Fourier Transforms

The infinite Fourier transform of a real valued function f(x) is defined by

$$F\big(f(x)\big) = \int_{-\infty}^{\infty} f(x)e^{iux}dx, provided\ the\ integral\ exists.$$

The inverse Fourier transform of F(u) denoted by

$$F^{-1}\big(F(u)\big) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(u)e^{-iux}du$$

- Applications:
- Identification of Earthquake induced damage areas using FT&SPOT, HRVIR pan images in civil.
- It is used to convert any periodical signal in time domain to frequency domain is used in random vibrations & turbulent flow situations
- Use od an oscillator to produce sinusoidal motions whereby the stiffness and the mass distribution are tuned such that a particular natural frequency coincides with excitation and response frequency.
- F(JW) is defined by F.T of f(t) , it provides a frequency domain representation of the non-periodic function.
- Fourier Transform can solve by MATLAB
- To find the system is stable and impulse response does not possess a F.T

- If f(x) is defined for all positive values of x, we define
- $F_c\big(f(x)\big) = \int_0^\infty f(x) cosuxdx$---Fourier cosine transform
- $f(x) = \frac{2}{\pi}\int_0^\infty F_c\big(f(x)\big) cosuxdx$---Inverse Fourier cosine transform

- $F_s\big(f(x)\big) = \int_0^\infty f(x) sinuxdx$---Fourier sine transform
- $f(x) = \frac{2}{\pi}\int_0^\infty F_c\big(f(x)\big) sinuxdx$---Inverse Fourier sine transform

- Find the Fourier transform of the function
- $f(x) = \begin{cases} 1, & |x| \le a \\ 0, & |x| > a \end{cases}, hence\ Evluate \int_0^{\infty} \frac{sinx}{x} dx.$
- Solution:
- $F\big(f(x)\big) = \int_{-\infty}^{\infty} f(x) e^{iux} dx,$
- $= \int_{-a}^{a} 1. e^{iux} dx$
- $= \left[\frac{e^{iux}}{iu}\right]_{-a}^{a} = \frac{e^{iua} - e^{-iua}}{iu} = \frac{2isinau}{iu} = \frac{2sinau}{u}.$
- $To\ evluate \int_0^{\infty} \frac{sinx}{x} dx$ apply inverse F.T
- $f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(u)\ e^{-iux} du,$
- $= \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{2sinau}{u} e^{-iux} du,$
- Put x=0 and a=1, we get
- $1 = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{sinu}{u} e^{-iu0} du,$
- F(u) is an even function and replace u as x
- $1 = \frac{2}{\pi} \int_0^{\infty} \frac{sinx}{x} dx,$
- $\frac{\pi}{2} = \int_0^{\infty} \frac{sinx}{x} dx$

- Find the Fourier transform of the function
- $f(x) = \begin{cases} 1 - x^2 & |x| \le 1 \\ 0, & |x| > 1 \end{cases}, hence\ Evluate \int_0^{\infty} \frac{xcosx - sinx}{x^3} cos\left(\frac{x}{2}\right) dx.$
- Solution:
- $F\big(f(x)\big) = \int_{-\infty}^{\infty} f(x) e^{iux} dx = \int_{-1}^{1} (1 - x^2) e^{iux} dx$
- $= \left[(1 - x^2)\frac{e^{iux}}{iu} - (-2x)\frac{e^{iux}}{i^2u^2} - 2\frac{e^{iux}}{i^3u^3}\right]_{-1}^{1}$
- $= \left[0 + 2\frac{e^{iu}}{-u^2} + 2\frac{e^{iu}}{iu^3} - (2\frac{e^{-iu}}{-u^2} - 2\frac{e^{-iu}}{iu^3}\right] = \frac{4cosu}{-u^2} + \frac{4sinu}{u^3}$
- $= \frac{-4(ucosu - sinu)}{u^3}$
- $To\ evluate \int_0^{\infty} \frac{xcosx - sinx}{x^3} cos\left(\frac{x}{2}\right) dx$ apply inverse F.T
- $\frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{-4(ucosu - sinu)}{u^3} e^{-iux} du = \begin{cases} 1 - x^2 & |x| \le 1 \\ 0, & |x| > 1 \end{cases},$
- Put $x = \frac{1}{2}$, we get
- $\frac{3}{4} = \frac{2}{\pi} \int_{-\infty}^{\infty} \frac{-4(ucosu - sinu)}{u^3} e^{\frac{-iu}{2}} du$
- $-\frac{3\pi}{8} = \int_{-\infty}^{\infty} \frac{4(ucosu - sinu)}{u^3} \left(cos\frac{u}{2} - isin\frac{u}{2}\right) du,$
- F(u) is an even function and replace u as x
- $\frac{-3\pi}{8} = 2\int_0^{\infty} \frac{4(xcosx - sinx)}{x^3} \left(cos\frac{x}{2}\right) dx,$
- $\frac{-3\pi}{16} = \int_0^{\infty} \frac{4(xcosx - sinx)}{x^3} \left(cos\frac{x}{2}\right) dx,$

