of radian frequency or angle around the unit circle with the point Z=-1 corresponding to half the sampling frequency. The least confusing point of view toward digital filter design is to consider the filter as being specified in terms of angle around the unit circle rather than in terms of analog frequencies.

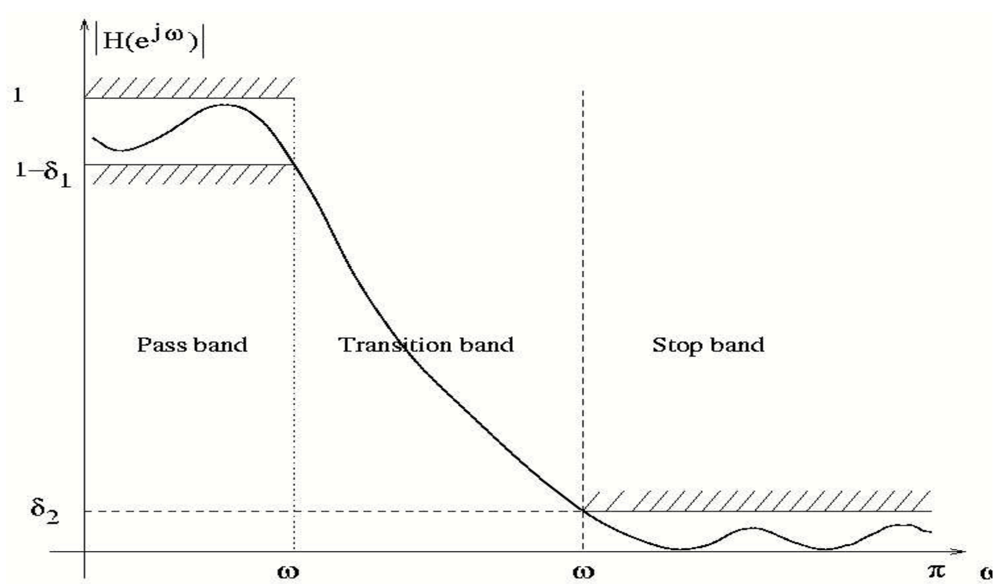

Figure 6.1: Tolerance limits for approximation of ideal low-pass filter

A separate problem is that of determining an appropriate set of specifications on the digital filter. In the case of a low pass filter, for example, the specifications often take the
form of a tolerance scheme, as shown in Fig. 4.1

$$1 - \delta_1 \leq | H(e^{j\omega}) | \leq 1, \quad | \omega | \leq \omega_p$$

$$| H(e^{j\omega}) | \leq \delta_2, \quad \omega_s \leq | \omega | \leq \pi$$

Many of the filters used in practice are specified by such a tolerance scheme, with no constraints on the phase response other than those imposed by stability and causality requirements; i.e., the poles of the system function must lie inside the unit circle. Given a set of specifications in the form of Fig. 7.1, the next step is to and a discrete time linear system whose frequency response falls within the prescribed tolerances. At this point the filter design problem becomes a problem in approximation. In the case of infinite impulse response (IIR) filters, we must approximate the desired frequency response by a rational function, while in the finite impulse response (FIR) filters case we are concerned with polynomial approximation.

4.2 Design of IIR Filters from Analog Filters

The traditional approach to the design of IIR digital filters involves the transformation of an analog filter into a digital filter meeting prescribed specifications. This is a reasonable approach because:

- The art of analog filter design is highly advanced and since useful results can be achieved, it is advantageous to utilize the design procedures already developed for analog filters.
- Many useful analog design methods have relatively simple closed-form design formulas.

Therefore, digital filter design methods based on analog design formulas are rather simple to implement.
An analog system can be described by the differential equation

$$\sum_{k=0}^{N} c_k \frac{d^k y_a(t)}{dt^k} = \sum_{k=0}^{M} d_k \frac{d^k x_a(t)}{dt^k}$$ --6.1

And the corresponding rational function is

$$H_a(s) = \frac{\sum_{k=0}^{M} d_k s^k}{\sum_{k=0}^{N} c_k s^k} = \frac{y_a(s)}{x_a(s)}$$

--6.2

The corresponding description for digital filters has the form

$$\sum_{k=0}^{N} a_k y(n-k) = \sum_{k=0}^{M} b_k x(n-k)$$

---6.3

and the rational function

$$H(z) = \frac{\sum_{k=0}^{M} b_k z^{-k}}{\sum_{k=0}^{N} a_k z^{-k}} = \frac{Y(z)}{X(z)}$$

