ATME COLLEGE OF ENGINEERING

13th KM Stone, Bannur Road, Mysore - 570 028

DEPARTMENT OF ELECTRICAL & ELECTRONICS ENGINEERING

NOTES

COURSE TITLE: SIGNALS & DIGITAL SIGNAL PROCESSING

COURSE CODE: BEE502

SEMESTER: V

MODULE-2: DISCRETE FOURIER TRANSFORMS

INSTITUTIONAL VISION AND MISSION

VISION:

- Development of academically excellent, culturally vibrant, socially responsible, and globally competent human resources.

MISSION:

- To keep pace with advancements in knowledge and make the students competitive and capable at the global level.
- To create an environment for the students to acquire the right physical, intellectual, emotional, and moral foundations and shine as torchbearers of tomorrow's society.
- To strive to attain ever-higher benchmarks of educational excellence.

Department Vision and Mission

Vision:

To produce Electrical & Electronics Engineers through greatest quality of technical education, technical skill training and intellectual capacity building of individuals.

Mission:

- To provide knowledge to students that builds a strong foundation in the basic principles of electrical engineering, problem solving abilities, analytical skills, soft skills and communication skills for their overall development.
- To offer outcome based technical education.
- To encourage faculty in training & development and to offer consultancy through research & industry interaction.

MODULE - 2: DISCRETE FOURIER TRANSFORMS

Structure

2.0 Objectives

1. To introduce the basic concepts and techniques for processing signals on a computer.
2. To study the conversion of analog signals to digital signals
3. To analyse the need for digital signal processing and importance of digital systems
4. To study the frequency domain analysis of digital signals
5. To study the importance of DFT and its applications

2.1 Introduction:

Before we introduce the DFT we consider the sampling of the Fourier transform of an aperiodic discrete-time sequence. Thus we establish the relation between the sampled Fourier transform and the DFT.A discrete time system may be described by the convolution sum, the Fourier representation and the z transform as seen in the previous chapter. If the signal is periodic in the time domain DTFS representation can be used, in the frequency domain the spectrum is discrete and periodic. If the signal is non-periodic or of finite duration the frequency domain representation is periodic and continuous this is not convenient to implement on the computer. Exploiting the periodicity property of DTFS representation the finite duration sequence can also be represented in the frequency domain, which is referred to as Discrete Fourier Transform DFT.

DFT is an important mathematical tool which can be used for the software implementation of certain digital signal processing algorithms .DFT gives a method to transform a given sequence to frequency domain and to represent the spectrum of the sequence using only k frequency values, where k is an integer that takes N values, K=0, 1, 2,…..N-1.

The advantages of DFT are:

1. It is computationally convenient.
2. The DFT of a finite length sequence makes the frequency domain analysis much simpler than continuous Fourier transform technique.

1.2 FREQUENCY DOMAIN SAMPLING AND RECONSTRUCTION OF DISCRETE TIME SIGNALS:

Consider an aperiodic discrete time signal x (n) with Fourier transform, an aperiodic finite energy signal has continuous spectra. For an aperiodic signal x[n] the spectrum is:

$$X[w] = \sum_{n=-\infty}^{\infty} x[n] e^{-jwn} \quad \text{……………………………….(1.1)}$$

Suppose we sample X[w] periodically in frequency at a sampling of δw radians between successive samples. We know that DTFT is periodic with 2π, therefore only samples in the fundamental frequency range will be necessary. For convenience we take N equidistant samples in the interval (0<=w<2π). The spacing between samples will be $\delta w = \frac{2\pi}{N}$ as shown below in Fig.1.1.

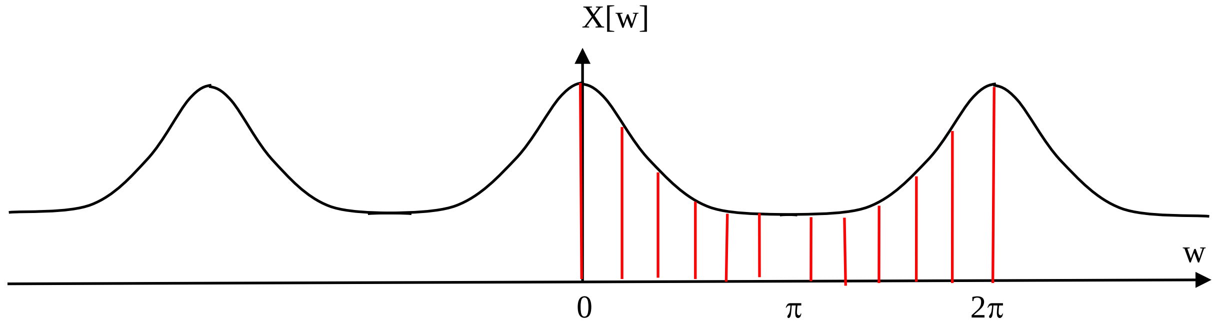

Fig 1.1 Frequency Domain Sampling

Let us first consider selection of N, or the number of samples in the frequency domain.

