Z-Transform

Defintion:

$$z_T(u_n) = u(z) = \sum_{n=0}^{\infty} \frac{u_n}{z^n} = \sum_{n=0}^{\infty} u_n z^{-n}$$

Application of Z-Transform

To identify the stability of the system by Region of converges using Z-Transform.

Damping Rule

If$z_T(u_n) = \bar{u}(z)$ $then\ for\ a\ constant\ a \neq 0$

$$i)\ z_T\,(a^{-n}u_n) = u(az)$$

$$ii)\ z_T\,(a^n u_n) = u(z/a)$$

Shifting Rules

$If\ k > 0\ then$

$$i)\ z_T\,(u_{n-k}) = z^{-k}z(u_n)\ for\ n \geq k\ , and$$

$$ii)\ z_T\,(u_{n+k}) = z^k\{z(u_n) - \sum_{n=0}^{\infty} u_n z^{-n}\}$$

Note:

i) $z_T\,(u_{n-1}) = \frac{1}{z}z(u_n)\ for\ n \geq 1$

$ii)\ z_T\,(u_{n-2}) = \frac{1}{z^2}z(u_n)\ for\ n \geq 2$

$iii)\ z_T\,(u_{n-3}) = \frac{1}{z^3}z(u_n)\ for\ n \geq 3$

iv) $z_T\,(u_{n+1}) = z\{z(u_n) - u_0\}$

$v)\ z_T\,(u_{n+2}) = z^2\left\{z(u_n) - u_0 - \frac{u_1}{z}\right\}$

$iv)\ z_T\,(u_{n+3}) = z^3\left\{z(u_n) - u_0 - \frac{u_1}{z} - \frac{u_1}{z^2}\right\}$

Initial Value Theorem

$$\text{If} z_T(u_n) = \bar{u}(z) then \lim_{z\to\infty} \bar{u}(z) = u_0$$

Note:

$$\lim_{z\to\infty}\{z[\bar{u}(z) - u_0]\} = u_1$$

$$\lim_{z\to\infty}\left\{z^2\left[\bar{u}(z) - u_0 - \frac{u_1}{z}\right]\right\} = u_2 \ \ and\ so\ on$$

Final Value Theorem

$$\text{If} z_T(u_n) = \bar{u}(z) then \lim_{z\to\infty} u_n = \lim_{z\to 1}(z-1)\,\bar{u}(z)$$

Determine the z- transform of the sequence $\{u_n\} = \{k^n\}$

Solution:

By definition

- $z_T(k^n) = u(z) = \sum_{n=0}^{\infty} \frac{k^n}{z^n} = \sum_{n=0}^{\infty} \left(\frac{k}{z}\right)^n$

$$= \left(\frac{k}{z}\right)^0 + \left(\frac{k}{z}\right)^1 + \left(\frac{k}{z}\right)^2 + \left(\frac{k}{z}\right)^3 + \ldots\ldots$$

$$= 1 + \left(\frac{k}{z}\right)^1 + \left(\frac{k}{z}\right)^2 + \left(\frac{k}{z}\right)^3 + \ldots\ldots = \left(1 - \frac{k}{z}\right)^{-1}$$