---6.4

In transforming an analog filter to a digital filter we must therefore obtain either H(z)or h(n) (inverse Z-transform of H(z) i.e., impulse response) from the analog filter design. In such transformations, we want the imaginary axis of the S-plane to map into the finite circle of the Z-plane, a stable analog filter should be transformed to a stable digital filter. That is, if the analog filter has poles only in the left-half of S-plane, then the digital filter must have poles only inside the unit circle. These constraints are basic to all the techniques discussed

4.3 IIR Filter Design by Impulse Invariance

This technique of transforming an analog filter design to a digital filter design corresponds to choosing the unit-sample response of the digital filter as equally spaced samples of the impulse response of the analog filter. That is,

$$h(n) = h_a(nT)$$

---6.5

Where T is the sampling period. Because of uniform sampling, we have

$$H(e^{j\omega T}) = \frac{1}{T} \sum_{k=-\infty}^{\infty} H_a(j\Omega + j\frac{2\pi}{T}k)$$

--6.6

Or

$$H(z)\mid_{z=e^{sT}} = \frac{1}{T} \sum_{k=-\infty}^{\infty} H_a(s + j\frac{2\pi}{T}k)$$

--6.7

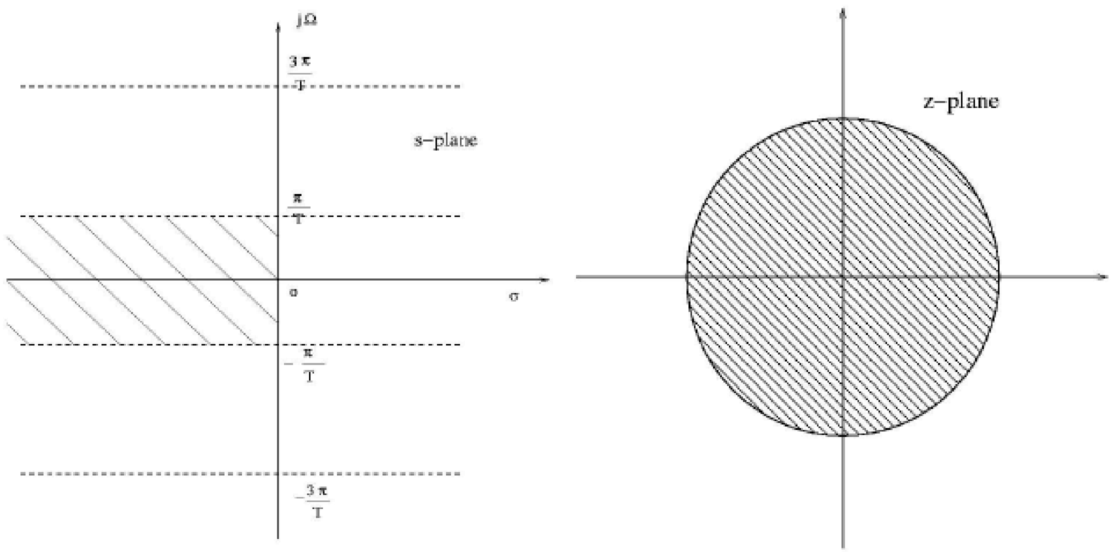

Figure 6.2: Mapping of s-plane into z-plane

Where s = jω and Ω=ω/T, is the frequency in analog domain and ω is the frequency in digital domain.
From the relationship Z = e^{ST} it is seen that strips of width 2π/T in the S-plane map into the entire Z-plane as shown in Fig. 7.2. The left half of each S-plane strip maps into interior of the unit circle, the right half of each S-plane strip maps into the exterior of the unit circle, and the imaginary axis of length 2π/T of S-plane maps on to once round the unit circle of Z-plane. Each horizontal strip of the S-plane is overlaid onto the Z-plane to form the digital filter function from analog filter function. The frequency response of the digital filter is related to the frequency response of the

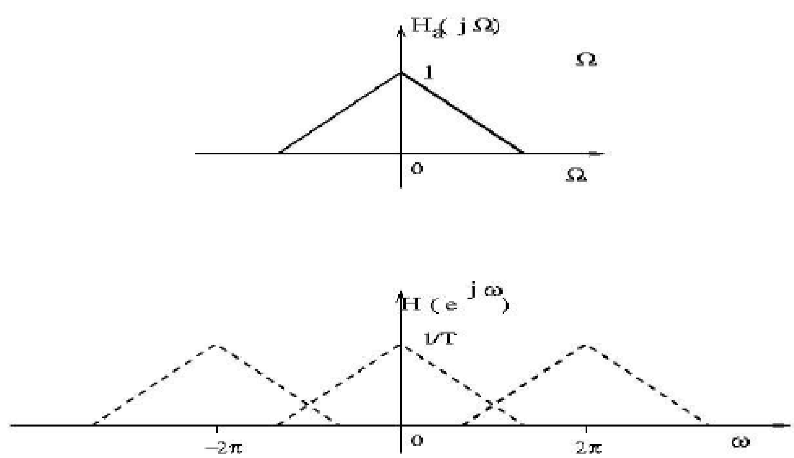

Figure 6.3: Illustration of the effects of aliasing in the impulse invariance techniqueanalog filter as