If we evaluate equation (1) at $w = \frac{2\pi k}{N}$

$$X\left[\frac{2\pi k}{N}\right] = \sum_{n=-\infty}^{\infty} x[n] e^{-j2\pi kn/N} \qquad k = 0,1,2,......,(N-1) \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1.2)$$

We can divide the summation in (1) into infinite number of summations where each sum contains N terms.

$$X\left[\frac{2\pi k}{N}\right] = ...+\sum_{n=-N}^{-1} x[n] e^{-j2\pi kn/N} + \sum_{n=0}^{N-1} x[n] e^{-j2\pi kn/N} + \sum_{n=N}^{2N-1} x[n] e^{-j2\pi kn/N}$$

$$= \sum_{l=-\infty}^{\infty} \sum_{n=lN}^{lN+N-1} x[n] e^{-j2\pi kn/N}$$

If we then change the index in the summation from n to n-l N and interchange the order of summations we get:

$$\left[\frac{2\pi k}{N}\right] = \sum_{n=0}^{N-1}\left[\sum_{l=-\infty}^{\infty} x[n-lN]\right] e^{-j2\pi kn/N} \qquad for\ \ k = 0,1,2,......,(N-1) \ldots\ldots.(1.3)$$

Denote the quantity inside the bracket as $x_p[n]$. This is the signal that is a repeating version of x[n] every N samples. Since it is a periodic signal it can be represented by the Fourier series.

$$x_p[n] = \sum_{k=0}^{N-1} c_k e^{j2\pi kn/N} \qquad n = 0,1,2,........,(N-1)$$

With FS coefficients:

$$c_k = \frac{1}{N}\sum_{n=0}^{N-1} x_p[n] e^{-j2\pi kn/N} \qquad k = 0,1,2,.......,(N-1) \ldots\ldots\ldots\ldots\ldots (1.4)$$

Comparing the expressions in equations (1.4) and (1.3) we conclude the following:

$$c_k = \frac{1}{N} X\left[\frac{2\pi}{N}k\right] \qquad k = 0,1,.......,(N-1) \ldots\ldots\ldots\ldots\ldots\ldots (1.5)$$

Therefore it is possible to write the expression $x_p[n]$ as below:

$$x_p[n] = \frac{1}{N}\sum_{k=0}^{N-1} X\left[\frac{2\pi}{N}k\right] e^{j2\pi kn/N} \qquad n = 0,1,.....,(N-1) \ldots\ldots\ldots (1.6)$$

The above formula shows the reconstruction of the periodic signal $x_p[n]$ from the samples of the spectrum X[w]. But it does not say if X[w] or x[n] can be recovered from the samples.

Let us have a look at that:

Since $x_p[n]$ is the periodic extension of x[n] it is clear that x[n] can be recovered from $x_p[n]$ if there is no aliasing in the time domain. That is if x[n] is time-limited to less than the period N of $x_p[n]$.This is depicted in Fig. 1.2 below:

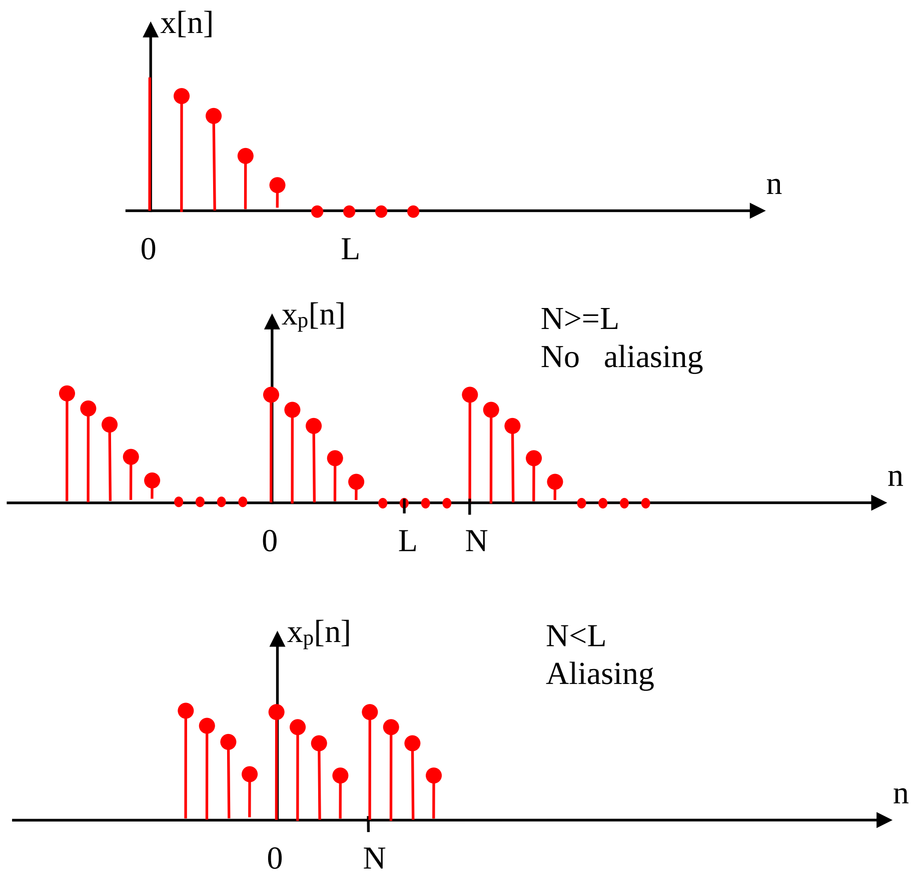

Fig. 1.2 Signal Reconstruction

Hence we conclude:
The spectrum of an aperiodic discrete-time signal with finite duration L can be exactly recovered from its samples at frequencies $w_k = \frac{2\pi k}{N}$ if N >= L.