- Find the Fourier sine transform of $e^{-|x|}$ *and hence show that* $\int_0^\infty \frac{xsinmx}{1+x^2}dx = \frac{\pi e^{-m}}{2}, m > 0$.
- Solution:
- X being positive interval$(0,\infty), e^{-|x|} = e^{-x}$
- $F_s\big(f(x)\big) = \int_0^\infty f(x)sinuxdx$
- $= \int_0^\infty e^{-x}sinuxdx$
- $= \left[\frac{e^{-x}}{1+u^2}(-sinux - ucosux)\right]_0^\infty$
- $= 0 - \frac{e^{-0}}{1+u^2}(-sinu0 - ucosu0)$
- $= -\frac{1}{1+u^2}(-u) = \frac{U}{1+u^2}$
- Apply inverse Fourier sine transform
- $f(x) = \frac{2}{\pi}\int_0^\infty F_s\big(f(x)\big)cosuxdu$
- $e^{-x} = \frac{2}{\pi}\int_0^\infty \frac{U}{1+u^2}cosuxdx$ put x = m
- $e^{-m} = \frac{2}{\pi}\int_0^\infty \frac{U}{1+u^2}cosumdu$, *Replace u as x*
- $\frac{\pi}{2}e^{-m} = \int_0^\infty \frac{x}{1+x^2}cosmxdx$

- Find the Fourier sine and cosine transform of $f(x) = \begin{cases} 1, & 0 \le x < 2 \\ 0, & x \ge 2 \end{cases}$
- Solution:
- $F_s\big(f(x)\big) = \int_0^\infty f(x)sinuxdx$
- $= \int_0^2 sinuxdx = \left[\frac{-cosux}{u}\right]_0^2 = \frac{-cos2u+cos0}{u} = \frac{1-cos2u}{u}$
- $F_c\big(f(x)\big) = \int_0^\infty f(x)cosuxdx$
- $= \int_0^2 cosuxdx = \left[\frac{sinux}{u}\right]_0^2 = \frac{sin2u-sin0}{u} = \frac{sin2u}{u}$
- Find the Fourier Cosine transform of $f(x) = \begin{cases} x, & 0 < x < 1 \\ 2-x, & 1 < x < 2 \\ 0, & x > 2 \end{cases}$
- Solution:
- $F_c\big(f(x)\big) = \int_0^\infty f(x)cosuxdx$
- $= \int_0^2 f(x)cosuxdx = \int_0^1 xcosuxdx + \int_1^2 (2-x)cosuxdx$
- $= \left[\frac{xsinux}{u} - \frac{-cosux}{u^2}\right]_0^1 + \left[\frac{(2-x)sinux}{u} - \frac{-(-cosux)}{u^2}\right]_1^2$
- $= \frac{sinu}{u} + \frac{cosu}{u^2} - 0 - \frac{cos0}{u^2} + \frac{(2-2)sin2u}{u} - \frac{cos2u}{u^2} - (\frac{(2-1)sinu}{u} - \frac{cosu}{u^2})$
- $= \frac{sinu}{u} + \frac{cosu}{u^2} - \frac{1}{u^2} - \frac{cos2u}{u^2} - \frac{sinu}{u} + \frac{cosu}{u^2}$
- $= 2\frac{cosu}{u^2} - \frac{cos2u}{u^2} - \frac{1}{u^2}$