$$H(e^{j\omega}) = \frac{1}{T}\sum_{k=-\infty}^{\infty} H_a(j\frac{\omega}{T} + j\frac{2\pi}{T}k)$$ --6.8

From the discussion of the sampling theorem it is clear that if and only if

$$H_a(j\Omega) = 0, \quad |\,\Omega\,| \geq \frac{\pi}{T}$$

Then

$$H(e^{j\omega}) = \frac{1}{T}H_a(j\frac{\omega}{T}), \quad |\,\omega\,| \leq \pi$$

Unfortunately, any practical analog filter will not be band limited, and consequently there is interference between successive terms in Eq. (7.8) as illustrated in Fig. 7.3. Because of the aliasing that occurs in the sampling process, the frequency response of the resulting digital filter will not be identical to the original analog frequency response. To get the filter design procedure, let us consider the system function of the analog filter expressed in terms of a partial-fraction expansion

$$H_a(s) = \sum_{k=1}^{N} \frac{A_k}{s - s_k}$$ ---6.9

The corresponding impulse response is

$$h_a(t) = \sum_{k=1}^{N} A_k e^{s_k t} U(t)$$ --- 6.10

And the unit-sample response of the digital filter is then

$$h(n) = h_a(nT) = \sum_{k=1}^{N} A_k e^{s_k nT} u(n) = \sum_{k=1}^{N} A_k (e^{s_k T})^n U(n)$$ --------------6.11

The system function of the digital filter H(z) is given by

$$H(z) = \sum_{k=1}^{N} \frac{A_k}{(1 - \exp^{s_k T} z^{-1})}$$ --6.12

In comparing Eqs. (7.9) and (7.12) we observe that a pole at s=sk in the S-plane transforms to a pole at $\exp^{skT}$ in the Z-plane. It is important to recognize that the impulse invariant design procedure does not correspond to a mapping of the S-plane to the Z-plane.

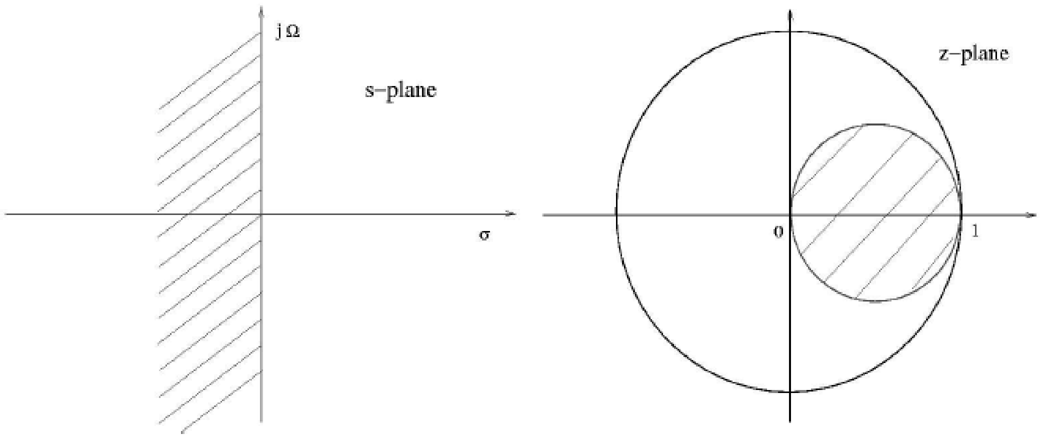

Figure 6.4: Mapping of s-plane to z-plane corresponding to first backward-difference approximation to the derivative

In contrast to the impulse invariance technique, decreasing the sampling period T, theoretically produces a better filter since the spectrum tends to be concentrated in a very small region of the unit circle. These two procedures are highly unsatisfactory for anything but low pass filters. An alternative approximation to the derivative is a forward difference and it provides a mapping into the unstable digital filters.