We compute $x_p[n]$ for n=0, 1,....., N-1 using equation (1.6)
Then X[w] can be computed using equation (1.1).

1.3 Discrete Fourier Transform:

The DTFT representation for a finite duration sequence is

$$X(j\omega) = \sum_{n=-\infty}^{\infty} x(n)\, e^{-j\omega n}$$

$$X(n) = 1/2\pi \int X(j\omega)\, e^{j\omega n}\, d\omega, \quad \text{Where } \omega = 2\pi k/n2\pi$$

Where x(n) is a finite duration sequence, X(jω) is periodic with period 2π.It is convenient sample X(jω) with a sampling frequency equal an integer multiple of its period =m that is taking N uniformly spaced samples between 0 and 2π.

Let $\omega_k = 2\pi k/n$, $0 \le k \le N-1$

Therefore $X(j\omega) = \sum_{n=-\infty}^{\infty} x(n)\, e^{-j2\pi kn/N}$

Since X(jω) is sampled for one period and there are N samples X(jω) can be expressed as

$$X(k) = X(j\omega)\Big|_{\omega=2\pi kn/N} = \sum_{n=0}^{N-1} x(n)\, e^{-j2\pi kn/N} \qquad 0 \le k \le N-1$$

1.4 Matrix relation of DFT

The DFT expression can be expressed as

[X] = [x(n)] [WN]

Where $[X] = [X(0), X(1), \ldots\ldots]^T$

[x] is the transpose of the input sequence. WN is a N x N matrix

$$WN = \begin{matrix} 1 & 1 & 1 & 1 & \ldots\ldots\ldots\ldots\ldots\ldots 1 \\ 1 & wn1 & wn2 & wn3 & \ldots\ldots\ldots\ldots\ldots\ldots wn\ n\text{-}1 \\ 1 & wn2 & wn4 & wn6 & \ldots\ldots\ldots\ldots\ldots wn2(n\text{-}1) \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ 1 & \ldots & \ldots & \ldots & \ldots wN\ (N\text{-}1)(N\text{-}1) \end{matrix}$$

ex;

4 pt DFT of the sequence 0,1,2,3

$$\begin{matrix} X(0) \\ X(1) \\ X(2) \\ X(3) \end{matrix} = \begin{matrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & -j \end{matrix}$$

Solving the matrix X(K) = 6 , -2+2j, -2 , -2-2j

1.5 Relationship of Fourier Transforms with other transforms

1.5.1 Relationship of Fourier transform with continuous time signal:

Suppose that $x_a(t)$ is a continuous-time periodic signal with fundamental period $T_p = 1/F_0$.The signal can be expressed in Fourier series as

$$x_a(t) = \sum_{k=-\infty}^{\infty} c_k e^{j2\pi k F_0}$$

Where {ck} are the Fourier coefficients. If we sample $x_a(t)$ at a uniform rate Fs = N/T_p = 1/T, we obtain discrete time sequence

$$x(n) \equiv x_a(nT) = \sum_{k=-\infty}^{\infty} c_k e^{j2\pi k F_0 n T} = \sum_{k=-\infty}^{\infty} c_k e^{j2\pi kn/N}$$

$$= \sum_{k=0}^{N-1} \left[\sum_{l=-\infty}^{\infty} c_{k-lN} \right] e^{j2\pi kn/N}$$

$$X(k) = N \sum_{l=-\infty}^{\infty} c_{k-lN} \equiv N\tilde{c}_k$$

Thus $\{c_{k'}\}$ is the aliasing version of $\{c_k\}$

1.5.2 Relationship of Fourier transform with z-transform

Let us consider a sequence x(n) having the z-transform

$$X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n}$$

With ROC that includes unit circle. If X(z) is sampled at the N equally spaced points on the unit circle $Z_k = e^{j2\pi k/N}$ for K= 0,1,2,………..N-1 we obtain

$$X(k) \equiv X(z)|_{z=e^{j2\pi nk/N}} \qquad k = 0, 1, \ldots, N-1$$

$$= \sum_{n=-\infty}^{\infty} x(n) e^{-j2\pi nk/N}$$

The above expression is identical to Fourier transform X(ω) evaluated at N equally spaced frequencies $\omega_k = 2\pi k/N$ for K= 0,1,2,………..N-1.

If the sequence x(n) has a finite duration of length N or less. The sequence can be recovered from its N-point DFT. Consequently X(z) can be expressed as a function of DFT as

$$X(z) = \sum_{n=0}^{N-1} x(n) z^{-n}$$

$$X(z) = \sum_{n=0}^{N-1} \left[\frac{1}{N} \sum_{k=0}^{N-1} X(k) e^{j2\pi kn/N} \right] z^{-n}$$

$$X(z) = \frac{1}{N} \sum_{k=0}^{N-1} X(k) \sum_{n=0}^{N-1} \left(e^{j2\pi k/N} z^{-1} \right)^n$$

$$X(z) = \frac{1 - z^{-N}}{N} \sum_{k=0}^{N-1} \frac{X(k)}{1 - e^{j2\pi k/N} z^{-1}}$$

Fourier transform of a continuous time signal can be obtained from DFT as

$$X(\omega) = \frac{1 - e^{-j\omega N}}{N} \sum_{k=0}^{N-1} \frac{X(k)}{1 - e^{-j(\omega - 2\pi k/N)}}$$

1. The first five points of the 8-point DFT of a real valued sequence are {0.25, 0.125-j0.318, 0, 0.125-j0.0518, 0}. Determine the remaining three points

Ans: Since x(n) is real, the real part of the DFT is even, imaginary part odd. Thus the remaining points are {0.125+j0.0518,0,0, 0.125+j0.318}.