$$= \left(\frac{z-k}{z}\right)^{-1} = \frac{z}{z-k}, \qquad |z| > |k|$$

- If K=1, $z_T(1^n) = \frac{z}{z-1}$

Transform of e^{an}

- $z_T(e^{an}) = z_T\left((e^a)^n\right) = \frac{z}{z-e^a}$

Transform of cosh$n\theta$ and$sinhn\theta$

- $\cosh n\theta = \frac{1}{2}\left(e^{n\theta} + e^{-n\theta}\right)$
- $sinhn\theta = \frac{1}{2}\left(e^{n\theta} - e^{-n\theta}\right)$
- $z_T(\cosh n\theta) = \frac{1}{2} z_T\left(e^{n\theta} + e^{-n\theta}\right)$
- $= \frac{1}{2}\left(\frac{z}{z-e^{\theta}} + \frac{z}{z-e^{-\theta}}\right)$
- $= \frac{1}{2}\left(\frac{z\left(z-e^{-\theta}\right)+z(z-e^{\theta})}{(z-e^{\theta})(z-e^{-\theta})}\right)$
- $= \frac{1}{2}\left(\frac{z^2-ze^{-\theta}+z^2-ze^{\theta}}{z^2-ze^{-\theta}-ze^{\theta}+1}\right)$
- $= \frac{z}{2}\left(\frac{2z-(e^{-\theta}+e^{\theta})}{z^2-z(e^{-\theta}+e^{\theta})+1}\right)$
- $= \frac{z(z-cosh\theta)}{z^2-2zcosh\theta+1}$
- $z_T(\sinh n\theta) = \frac{zsinh\theta}{z^2-2zcosh\theta+1}$

Transform of cos$n\theta$ and$sinn\theta$

- $\cos n\theta = \frac{1}{2}\left(e^{in\theta} + e^{-in\theta}\right)$
- $sinn\theta = \frac{1}{2}\left(e^{in\theta} - e^{-in\theta}\right)$
- $z_T(\cos n\theta) = \frac{1}{2} z_T\left(e^{in\theta} + e^{-in\theta}\right)$
- $= \frac{z(z-cos\theta)}{z^2-2zcos\theta+1}$
- $z_T(sinn\theta) = \frac{1}{2} z_T\left(e^{in\theta} - e^{-in\theta}\right)$
- $= \frac{zsin\theta}{z^2-2zcos\theta+1}$

By putting $\theta = \frac{\pi}{2}$ *, we get*

- $Z\left(\cos n\frac{\pi}{2}\right) = \frac{z^2}{z^2+1}$
- $Z\left(sinn\frac{\pi}{2}\right) = \frac{z}{z^2+1}$

- $Transform\ of\ n^p$
- Let p be the positive integer, show that $z_T(n^p) = -z\frac{d}{dz}\{z_T(n^{p-1})\}$

Proof:

- Using R.H.S$z_T(n^{p-1}) = \sum_{n=0}^{\infty} n^{p-1} z^{-n}$
- $\frac{d}{dz}\{z_T(n^{p-1})\} = \sum_{n=0}^{\infty} n^{p-1}(-n)z^{-n-1}$
- Multiply –z both sides we get,
- $-z\frac{d}{dz}\{z_T(n^{p-1})\} = \sum_{n=0}^{\infty} n^p z^{-n}$
- $z(n^p) = -z\frac{d}{dz}\{z_T(n^{p-1})\}$
- This is a recurrence relation from which $z_T(n^p) can\ be\ computed\ if z_T(n^{p-1})$ is known.

- For p=1, $z(n) = -z\frac{d}{dz}\{z(1)\}$
- $= -z\frac{d}{dz}\left(\frac{z}{z-1}\right) = -z\left(\frac{1.(z-1)-z.1}{(z-1)^2}\right)$
- $= \frac{z}{(z-1)^2}$
- For p=2, $z(n^2) = -z\frac{d}{dz}\{z_T(n)\}$
- $= -z\frac{d}{dz}\left(\frac{z}{(z-1)^2}\right) = \frac{z(z+1)}{(z-1)^3}$
- For p=3, $z(n^3) = -z\frac{d}{dz}\{z(n^2)\}$
- $= -z\frac{d}{dz}\left(\frac{z(z+1)}{(z-1)^3}\right) = \frac{z\left(z^2+4z+1\right)}{(z-1)^4}$
- And so on

u_n	$z(u_n) = \sum_{n=0}^{\infty} u_n z^{-n}$
1	$\frac{z}{z-1}$
$(-1)^n$	$\frac{z}{z+1}$
$(-1)^n(a)^n$	$\frac{z}{z+a}$
$(-1)^n(e)^{an}$	$\frac{z}{z-e^a}$
$(a)^n$	$\frac{z}{z-a}$
$coshn\theta$	$\frac{z(z-cosh\theta)}{z^2-2zcosh\theta+1}$
$sinhn\theta$	$\frac{zsinh\theta}{z^2-2zcosh\theta+1}$
$cosn\theta$	$\frac{z(z-cos\theta)}{z^2-2zcos\theta+1}$