4.5 IIR Filter Design By The Bilinear Transformation

In the previous section a digital filter was derived by approximating derivatives by differences. An alternative procedure is based on integrating the differential equation and then using a numerical approximation to the integral. Consider the first - order equation

$$c_1 y_a^{'}(t) + c_0 y_a(t) = d_0 x_a(t)$$

--6.20

Where y'$_a$(t) is the first derivative of ya(t). The corresponding analog system function is

$$H_a(s) = \frac{d_0}{c_0 + c_1 s}$$

We can write ya(t) as an integral of y'$_a$(t), as in

$$y_a(t) = \int_{t_0}^{t} y_a^{'}(t)dt + y_a(t_0)$$

In particular, if t = nT and t_0 = (n - 1)T,

$$y_a(nT) = \int_{(n-1)T}^{nT} y_a'(\tau)d\tau + y_a((n-1)T)$$

If this integral is approximated by a trapezoidal rule, we can write

$$y_a(nT) = y_a((n-1)T) + \frac{T}{2}[y_a'(nT) + y_a'((n-1)T)]$$ ----------------------6.21

However, from Eq. (7.20),

$$y_a'(nT) = -\frac{c_0}{c_1}y_a(nT) + \frac{d_0}{c_1}x_a(nT)$$

Substituting into Eq. (4.21) we obtain

$$[y(n) - y(n-1)] = \frac{T}{2}[-\frac{c_0}{c_1}(y(n) + y(n-1)) + \frac{d_0}{c_1}(x(n) + x(n-1))]$$

Where y(n) = y(nT) and x(n) = x(nT). Taking the Z-transform and solving for H(z) gives

$$H(z) = \frac{Y(z)}{X(z)} = \frac{d_0}{c_0 + c_1\frac{2}{T}\frac{1-z^{-1}}{1+z^{-1}}}$$ --6.22

From Eq. (7.22) it is clear that H(z) is obtained from Ha(s) by the substitution

$$s = \frac{2}{T}\frac{1-z^{-1}}{1+z^{-1}}$$ ---6.23

That is,

$$H(z) = H_a(s)\,|_{s=\frac{2}{T}\frac{1-z^{-1}}{1+z^{-1}}}$$ --6.24

This can be shown to hold in general since an N^{th} - order differential equation of the form of Eq. (6.1) can be written as a set of N first-order equations of the form of Eq. (6.20). Solving Eq. (6.23) for z gives

$$z = \frac{1 + \frac{T}{2}s}{1 - \frac{T}{2}s}$$ --6.25

The invertible transformation of Eq. (7.23) is recognized as a bilinear transformation. To see that this mapping has the property that the imaginary axis in the s-plane maps onto the unit circle in the z-plane, consider $z = e^{j\omega}$, then from Eq. (7.23), s is given by

$$\begin{aligned} s &= \frac{2}{T}\frac{1 - e^{-j\omega}}{1 + e^{-j\omega}} \\ &= \frac{2}{T}\frac{j\sin(\omega/2)}{\cos(\omega/2)} \\ &= \frac{2}{T}j\tan(\omega/2) \\ &= \sigma + j\Omega \end{aligned}$$

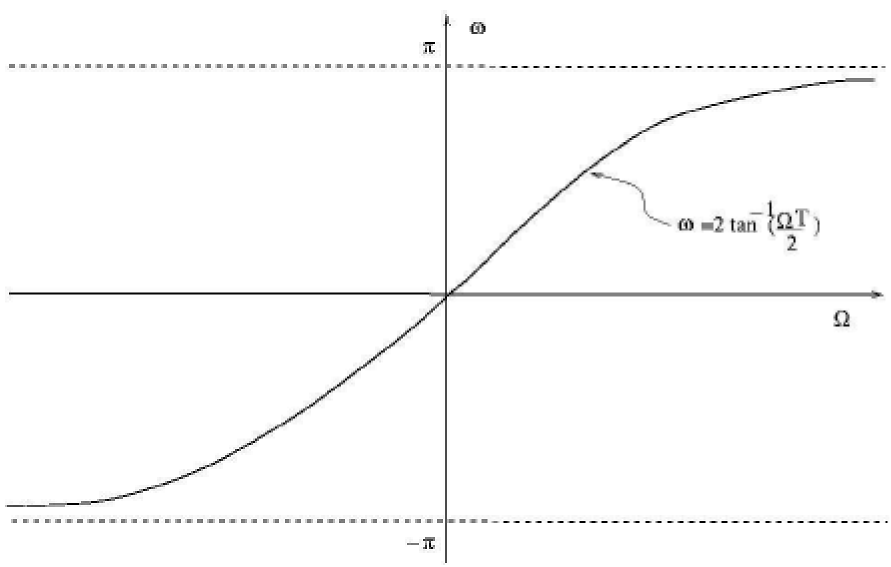

Figure 6.5: Mapping of analog frequency axis onto the unit circle using the bilinear Transformation