2. Compute the eight-point DFT circular convolution for the following sequences.
   $x_2(n) = \sin 3\pi n/8$

Ans:

(a)

$$\begin{aligned}
\tilde{x}_2(l) &= x_2(l), \quad 0 \le l \le N-1 \\
&= x_2(l+N), \quad -(N-1) \le l \le -1 \\
\tilde{x}_2(l) &= sin(\frac{3\pi}{8} l), \quad 0 \le l \le 7 \\
&= sin(\frac{3\pi}{8}(l+8)), \quad -7 \le l \le -1 \\
&= sin(\frac{3\pi}{8}|l|), \quad |l| \le 7
\end{aligned}$$

$$\begin{aligned}
\text{Therefore, } x_1(n) \, ⑧ \, x_2(n) &= \sum_{m=0}^{3} \tilde{x}^2(n-m) \\
&= sin(\frac{3\pi}{8}|n|) + sin(\frac{3\pi}{8}|n-1|) + \ldots + sin(\frac{3\pi}{8}|n-3|) \\
&= \{1.25, 2.55, 2.55, 1.25, 0.25, -1.06, -1.06, 0.25\}
\end{aligned}$$

3. Compute the eight-point DFT circular convolution for the following sequence
$X_3(n) = \cos 3\pi n/8$

$$\tilde{x}_2(n) = \cos(\frac{3\pi}{8}n), \quad 0 \le l \le 7$$
$$= -\cos(\frac{3\pi}{8}n), \quad -7 \le l \le -1$$
$$= [2u(n)-1]\cos(\frac{3\pi}{8}n), \quad |n| \le 7$$

$$\text{Therefore, } x_1(n) \, (8) \, x_2(n) = \sum_{m=0}^{3} \left(\frac{1}{4}\right)^m \tilde{x}^2(n-m)$$
$$= \{0.96, 0.62, -0.55, -1.06, -0.26, -0.86, 0.92, -0.15\}$$

4. Define DFT. Establish a relation between the Fourier series coefficients of a continuous time signal and DFT

Solution

The DTFT representation for a finite duration sequence is

$$X(j\omega) = \sum_{n=-\infty}^{\infty} x(n)\, e^{-j\omega n}$$

$$X(n) = 1/2\pi \int_{2\pi} X(j\omega)\, e^{j\omega n}\, d\omega, \quad \text{Where } \omega = 2\pi k/n$$

Where x(n) is a finite duration sequence, $X(j\omega)$ is periodic with period 2π.It is convenient sample $X(j\omega)$ with a sampling frequency equal an integer multiple of its period =m that is taking N uniformly spaced samples between 0 and 2π.

Let $\omega_k = 2\pi k/n$, $0 \le k \le N$

$$\text{Therefore } X(j\omega) = \sum_{n=-\infty}^{\infty} x(n)\, e^{-j2\pi kn/N}$$

Since $X(j\omega)$ is sampled for one period and there are N samples $X(j\omega)$ can be expressed as

$$X(k) = X(j\omega)\big|_{\omega=2\pi kn/N} = \sum_{n=0}^{N-1} x(n)\, e^{-j2\pi kn/N} \quad 0 \le k \le N-1$$

5.7 If $X(k)$ is the DFT of the sequence $x(n)$, determine the N-point DFTs of the sequences

$$x_c(n) = x(n)\cos\frac{2\pi kn}{N} \qquad 0 \le n \le N-1$$

and

$$x_s(n) = x(n)\sin\frac{2\pi kn}{N} \qquad 0 \le n \le N-1$$

in terms of $X(k)$.

Solution:-

$$\begin{aligned}
X_c(k) &= \sum_{n=0}^{N-1}\frac{1}{2}x(n)\left(e^{j\frac{2\pi k_0 n}{N}} + e^{-j\frac{2\pi k_0 n}{N}}\right)e^{-\frac{2\pi kn}{N}} \\
&= \frac{1}{2}\sum_{n=0}^{N-1} x(n)e^{-j\frac{2\pi(k-k_0)n}{N}} + \frac{1}{2}\sum_{n=0}^{N-1} x(n)e^{-j\frac{2\pi(k+k_0)n}{N}} \\
&= \frac{1}{2}X(k-k_0)_{\text{mod}N} + \frac{1}{2}X(k+k_0)_{\text{mod}N} \\
\text{similarly, } X_s(k) &= \frac{1}{2j}X(k-k_0)_{\text{mod}N} - \frac{1}{2j}X(k+k_0)_{\text{mod}N}
\end{aligned}$$