$sinn\theta$	$\frac{zsin\theta}{z^2-2zcos\theta+1}$
n	$\frac{z}{(z-1)^2}$
n^2	$\frac{z(z+1)}{(z-1)^3}$
n^3	$\frac{z(z^2+4z+1)}{(z-1)^4}$
$\cos\frac{n\pi}{2}$	$\frac{z^2}{z^2+1}$
$sin\frac{n\pi}{2}$	$\frac{z}{z^2+1}$
$(k)^n n$	$\frac{kz}{(z-k)^2}$
$(k)^n n^2$	$\frac{k^2z+z^2k}{(z-k)^3}$

Find the Z- transforms of the following:

$$i)\ u_n = \frac{1}{n}, n > 0 \quad ii)\ u_n = \frac{1}{n+1}, n \geq 0$$

$$iii)\ u_n = \frac{1}{n(n+1)}, n > 0, \qquad iv)\ u_n = \frac{1}{n!}, n \geq 0$$

- i) for n>0,
- $z_T\left(\frac{1}{n}\right) = \sum_{n=1}^{\infty} \frac{1}{n} z^{-n}$
- $= z^{-1} + \frac{1}{2} z^{-2} + \frac{1}{3} z^{-3} + \frac{1}{4} z^{-4} + \ldots.$
- $= -\frac{\left(-\frac{1}{z}\right)}{1} + \frac{\left(-\frac{1}{z}\right)^2}{2} - \frac{\left(-\frac{1}{z}\right)^3}{3} + \frac{\left(-\frac{1}{z}\right)^4}{4} + \ldots$
- $= -log\left(1 - \frac{1}{z}\right)$
- $= log\left(\frac{z-1}{z}\right)^{-1} = log\left(\frac{z}{z-1}\right)$

- ii) for n ≥ 0,
- $z_T\left(\frac{1}{n+1}\right) = \sum_{n=0}^{\infty} \frac{1}{n+1} z^{-n}$
- $= 1 + \frac{1}{2} z^{-1} + \frac{1}{3} z^{-2} + \frac{1}{4} z^{-3} + \ldots.$
- $= z\left(z^{-1} + \frac{1}{2} z^{-2} + \frac{1}{3} z^{-3} + \frac{1}{4} z^{-4} + \ldots.\right)$
- Using previous problem, the above step can be written as
- $= zlog\left(\frac{z}{z-1}\right)$

- iii) for n > 0,
- $z_T\left(\frac{1}{n(n+1)}\right) = z_T\left(\frac{(n+1)-n}{n(n+1)}\right)$
- $= z_T\left(\frac{(n+1)}{n(n+1)} - \frac{n}{n(n+1)}\right)$
- $= z_T\left(\frac{1}{n} - \frac{1}{n+1}\right)$
- $= z_T\left(\frac{1}{n}\right) - z_T\left(\frac{1}{n+1}\right)$
- =$log\left(\frac{z}{z-1}\right) - zlog\left(\frac{z}{z-1}\right)$
- =$(1-\text{z})log\left(\frac{z}{z-1}\right)$
- $iv)\ for\ n \geq 0, z_T\left(\frac{1}{n!}\right) = \sum_{n=1}^{\infty}\frac{1}{n!}z^{-n} = e^{\frac{1}{z}}$

- Find the z-transforms of
- $i)\ \left(\frac{1}{2}\right)^n + \left(-\frac{1}{3}\right)^n + (-a)^{3n-2}$
- Solution:
- $z_T\left(\frac{1}{2}\right)^n + z\left(-\frac{1}{3}\right)^n + \left((-a)^3\right)^n(-a)^{-2}$
- $= \frac{z}{z-\frac{1}{2}} + \frac{z}{z-\left(-\frac{1}{3}\right)} + \frac{a^{-2}z}{z-(-a)^3}$
- $= \frac{2z}{2z-1} + \frac{3z}{3z+1} + \frac{a^{-2}z}{z+a^3}$