Thus for z on the unit circle, $\sigma = 0$ and Ω and ω are related by

$$T\,\Omega/2 = \tan(\omega/2)$$
or
$$\omega = 2\tan^{-1}(T\,\Omega/2)$$

This relationship is plotted in Fig. (6.5), and it is referred as frequency warping. From the _gure it is clear that the positive and negative imaginary axis of the s-plane are mapped, respectively, into the upper and lower halves of the unit circle in the z-plane. In addition to the fact that the imaginary axis in the s-plane maps into the unit circle in the z-plane, the left half of the s-plane maps to the inside of the unit circle and the right half of the s-plane maps to the outside of the unit circle, as shown in Fig. (6.6). Thus we see that the use of the bilinear transformation yields stable digital filter from analog filter. Also this transformation avoids the problem of aliasing encountered with the use of impulse invariance, because it maps the entire imaginary axis in the s-plane onto the unit circle in the z-plane. The price paid for this, however, is the introduction of a distortion in the frequency axis.

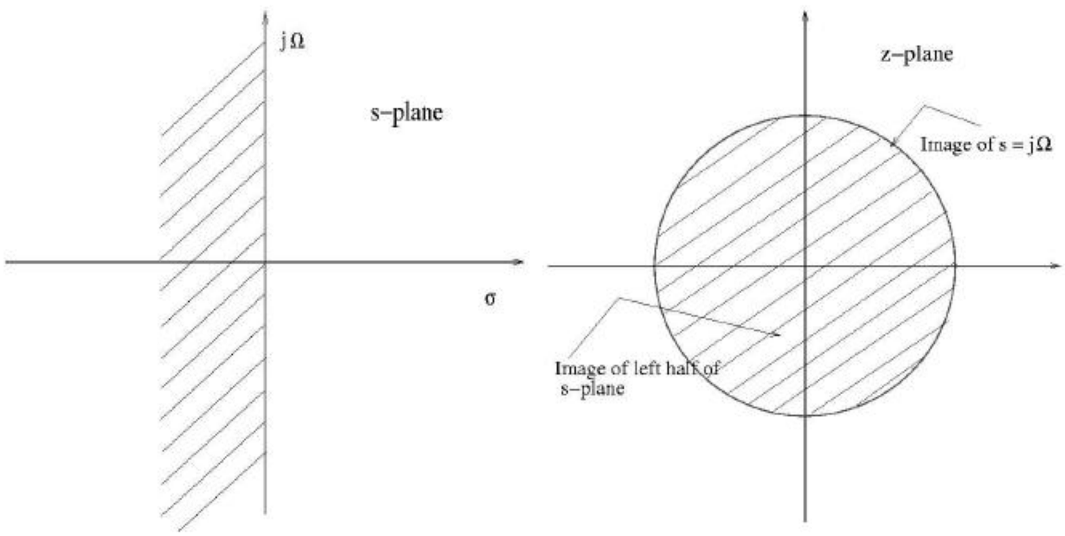

Figure 6.6: Mapping of the s-plane into the z-plane using the bilinear transformation

Recommended questions with solution

Q1

Design a digital band pass filter from a 2^{nd} order analog low pass Butterworth prototype filter using bilinear transformation. The lower and upper cut-off frequencies for band pass filter are $5\pi/12$ and $7\pi/12$. Assume $T = 2$ sec. **[12]**

Sol. :

$$\omega_l = \frac{5\pi}{12}$$

$$\Omega_l = \frac{2}{T}\tan\frac{\omega_l}{2}$$

For

$$T = 2$$

$$\Omega_l = \tan\frac{\omega_l}{2}$$

$$\omega_u = \frac{7\pi}{12}$$

$\therefore$

$$\Omega_u = \tan\frac{\omega_u}{2}$$

Analog low pass to band pass

$$s \rightarrow \frac{s^2 + \Omega_l \Omega_u}{s(\Omega_u - \Omega_l)} \quad \text{... (1)}$$