5. Find the 4-point DFT of sequence x(n) = 6+ sin(2πn/N), n= 0,1,………N-1

Solution :-

$$\text{Here } x(n) = 6+\sin\left(\frac{2\pi n}{N}\right), \qquad \textit{with } N=4$$

$$\begin{aligned}
x(n) &= 6+\sin\left(\frac{2\pi n}{4}\right), n = 0,1,2,3. \\
&= 6+\sin\left(\frac{\pi}{2}n\right), n = 0,1,2,3. \\
&= \{6, 7, 6, 5\}.
\end{aligned}$$

6. **Compute the N-point DFTs of the signal**

$$x(n) = \cos\frac{2\pi}{N}k_0 n \quad 0 \le n \le N-1$$

Solution

$$\begin{aligned} X(k) &= \sum_{n=0}^{N-1} e^{j\frac{2\pi}{N}nk_0} e^{-j\frac{2\pi}{N}kn} \\ &= \sum_{n=0}^{N-1} e^{-j\frac{2\pi}{N}(k-k_0)n} \\ &= N\delta(k-k_0) \end{aligned}$$

$$x(n) = \frac{1}{2}e^{j\frac{2\pi}{N}nk_0} + \frac{1}{2}e^{-j\frac{2\pi}{N}nk_0}$$

From (e) we obtain $X(k) = \frac{N}{2}\left[\delta(k-k_0) + \delta(k-N+k_0)\right]$

2.1Properties of DFT

The DFT and IDFT for an N-point sequence x(n) are given as

$$\text{DFT: } X(k) = \sum_{n=0}^{N-1} x(n) W_N^{kn} \qquad k = 0, 1, \ldots, N-1$$

$$\text{IDFT: } x(n) = \frac{1}{N} \sum_{k=0}^{N-1} X(k) W_N^{-kn} \qquad n = 0, 1, \ldots, N-1$$

where W_N is defined as

$$W_N = e^{-j2\pi/N}$$

In this section we discuss about the important properties of the DFT. These properties are helpful in the application of the DFT to practical problems.

The notation used below to denote the N-point DFT pair $x(n)$ and $X(k)$ is

$$x(n) \underset{N}{\overset{\text{DFT}}{\longleftrightarrow}} X(k)$$

2.1.1 Periodicity

If $x(n)$ and X(k) are an N-point DFT pair, then

$$x(n+N) = x(n) \quad \text{for all } n$$

$$X(k+N) = X(k) \quad \text{for all } k$$

2.1.2 Linearity

If $x_1(n) \underset{N}{\overset{\text{DFT}}{\longleftrightarrow}} X_1(k)$

$x_2(n) \underset{N}{\overset{\text{DFT}}{\longleftrightarrow}} X_2(k)$

Then A x1 (n) + b x2 (n) ←→ a X1(k) + b X2(k)

2.1.3 Circular shift

In linear shift, when a sequence is shifted the sequence gets extended. In circular shift the number of elements in a sequence remains the same. Given a sequence x (n) the shifted version x (n-m) indicates a shift of m. With DFTs the sequences are defined for 0 to N-1.

If x (n) = x (0), x (1), x (2), x (3)

X (n-1) = x (3), x (0), x (1).x (2)

X (n-2) = x (2), x (3), x (0), x (1)

2.1.4 Time shift

If $x(n) \longleftrightarrow X(k)$

Then $x(n-m) \longleftrightarrow W_N^{mk} X(k)$

2.1.5Frequency shift

If $x(n) \longleftrightarrow X(k)$

$W_n^{+nok} x(n) \longleftrightarrow X(k+no)$

Consider $x(k) = \sum_{n=0}^{N-1} x(n) W_n^{kn}$

$$X(k+no) = \sum_{n=0}^{N-1} x(n) W_N^{(k+no)n}$$

$$= \sum x(n) W_N^{kn} W_N^{non}$$

$\therefore X(k+no) \longleftrightarrow x(n) W_N^{non}$

2.1.6Symmetry

For a realsequence, if $x(n) \longleftrightarrow X(k)$

$$X(N-K) = X^*(k)$$

For a complex sequence
DFT(x*(n)) = X*(N-K)

If x(n)	then X(k)
Real and even	real and even
Real and odd	imaginary and odd
Odd and imaginary	real odd
Even and imaginary	imaginary and even

2.1.7 Convolution theorem

Circular convolution in time domain corresponds to multiplication of the DFTs

If y(n) = x(n) ⊗ h(n) then Y(k) = X(k) H(k)

Ex let x(n) = 1,2,2,1 and h(n) = 1,2,2,1
Then y (n) = x(n) ⊗ h(n)

Y(n) = 9,10,9,8

N pt DFTs of 2 real sequences can be found using a single DFT

If g(n) & h(n) are two sequences then let x(n) = g(n) +j h(n)

G(k) = ½ (X(k) + X*(k))

H(k) = 1/2j (X(K) +X*(k))

2N pt DFT of a real sequence using a single N pt DFT

Let x(n) be a real sequence of length 2N with y(n) and g(n) denoting its N pt DFT

Let y(n) = x(2n) and g(2n+1)

$X(k) = Y(k) + WN^{k} G(k)$

Using DFT to find IDFT

The DFT expression can be used to find IDFT
X(n) = 1/N [DFT(X*(k)]*

2.2 Digital filtering using DFT

In a LTI system the system response is got by convoluting the input with the impulse response. In the frequency domain their respective spectra are multiplied. These spectra are continuous and hence cannot be used for computations. The product of 2 DFT s is equivalent to the circular convolution of the corresponding time domain sequences. Circular convolution cannot be used to determine the output of a linear filter to a given input sequence. In this case a frequency domain methodology equivalent to linear convolution is required. Linear convolution can be implemented using circular convolution by taking the length of the convolution as N >= n1+n2-1 where n1 and n2 are the lengths of the 2 sequences.