- Find the z-transforms of
- $i)\ (3n+5)^2\ ii)\left(2n-\frac{1}{3}\right)^3$
- Solution:
- $i) z_T(9n^2+30n+25)=9\frac{z(z+1)}{(z-1)^3}+30\frac{z}{(z-1)^2}+25\frac{z}{z-1}$
- $z_T\left(2n-\frac{1}{3}\right)^3=z\left(8n^3-4n^2+\frac{4n}{3}-\frac{1}{27}\right)$
- $=8\frac{z(z^2+4z+1)}{(z-1)^4}-4\frac{z(z+1)}{(z-1)^3}+\frac{4}{3}\frac{z}{(z-1)^2}-\frac{1}{27}\frac{z}{z-1}$
- Find the z-transforms of
- $i) n-sin\left(\frac{n\pi}{2}+\frac{\pi}{4}\right)+\left(-\frac{1}{3}\right)^3$
- Solution:
- $i) z_T\left(n-sin\left(\frac{n\pi}{2}+\frac{\pi}{4}\right)+\left(-\frac{1}{3}\right)^3\ \right)=z_T\left[(n-sin\frac{n\pi}{2}cos\frac{\pi}{4}-cos\frac{n\pi}{2}sin\frac{\pi}{4}+\left(-\frac{1}{3}\right)^3\right]$
- $=\frac{z}{(z-1)^2}-\frac{zsin\frac{\pi}{2}cos\frac{\pi}{4}}{z^2-2zcos\frac{\pi}{2}+1}-\frac{z(z-cos\frac{\pi}{2})sin\frac{\pi}{4}}{z^2-2zcos\frac{\pi}{2}+1}+\frac{\left(-\frac{1}{3}\right)^3 z}{z-1}=\frac{z}{(z-1)^2}-\frac{\frac{z}{\sqrt{2}}}{z^2+1}-\frac{z^2\frac{1}{\sqrt{2}}}{z^2+1}+\frac{\left(-\frac{1}{3}\right)^3 z}{z-1}$
- $=\frac{z}{(z-1)^2}-\frac{z}{\sqrt{2}(z^2+1)}-\frac{z^2}{\sqrt{2}(z^2+1)}+\frac{\left(-\frac{1}{3}\right)^3 z}{z-1}$

- Find the z-transforms of $a^n \cos n\theta + e^{-an} sinn\theta + e^{-an} n^2 + a^{-n} n$
- Solution:
- *By damping rule* $z_T\ (a^n u_n) = u(az), z_T\ (a^n u_n) = u(z/a)$
- $z_T(a^n \cos n\theta + e^{-an} sinn\theta + e^{-an} n^2 + a^{-n} n)$
- $= \frac{z(z-cos\theta)}{z^2-2zcos\theta+1}_{z\to\frac{z}{a}} - \frac{zsin\theta}{z^2-2zcos\theta+1}_{z\to ze^a} + \frac{z(z+1)}{(z-1)^3}_{z\to ze^a} + \frac{z}{(z-1)^2}_{z\to az}$
- $= \frac{\frac{z}{a}\left(\frac{z}{a}-cos\theta\right)}{\left(\frac{z}{a}\right)^2-2\frac{z}{a}cos\theta+1} - \frac{ze^a sin\theta}{(ze^a)^2-2ze^a cos\theta+1} + \frac{ze^a(ze^a+1)}{(ze^a-1)^3} + \frac{az}{(az-1)^2}$
- $= \frac{z\left(z-a^2cos\theta\right)}{z^2-2azcos\theta+a^2} - \frac{ze^a sin\theta}{(ze^a)^2-2ze^a cos\theta+1} + \frac{ze^a(ze^a+1)}{(ze^a-1)^3} + \frac{az}{(az-1)^2}$