Analog prototype is,

$$H(s) = \frac{1}{s^2 + \sqrt{2s} + 1} \quad \text{... (2)}$$

Putting equation (1) in equation (2) and then to get bilinear analog to digital

$$s \rightarrow \frac{z-1}{z+1} = \frac{1-z^{-1}}{1+z^{-1}}$$

Combining above two steps we get

$$s \rightarrow \frac{\left(\frac{1-z^{-1}}{1+z^{-1}}\right)^2 + \Omega_l \Omega_u}{\left(\frac{1-z^{-1}}{1+z^{-1}}\right)(\Omega_u - \Omega_l)} = \frac{(1-z^{-1})^2 + \Omega_l \Omega_u (1+z^{-1})^2}{(1-z^{-2})(\Omega_u - \Omega_l)}$$

$$\therefore \quad H(z) = \frac{1}{\left[\frac{(1-z^{-1})^2 + \Omega_l\Omega_u(1+z^{-1})^2}{(1-z^{-2})(\Omega_u-\Omega_l)}\right]^2 + \sqrt{2}\left[\frac{(1-z^{-1})^2 + \Omega_l\Omega_u(1+z^{-1})^2}{(1-z^{-2})(\Omega_u-\Omega_l)}\right] + 1}$$

Q2

Show that the bilinear transformation maps.

i) The $j\Omega$ axis in s-plane onto the unit circle, $|z| = 1$.

ii) The left half s-plane, Re (s) < 0 inside the unit circle, $|z < 1|$.

Sol. :

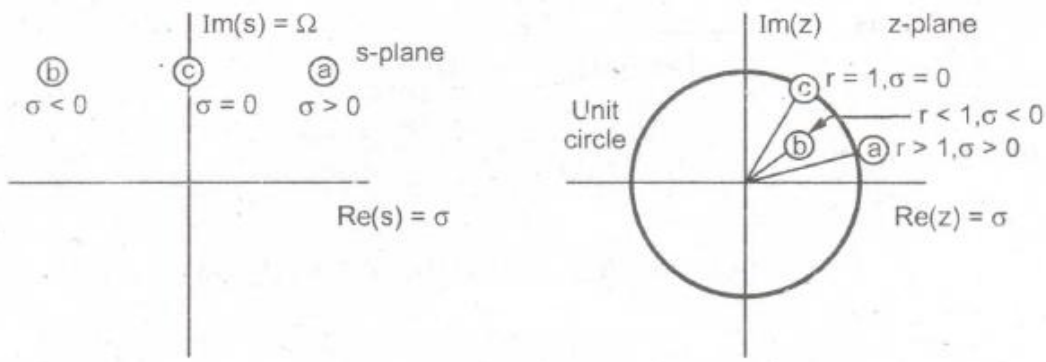

Fig. 3

Here $\quad s = \sigma + j\Omega \quad$ and $z = re^{j\omega}$

Q3

Fig. 4 shows the frequency response of an infinite-length ideal multi-band real filter. Find. h(n), impulse response of this filter. Present the sketch of implementation of $\omega(n)\ h(n)$ (Truncated impulse response of this filter) via block diagram. Where $\omega(n)$ is a finite length window sequence ? **[12]**

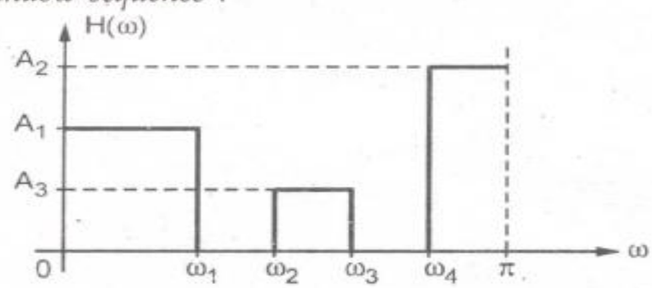

Fig. 4

Q4

We are interested to design an FIR filter with a stopband attenuation of 64 dB and $\Delta\omega = 0.05\pi$ using windows. Provide the means to achieve precisely this attenuation using suitable window function. **[3]**

Sol. : Hamming window will satisfy the stopband attenuation requirement i.e. 64 dB. Because it has lower transition width.

Hamming window function is given by :

$$\omega(n) = 0.54 - 0.46\cos\left(\frac{2\pi n}{N-1}\right) \qquad 0 \le n \le N-1$$

Q5

The transfer function of analog low pass filter is given by $H(s) = \dfrac{(s-1)}{(s^2-1)(s^2+s+1)}$.