2.2.1 Overlap and add

In order to convolve a short duration sequence with a long duration sequence x(n) ,x(n) is split into blocks of length N x(n) and h(n) are zero padded to length L+M-1 . circular convolution is performed to each block then the results are added. These data blocks may be represented as

$$x_1(n) = \{x(0), x(1), \ldots, x(L-1), \underbrace{0, 0, \ldots, 0}_{M-1 \text{ zeros}}\}$$

$$x_2(n) = \{x(L), x(L+1), \ldots, x(2L-1), \underbrace{0, 0, \ldots, 0}_{M-1 \text{ zeros}}\}$$

$$x_3(n) = \{x(2L), \ldots, x(3L-1), \underbrace{0, 0, \ldots, 0}_{M-1 \text{ zeros}}\}$$

The two N-point DFTs are multiplied together to form

$$Y_m(k) = H(k)X_m(k) \qquad k = 0, 1, \ldots, N-1$$

The IDFT yields data blocks of length N that are free of aliasing since the size of the DFTs and IDFT is N = L+M -1 and the sequences are increased to N-points by appending zeros to each block. Since each block is terminated with M-1 zeros, the last M-1 points from each output block must be overlapped and added to the first M-1 points of the succeeding block. Hence this method is called the overlap method. This overlapping and adding yields the output sequences given below.

$$y(n) = \{y_1(0), y_1(1), \ldots, y_1(L-1), y_1(L) + y_2(0), y_1(L+1) + y_2(1), \ldots, y_1(N-1) + y_2(M-1), y_2(M), \ldots\}$$

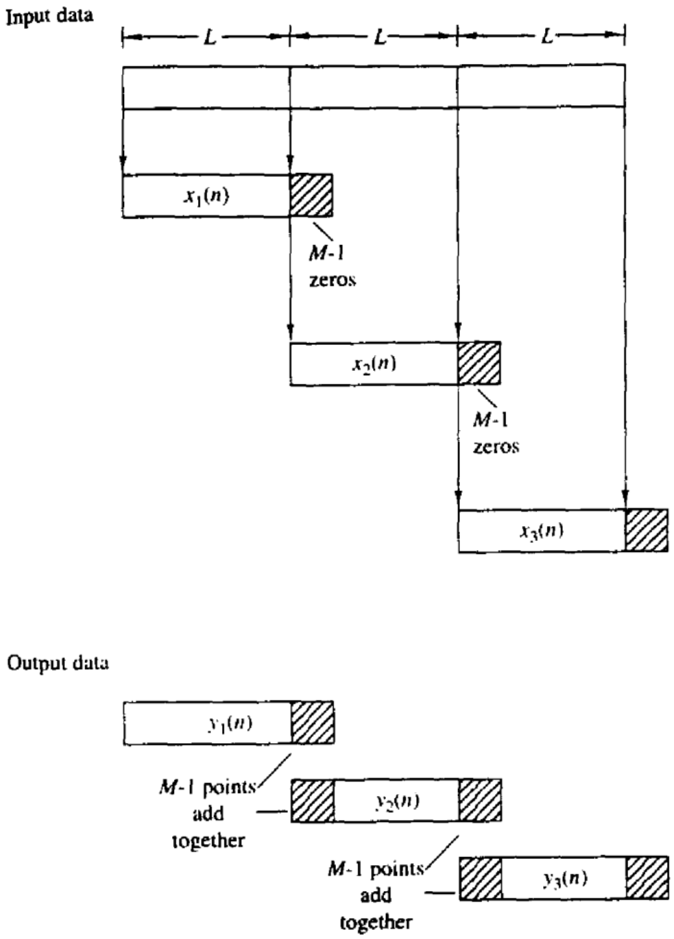

Figure 5.11 Linear FIR filtering by the overlap-add method.

2.2.2 Overlap and save method

In this method x (n) is divided into blocks of length N with an overlap of k-1 samples. The first block is zero padded with k-1 zeros at the beginning. H (n) is also zero padded to length N. Circular convolution of each block is performed using the N length DFT .The output signal is obtained after discarding the first k-1 samples the final result is obtained by adding the intermediate results.

In this method the size of the I/P data blocks is N= L+M-1 and the size of the DFts and IDFTs are of length N. Each data block consists of the last M-1 data points of the previous data block followed by L new data points to form a data sequence of length N= L+M-1. An N-point DFT is computed from each data block. The impulse response of the FIR filter is increased in length by appending L-1 zeros and an N-point DFT of the sequence is computed once and stored.