- Obtain the inverse Z transform of $\frac{3z^2+z}{(5z-1)(5z+2)}$
- Solution:
- $\frac{3z^2+z}{(5z-1)(5z+2)} = \frac{Az}{5z-1} + \frac{Bz}{5z+2}$
- $3z^2 + z = Az(5z+2) + Bz(5z-1)$
- $3z + 1 = A(5z+2) + B(5z-1)$
- $put\ z = \frac{1}{5}, we\ get \frac{3}{5} + 1 = A(1+2)$
- $\frac{8}{5} = 3A\ implies\ A = \frac{8}{15}$
- $put\ z = \frac{-2}{5}, we\ get \frac{3*(-2)}{5} + 1 = B(-2-1)$
- $\frac{1}{5} = 3B\ implies\ B = \frac{1}{15}$
- $\frac{3z^2+z}{(5z-1)(5z+2)} = \frac{\frac{8}{15}}{5(z-\frac{1}{5})} + \frac{\frac{1}{15}}{5(z+\frac{2}{5})}$
- Apply inverse Z-Transform, we get
- $u_n = \frac{8}{75}\left(\frac{1}{5}\right)^n + \frac{2}{75}\left(-\frac{2}{5}\right)^n$

- Obtain the inverse Z transform of $\frac{2z^2-7z+7}{(z-1)^2(z-2)}$
- Solution:
- $\frac{2z^2-7z+7}{(z-1)^2(z-2)} = \frac{A}{z-1} + \frac{B}{(z-1)^2} + \frac{C}{z-2}$
- $2z^2 - 7z + 7 = A(z-2)(z-1) + B(z-2) + C((z-1)^2$
- $put\ z = 2, we\ get\ 1 =$C
- $put\ z = 1, we\ get\ -2 =$B,
- Equating coefficient of z^2, we get 2=A + C implies A =1
- $\frac{2z^2-7z+7}{(z-1)^2(z-2)} = \frac{1}{z-1} + \frac{-2}{(z-1)^2} + \frac{1}{z-2} = \frac{1}{z}\frac{z}{z-1} + \frac{1}{z}\frac{-2z}{(z-1)^2} + \frac{1}{z}\frac{z}{z-2}$
- Apply inverse Z transform, we get
- $z_T^{-1}\left(\frac{2z^2-7z+7}{(z-1)^2(z-2)}\right) = 1^{n-1} - 2n(1^{n-1}) + (2^{n-1}) = \begin{cases} 0, & n = 0 \\ 1 - 2n + 2^{n-1} & n \geq 1 \end{cases}$

- Using Z-transform methods,
- solve the difference equation $y_{n+2} - 2y_{n+1} + y_n = 0$subject to $y_0 = 0, y_1 = 1$
- Solution:
- Apply Z-Transform
- $z_T(y_{n+2} - 2y_{n+1} + y_n) = 0$
- $z^2\left\{\overline{y(z)} - y_0 - \frac{y_1}{z}\right\} - 2z\{\overline{y(z)} - y_0\} + \overline{y(z)} = 0$
- $sub \quad y_0 = 0, y_1 = 1$
- $z^2\left\{\overline{y(z)} - \frac{1}{z}\right\} - 2z\{\overline{y(z)}\} + \overline{y(z)} = 0$
- $\overline{y(z)}(z^2 - 2z + 1) - z = 0$
- $\overline{y(z)} = \frac{z}{(z-1)^2}$
- Apply inverse Z-Transform, we get $y_n = n$