Find H(z) using impulse invariance method. Take T = 1 sec. **[6]**

Sol. :

$$H(s) = \frac{(s-1)}{(s^2-1)(s^2+s+1)}$$

$$= \frac{(s-1)}{(s+1)(s-1)(s^2+s+1)}$$

$$= \frac{1}{(s+1)(s^2+s+1)}$$

$$= \frac{1}{(s+1)(s+0.5-j\,0.866)(s+0.5+j\,0.866)}$$

$$= \frac{C_1}{s+1} + \frac{C_2}{s+0.5-j\,0.866} + \frac{C_2^*}{s+0.5+j\,0.866}$$

Using practical fraction expansion, we get

$$C_1 = 1, \quad C_2 = 0.577e^{-j2.62} \quad \text{and} \quad C_2^* = 0.577e^{j2.62}$$

$$\therefore \qquad H(s) = \frac{1}{s+1} + \frac{0.577e^{-j2.62}}{s+0.5-j\,0.866} + \frac{0.577e^{j2.62}}{s+0.5+j\,0.866}$$

The three poles are :

$$s_1 = -1, \quad s_2 = -0.5 + j\,0.866 \quad \text{and } s_3 = -0.5 - j\,0.866$$

We know that,

$$H(z) = \sum_{i=1}^{3} \frac{C_i}{1-e^{S_i T} z^{-1}}$$

$$= \frac{C_1}{1-e^{s_1 T} z^{-1}} + \frac{C_2}{1-e^{s_2 T} z^{-1}} + \frac{C_3}{1-e^{s_3 T} z^{-1}}$$

Here $C_3 = C_2^*$

$$\therefore \quad H(z) = \frac{1}{1-e^{-T} z^{-1}} + \frac{0.577 e^{-j2.62}}{1-e^{(-0.5+j0.866)T} z^{-1}} + \frac{0.577 e^{j2.62}}{1-e^{(0.5-j0.866)T} z^{-1}}$$

$$= \frac{1}{1-e^{-T} z^{-1}} + \frac{0.577 e^{-j2.62}}{1-e^{-0.5T} e^{j0.866T} z^{-1}} + \frac{0.577 e^{j2.62}}{1-e^{-0.5T} e^{-j0.866T} z^{-1}}$$

$$= \frac{1}{1-e^{-T} z^{-1}} + \frac{2(0.577)\cos(-2.62) - 2(0.577) e^{-0.5T} z^{-1} \cos(-2.62 - 0.866T)}{1 - 2e^{-0.5T} \cos(0.866T) z^{-1} + e^{-T} z^{-2}}$$

Multiplying the numerator and denominator of first term on RHS by z and by τ^2 for second term on RHS, above equation becomes,

$$H(z) = \frac{z}{z-e^{-T}} + \frac{-z^2 - 1.154 e^{-0.5T} \cos\left(\frac{5\pi}{6} + 0.866T\right) z}{z^2 - 2e^{-0.5T} \cos(0.866T) z + e^{-T}}$$

In terms of sampling interval T = 1, transfer function is,

$$H(z) = \frac{b_0 z^{-1} + b_1 z^{-2}}{1 - a_1 z^{-1} - a_2 z^{-2} - a_3 z^{-3}}$$

Where $b_0 = -2e^{-0.5T} \cos(0.866T) + e^{-T} + 1.154 e^{-0.5T} \cos\left(\frac{5\pi}{6} + 0.866T\right) = 1.0773$

$$b_1 = e^{-T} + 1.154 e^{-1.5T} \cos\left(\frac{5\pi}{6} + 0.866T\right) = 0.1254$$

$$a_1 = e^{-T} + 2e^{-0.5T} \cos(0.866T) = 1.1538$$

$$a_2 = -e^{-T} - 2e^{-1.5T} \cos(0.866T) = -0.657$$

$$a_3 = e^{-2T} = 0.1353$$

Q6

Design a linear phase high pass filter using the Hamming window for the following desired frequency response.

$$H_d(\omega) = \begin{cases} e^{-j3\omega} & \frac{\pi}{6} \le |\omega| \le \pi \\ 0 & |\omega| < \frac{\pi}{6} \end{cases}$$

$\omega(n) = 0.54 - 0.46\cos\left(\frac{2\pi n}{N-1}\right)$, *where N is the length of the Hamming window.*

Sol. :

$$h_d(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} H_d(\omega)e^{j\omega n}d\omega$$

$$= \frac{1}{2\pi}\int_{-\pi}^{-\pi/6} e^{-j3\omega}e^{j\omega n}d\omega + \frac{1}{2\pi}\int_{\pi/6}^{\pi} e^{-j3\omega}e^{j\omega n}d\omega$$

$$= \frac{1}{\pi(n-3)}\left[\sin[\pi(n-3)] - \sin\left[\frac{\pi}{6}(n-3)\right]\right] \quad n \ne 3$$

Also, $h_d(3) = \frac{1}{2\pi}\left(\frac{5\pi}{6} + \frac{5\pi}{6}\right) \frac{5}{6}$