The multiplication of two N-point DFTs {H(k)} and {Xm(k)} for the mth block of data yields

$$\hat{Y}_m(k) = H(k)X_m(k) \qquad k = 0, 1, \ldots, N-1$$

Then the *N*-point IDFT yields the result

$$\hat{Y}_m(n) = \{\hat{y}_m(0)\hat{y}_m(1)\cdots\hat{y}_m(M-1)\hat{y}_m(M)\cdots\hat{y}_m(N-1)\}$$

Since the data record is of the length N, the first M-1 points of Ym(n) are corrupted by aliasing and must be discarded. The last L points of Ym(n) are exactly the same as the result from linear convolution and as a consequence we get

$$\hat{y}_m(n) = y_m(n), n = M, M+1, \ldots, N-1$$

$$x_1(n) = \{\underbrace{0, 0, \ldots, 0,}_{M-1 \text{ points}} x(0), x(1), \ldots, x(L-1)\}$$

$$x_2(n) = \{\underbrace{x(L-M+1), \ldots, x(L-1),}_{\substack{M-1 \text{ data points} \\ \text{from } x_1(n)}} \underbrace{x(L), \ldots, x(2L-1)}_{L \text{ new data points}}\}$$

$$x_3(n) = \{\underbrace{x(2L-M+1), \ldots, x(2L-1),}_{\substack{M-1 \text{ data points} \\ \text{from } x_2(n)}} \underbrace{x(2L), \ldots, x(3L-1)}_{L \text{ new data points}}\}$$

and so forth. The resulting data sequences from the IDFT are given by (5.3.8), where the first $M-1$ points are discarded due to aliasing and the remaining L points constitute the desired result from linear convolution. This segmentation of the input data and the fitting of the output data blocks together to form the output sequence are graphically illustrated in Fig. 5.10.

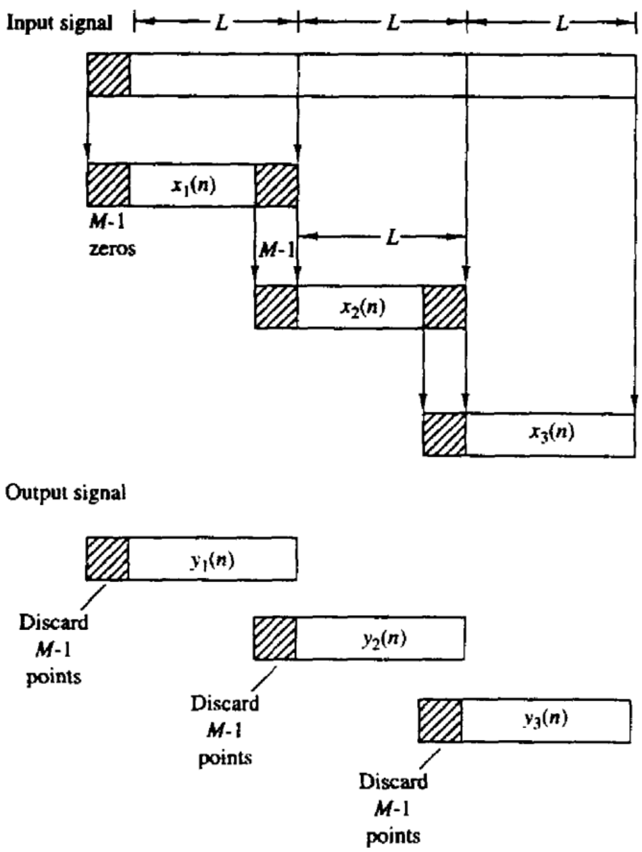

Figure 5.10 Linear FIR filtering by the overlap-save method.

Recommended Questions with solutions

1. State and Prove the Time shifting Property of DFT

Solution

The DFT and IDFT for an N-point sequence x(n) are given as

$$\text{DFT: } X(k) = \sum_{n=0}^{N-1} x(n) W_N^{kn} \qquad k = 0, 1, \ldots, N-1$$

$$\text{IDFT: } x(n) = \frac{1}{N} \sum_{k=0}^{N-1} X(k) W_N^{-kn} \qquad n = 0, 1, \ldots, N-1$$

where W_N is defined as

$$W_N = e^{-j2\pi/N}$$

Time shift:

If x (n) ←→ X (k)

Then x (n-m) ←→ WN^{mk} X (k)

2. **State and Prove the: (i) Circular convolution property of DFT; (ii) DFT of Real and even sequence.**

Solution

(i) Convolution theorem

Circular convolution in time domain corresponds to multiplication of the DFTs
If y(n) = x(n) ⊗ h(n) then Y(k) = X(k) H(k)
Ex let x(n) = 1,2,2,1 and h(n) = 1,2,2,1 Then y (n) = x(n) ⊗ h(n)

Y(n) = 9,10,9,8
N pt DFTs of 2 real sequences can be found using a single DFT
If g(n) & h(n) are two sequences then let x(n) = g(n) +j h(n)
G(k) = ½ (X(k) + X*(k))
H(k) = 1/2j (X(K) +X*(k))
2N pt DFT of a real sequence using a single N pt DFT
Let x(n) be a real sequence of length 2N with y(n) and g(n) denoting its N pt DFT
Let y(n) = x(2n) and g(2n+1)

$X(k) = Y(k) + W_N^k G(k)$

Using DFT to find IDFT

The DFT expression can be used to find IDFT

X(n) = 1/N [DFT(X*(k)]*

(ii)DFT of Real and even sequence.