- Using Z-transform methods,
- solve the difference equation $2y_{n+2} - 5y_{n+1} - 3y_n = 0$subject to $y_0 = 3, y_1 = 2$
- Solution:
- Apply Z-Transform
- $z_T(2y_{n+2} - 5y_{n+1} - 3y_n) = 0$
- $2z^2\left\{\overline{y(z)} - y_0 - \frac{y_1}{z}\right\} - 5z\{\overline{y(z)} - y_0\} - 3\overline{y(z)} = 0$
- $sub \quad y_0 = 3, y_1 = 2$
- $2z^2\left\{\overline{y(z)} - 3 - \frac{2}{z}\right\} - 5z\{\overline{y(z)} - 3\} - 3\overline{y(z)} = 0$
- $\overline{y(z)}(2z^2 - 5z - 3) - 6z^2 - 4z + 15z = 0$
- $\overline{y(z)}(2z + 1)(z - 3) = 6z^2 - 11z$
- $\overline{y(z)} = \frac{6z^2-11z}{(2z+1)(z-3)} = \frac{Az}{2z+1} + \frac{Bz}{z-3}$
- Dividing Z
- $\frac{6z-11}{(2z+1)(z-3)} = \frac{A}{2z+1} + \frac{B}{z-3}$
- $6z - 11 = A(z - 3) + B(2z + 1)$
- Putting z=3, $7 = 7\text{B} \rightarrow B = 1$
- $put\ z = \frac{-1}{2}, we\ get\ -3 - 11 = A(\frac{-1}{2} - 3)$
- $-14 = -\frac{7}{2}A\ implies\ A = 4$
- $\frac{6z^2-11z}{(2z+1)(z-3)} = \frac{4z}{2z+1} + \frac{z}{z-3}$
- $\overline{y(z)} = \frac{6z^2-11z}{(2z+1)(z-3)} = \frac{4z}{2(z+\frac{1}{2})} + \frac{z}{z-3}$
- Apply inverse Z-Transform, we get
- $y_n = 2\left(\frac{-1}{2}\right)^n + (3)^n$

- Using Z-transform methods,
- solve the difference equation $y_{n+2} - 5y_{n+1} + 6y_n = \left(\frac{1}{2}\right)^n$ subject to $y_0 = 0, y_1 = 0$
- Solution:
- Apply Z-Transform
- $z_T(y_{n+2} - 5y_{n+1} + 6y_n) = z_T\left(\frac{1}{2}\right)^n$
- $z^2\left\{\overline{y(z)} - y_0 - \frac{y_1}{z}\right\} - 5z\left\{\overline{y(z)} - y_0\right\} + 6\overline{y(z)} = \frac{z}{z-\frac{1}{2}}$
- Sub $y_0 = 0, y_1 = 0$
- $z^2\left\{\overline{y(z)}\right\} - 5z\left\{\overline{y(z)}\right\} + 6\overline{y(z)} = \frac{z}{z-\frac{1}{2}}$
- $\overline{y(z)} = \frac{z}{(z-\frac{1}{2})(z-3)(z-2)}$
- $\frac{z}{(z-\frac{1}{2})(z-3)(z-2)} = \frac{AZ}{(z-\frac{1}{2})} + \frac{BZ}{(z-3)} + \frac{CZ}{(z-2)}$
- Dividing Z
- $\frac{1}{(z-\frac{1}{2})(z-3)(z-2)} = \frac{A}{(z-\frac{1}{2})} + \frac{B}{(z-3)} + \frac{C}{(z-2)}$

$$1 = A(z-3)(z-2) + B\left(z-\frac{1}{2}\right)(z-2) + C\left(z-\frac{1}{2}\right)(z-3)$$

Sub z=1/2,

$$1 = A\left(\frac{1}{2}-3\right)\left(\frac{1}{2}-2\right) = A\frac{-5}{2} * \frac{-3}{2}$$

$$\frac{4}{15} = A$$

Sub z=2,

$$1 = C\left(2-\frac{1}{2}\right)(2-3) = C\frac{3}{2} * -1$$

$$-\frac{2}{3} = C$$

Sub z=3,

$$1 = B\left(3-\frac{1}{2}\right)(3-2) = B\frac{5}{2} * 1$$

$$\frac{2}{5} = B$$

$$\overline{y(z)} = \frac{z}{(z-\frac{1}{2})(z-3)(z-2)} = \frac{\frac{4}{15}Z}{(z-\frac{1}{2})} + \frac{\frac{2}{5}Z}{(z-3)} - \frac{\frac{2}{3}Z}{(z-2)}$$