- Find the Fourier cosine transform of $e^{-2x} + 4e^{-3x}$
- Solution:
- $F_c\big(f(x)\big) = \int_0^\infty f(x) cosuxdx$
- $= \int_0^\infty (e^{-2x} + 4e^{-3x}\ ) cosuxdx$
- $= \left[\frac{e^{-2x}}{(-2)^2+u^2}\left(-2cosux + usinux) + \frac{4e^{-3x}}{(-3)^2+u^2}(-3cosux + usinux)\right)\right]_0^\infty$
- $= 0 - \left[\frac{e^{-0}}{(-2)^2+u^2}\left(-2cosu0 + usin0) + \frac{4e^{-0}}{(-3)^2+u^2}(-3cosu0 + usinu0)\right)\right]$
- $= -\left[\frac{-2}{(-2)^2+u^2} + \frac{4}{(-3)^2+u^2}(-3)\right]$
- $= \left[\frac{2}{(-2)^2+u^2} + \frac{12}{(-3)^2+u^2}\right]$
- $= \left[\frac{2}{4+u^2} + \frac{12}{9+u^2}\right]$

- Find the Fourier cosine transform of $f(x) = \begin{cases} x \; for \; 0 < x < 1 \\ 2 - x \; for \; 1 < x < 2 \\ 0 \; for \; x > 2 \end{cases}$
- Solution:
- Fourier cosine transform of f(x)
- $F_c\big(f(x)\big) = \int_0^\infty f(x) cosuxdx$
- $= \int_0^1 xcosuxdx + \int_1^2 (2 - xcosuxdx + \int_2^\infty 0cosuxdx$
- $= \int_0^1 xcosuxdx + \int_1^2 (2 - x)cosuxdx + \int_2^\infty 0cosuxdx$
- $= \left[x\frac{sinux}{u} - \frac{-cosux}{u^2}\right]_0^1 + \left[(2-x)\frac{sinux}{u} - (-1)\frac{-cosux}{u^2}\right]_1^2$
- $= \frac{sinu}{u} + \frac{cosu}{u^2} - \frac{cos0}{u^2} + (2-2)\frac{sin2u}{u} - \frac{cos2u}{u^2} - (\frac{sinu}{u} - \frac{cosu}{u^2})$
- $= \frac{sinu}{u} + \frac{cosu}{u^2} - \frac{1}{u^2} - \frac{cos2u}{u^2} - \frac{sinu}{u} + \frac{cosu}{u^2}$
- $= \frac{2cosu}{u^2} - \frac{1}{u^2} - \frac{cos2u}{u^2}$

- Solve the integral equation $\int_0^\infty f(\theta)cos\alpha\theta d\theta = \begin{cases} 1-\alpha, & 0 \le \alpha \le 1 \\ 0, & \alpha > 1 \end{cases}$ $Hence\ evaluate\ \int_0^\infty \frac{sin^2 t}{t^2} dt$
- Solution:
- $\int_0^\infty f(\theta)cos\alpha\theta d\theta = F_c(\alpha) = \begin{cases} 1-\alpha, & 0 \le \alpha \le 1 \\ 0, & \alpha > 1 \end{cases} \quad ----(1)$
- By inverse formula , we have
- $f(\theta) = \frac{2}{\pi}\int_0^\infty F_c(\alpha)cos\alpha\theta d\alpha = \frac{2}{\pi}\int_0^1 (1-\alpha)\ cos\alpha\theta d\alpha$
- $= \frac{2}{\pi}\left[(1-\alpha)\frac{sin\alpha\theta}{\theta} + \frac{-cos\alpha\theta}{\theta^2}\right]_0^1$
- $= \frac{2}{\pi}\left[(1-1)\frac{sin\theta}{\theta} + \frac{-cos\theta}{\theta^2} - \left[\frac{sin0\theta}{\theta} + \frac{-cos0\theta}{\theta^2}\right]\right]$
- $= \frac{2}{\pi}\left[\frac{1-cos\theta}{\theta^2}\right] = \frac{2}{\pi}\left[\frac{2sin^2\left(\frac{\theta}{2}\right)}{\theta^2}\right] = \frac{1}{\pi}\left[\frac{sin^2\left(\frac{\theta}{2}\right)}{\left(\frac{\theta}{2}\right)^2}\right] -------(2)$
- Sub (2) in (1)
- $\int_0^\infty \frac{1}{\pi}\left[\frac{sin^2\left(\frac{\theta}{2}\right)}{\left(\frac{\theta}{2}\right)^2}\right] cos\alpha\theta d\theta = F_c(\alpha) = \begin{cases} 1-\alpha, & 0 \le \alpha \le 1 \\ 0, & \alpha > 1 \end{cases}$
- let $\alpha = 0$ and $\frac{\theta}{2} = t\ and\ 2dt = d\theta$
- $\int_0^\infty \left[\frac{sin^2(t)}{t^2}\right] dt = \frac{\pi}{2}$