For a real sequence, if x(n)←→ X(k)

X (N-K) = X* (k)

For a complex sequence

DFT(x*(n)) = X*(N-K)

If x(n)	then X(k)
Real and even	real and even
Real and odd	imaginary and odd
Odd and imaginary	real odd
Even and imaginary	imaginary and even

3. Distinguish between circular and linear convolution

Solution

1) Circular convolution is used for periodic and finite signals while linear convolution is used for aperiodic and infinite signals.
2) In linear convolution we convolved one signal with another signal where as in circular convolution the same convolution is done but in circular pattern depending upon the samples of the signal
3) Shifts are linear in linear in linear convolution, whereas it is circular in circular convolution.

For the sequences

$$x_1(n) = \cos\frac{2\pi}{N}n \qquad x_2(n) = \sin\frac{2\pi}{N}n \qquad 0 \le n \le N-1$$

determine the N-point:

(a) Circular convolution $x_1(n)$ Ⓝ $x_2(n)$

(b) Circular correlation of $x_1(n)$ and $x_2(n)$

(c) Circular autocorrelation of $x_1(n)$

(d) Circular autocorrelation of $x_2(n)$

Solution(a)

$$
\begin{aligned}
x_1(n) &= \frac{1}{2}\left(e^{j\frac{2\pi}{N}n} + e^{-j\frac{2\pi}{N}n}\right) \\
X_1(k) &= \frac{N}{2}\left[\delta(k-1) + \delta(k+1)\right] \\
\text{also } X_2(k) &= \frac{N}{2j}\left[\delta(k-1) - \delta(k+1)\right] \\
\text{So } X_3(k) &= X_1(k)X_2(k) \\
&= \frac{N^2}{4j}\left[\delta(k-1) - \delta(k+1)\right] \\
\text{and } x_3(n) &= \frac{N}{2}sin(\frac{2\pi}{N}n)
\end{aligned}
$$

Solution(b)

$$
\begin{aligned}
\tilde{R}_{xy}(k) &= X_1(k)X_2^*(k) \\
&= \frac{N^2}{4j}\left[\delta(k-1) - \delta(k+1)\right] \\
\Rightarrow \quad \tilde{r}_{xy}(n) &= -\frac{N}{2}sin(\frac{2\pi}{N}n)
\end{aligned}
$$

Solution(c)

$$
\begin{aligned}
\tilde{R}_{xx}(k) &= X_1(k)X_1^*(k) \\
&= \frac{N^2}{4}\left[\delta(k-1) + \delta(k+1)\right] \\
\Rightarrow \quad \tilde{r}_{xx}(n) &= \frac{N}{2}cos(\frac{2\pi}{N}n)
\end{aligned}
$$

Solution(d)

$$
\begin{aligned}
\tilde{R}_{yy}(k) &= X_2(k)X_2^*(k) \\
&= \frac{N^2}{4}\left[\delta(k-1) + \delta(k+1)\right] \\
\Rightarrow \quad \tilde{r}_{yy}(n) &= \frac{N}{2}cos(\frac{2\pi}{N}n)
\end{aligned}
$$

Q5.

Use the four-point DFT and IDFT to determine the sequence

$$x_3(n) = x_1(n) \; Ⓝ \; x_2(n)$$

where $x_1(n)$ and $x_2(n)$ are the sequence given

$$x_1(n) = \{\underset{\uparrow}{1}, 2, 3, 1\}$$

$$x_2(n) = \{\underset{\uparrow}{4}, 3, 2, 2\}$$

$$\begin{aligned} y(n) &= x_1(n)4x_2(n) \\ &= \sum_{m=0}^{3} x_1(m)_{\text{mod}4} x_2(n-m)_{\text{mod}4} \\ &= \{17, 19, 22, 19\} \end{aligned}$$

$$\begin{aligned} X_1(k) &= \{7, -2-j, 1, -2+j\} \\ X_2(k) &= \{11, 2-j, 1, 2+j\} \\ \Rightarrow X_3(k) &= X_1(k)X_2(k) \\ &= \{17, 19, 22, 19\} \end{aligned}$$

A linear time-invariant system with frequency response $H(\omega)$ is excited with the periodic input

$$x(n) = \sum_{k=-\infty}^{\infty} \delta(n - kN)$$

Suppose that we compute the N-point DFT $Y(k)$ of the samples $y(n)$, $0 \le n \le N-1$ of the output sequence. How is $Y(k)$ related to $H(\omega)$?

Solution

$$\begin{aligned} x(n) &= \sum_{i=-\infty}^{\infty} \delta(n - iN) \\ y(n) &= \sum_m h(m)x(n-m) \\ &= \sum_m h(m)\left[\sum_i \delta(n-m-iN)\right] \\ &= \sum_i h(n-iN) \end{aligned}$$

Therefore, $y(.)$ is a periodic sequence with period N. So

$$\begin{aligned} Y(k) &= \sum_{n=0}^{N-1} y(n) W_N^{kn} \\ &= H(w)|_{w=\frac{2\pi}{N}k} \end{aligned}$$

$$Y(k) = H\left(\frac{2\pi k}{N}\right) \quad k = 0, 1, \ldots, N-1$$

Further Readings

1. http://eeweb.poly.edu/iselesni/EL713/zoom/dftprop.pdf
2. https://engineering.purdue.edu/~ee538/DFT_Properties.pdf
3. http://www.nptel.ac.in/courses/108101039/download/Lecture-34.pdf
4. http://www.ecsutton.ece.ufl.edu/dip/handouts/convexample.pdf
5. http://sist.sysu.edu.cn/uploaded/file/Chpt03.pdf