Let $N = 7$

Impulse response of FIR filter is :

$$h(n) = h_d(n)\omega(n)$$

$$= \begin{cases} \left\{\left\{\frac{1}{\pi(n-3)}\left[\sin[\pi(n-3)] - \sin\left[\frac{\pi}{6}(n-3)\right]\right]\right\}\left\{0.54 - 0.46\cos\left(\frac{2\pi n}{6}\right)\right\} & n \neq 3 \\ \frac{5}{6}\left[0.54 - 0.46\cos\left(\frac{2\pi n}{6}\right)\right] & n = 3 \end{cases}$$

n	$h_d(n)$	$\omega(n)$	h(n)
0	− 0.1061	0.08	0.0085
1	− 0.1378	0.31	0.0427
2	− 0.1592	0.77	0.1226
3	0.8333	1	0.8333
4	0.1592	0.77	0.1226
5	0.1378	0.31	0.0427
6	0.1061	0.08	0.0085

7.

Design a digital lowpass Butterworth filter using bilinear transformation method to meet the following specifications. Take T = 2 sec.

Passband ripple ≤ 1.25 dB

Passband edge = 200 Hz

Stopband attenuation ≥ 15 dB

Stopband edge = 400 Hz

Sampling frequency = 2 kHz **[12]**

Sol. :

$$\Omega_P = 2\pi \times 200 = 400\,\pi\, rad/\sec$$

$$\Omega_s = 2\pi \times 400 = 800\,\pi\, rad/\sec$$

$$T_s = \frac{1}{f_s} = \frac{1}{2000}\sec$$

$$\omega_P = \Omega_P T_s = 400\pi \times \frac{1}{2000} = 0.2\,\pi \text{ rad}$$

$$\omega_s = \Omega_P T_s = 800\pi \times \frac{1}{2000} = 0.4\,\pi \text{ rad}$$

Given : T = 2 sec.

$$\Omega'_P = \frac{2}{T}\tan\left(\frac{\omega_P}{2}\right) = \tan\left(\frac{0.2\pi}{2}\right) = 0.3249$$

$$\Omega'_s = \frac{2}{T}\tan\left(\frac{\omega_s}{2}\right) = \tan\left(\frac{0.4\pi}{2}\right) = 0.7265$$

$$N = \frac{log\left[(10^{-k_P/10}-1)/(10^{-k_s/10}-1)\right]}{2log\,(\Omega'_P/\Omega'_s)}$$

$$= \frac{log(0.3335/30.6228)}{2\,log\,(0.3249/0.7265)} = 2.8083 \cong 3$$

$$\Omega_c = \frac{\Omega'_P}{(10^{-k_P/10}-1)^{1/2N}} = 1.7688$$

Referring to normalized lowpass butterworth filter tables

$$H_3(s) = \frac{1}{(s^2+s+1)(s+1)}$$

The required prewarped analog filter is obtained by applying lowpass to lowpass transformation.

$$H_a(s) = H_3(s)\Big|_{s\to\frac{s}{2}}$$

$$= \frac{1}{(s^2+s+1)(s+1)}\Bigg|_{s\to\frac{s}{2}}$$

$$= \frac{1}{\left(\frac{s^2}{4}+\frac{s}{2}+1\right)\left(\frac{s}{2}+1\right)}$$

$$= \frac{1}{\left(\frac{s^2}{4}+\frac{s+2}{2}\right)\left(\frac{s+2}{2}\right)}$$

$$= \frac{1}{\left(\frac{2s^2+4s+8}{8}\right)\left(\frac{s+2}{2}\right)}$$

$$= \frac{1}{\left(\frac{s^2+2s+4}{4}\right)\left(\frac{s+2}{2}\right)}$$

$$= \frac{8}{(s^2+2s+4)(s+2)} = \frac{8}{s^3+2s^2+2s^2+4s+8}$$

$$= \frac{8}{s^3+4s^2+8s+8}$$

Applying bilinear transformation to $H_a(s)$

$$H(z) = H_3(s)\Big|_{s \to \frac{2}{T}\left(\frac{1-z^{-1}}{1+z^{-1}}\right)}$$

$$\therefore \quad H(z) = \frac{8}{\left(\frac{1-z^{-1}}{1+z^{-1}}\right)^3 + 4\left(\frac{1-z^{-1}}{1+z^{-1}}\right)^2 + 8\left(\frac{1-z^{-1}}{1+z^{-1}}\right) + 8}$$

Further Readings

1. nptel.ac.in/video.php?subjectId=117102060
2. www.journals.elsevier.com/digital-signal-processing
3. www.dspguide.com/whatdsp.htm