Apply inverse Z-Transform, we get

$$y_n = \frac{4}{15}\left(\frac{1}{2}\right)^n + \frac{2}{5}(3)^n - \frac{2}{3}2^n$$

- Using Z-transform methods,
- solve the difference equation $2y_{n+2} - 3y_{n+1} - 2y_n = 6n + 1$ subject to $y_0 = 1, y_1 = 2$
- Solution:
- Apply Z-Transform
- $z_T(2y_{n+2} - 3y_{n+1} - 2y_n) = z_T(6n + 1)$
- $2z^2\left\{\overline{y(z)} - y_0 - \frac{y_1}{z}\right\} - 3z\left\{\overline{y(z)} - y_0\right\} - 2\overline{y(z)} = 6\frac{z}{(z-1)^2} + \frac{z}{z-1}$
- Sub $y_0 = 1, y_1 = 2$
- $2z^2\left\{\overline{y(z)} - 1 - \frac{2}{z}\right\} - 3z\left\{\overline{y(z)} - 1\right\} - 2\overline{y(z)} = \frac{6z+z^2-z}{(z-1)^2}$
- $\overline{y(z)}\,(2z^2 - 3z - 2) - 2z^2 - 4z + 3z = \frac{5z+z^2}{(z-1)^2}$
- $\overline{y(z)}\,(2z^2 - 3z - 2) = \frac{5z+z^2+(2z^2+z\,)(z-1)^2}{(z-1)^2}$
- $\overline{y(z)} = \frac{5z+z^2+(2z^2+z\,)(z^2-2z+1)}{(z-2)(2z+1)(z-1)^2}$
- $\overline{y(z)} = \frac{5z+z^2+(2z^4-4z^3+2z^2+z^3-2z^2+z\,)}{(z-2)(2z+1)(z-1)^2}$
- $\overline{y(z)} = \frac{2z^4-3z^3+z^2+6z}{(z-2)(2z+1)(z-1)^2}$
- Dividing Z
- $\frac{\overline{y(z)}}{z} = \frac{2z^3-3z^2+z+6}{(z-2)(2z+1)(z-1)^2} = \frac{A}{(z-1)} + \frac{B}{(z-1)^2} + \frac{C}{(z-2)} + \frac{D}{(2z+1)}$
- $2z^3 - 3z^2 + z + 6 = A(z-2)(2z+1)(z-1) + B(z-2)(2z+1) + C(z -$

- Using Z-transform methods,
- solve the difference equation $y_{n+2} + 2y_n = 0$ subject to $y_0 = 1, y_1 = \sqrt{2}$
- Solution:
- Apply Z-Transform
- $z_T(y_{n+2} + 2y_n) = z_T(0)$
- $z^2\left\{\overline{y(z)} - y_0 - \frac{y_1}{z}\right\} + 2\overline{y(z)} = 0$
- Sub $y_0 = 1, y_1 = \sqrt{2}$
- $z^2\left\{\overline{y(z)} - 1 - \frac{\sqrt{2}}{z}\right\} + 2\overline{y(z)} = 0$
- $\overline{y(z)}\,(z^2 + 2) = z^2 + \sqrt{2}z$
- $\overline{y(z)} = \frac{z^2}{z^2+2} + \frac{\sqrt{2}z}{z^2+2}$
- *We know that* $z_T\left(\cos\frac{n\pi}{2}\right) = \frac{z^2}{z^2+1}$ *and* $z_T\left(\sin\frac{n\pi}{2}\right) = \frac{z}{z^2+1}$
- $z_T\left(\sqrt{2}^n \cos\frac{n\pi}{2}\right) = \frac{z^2}{z^2+1}_{z\to z/\sqrt{2}} = \frac{z^2}{z^2+2}$, *and* $z_T\left(\sqrt{2}^n \sin\frac{n\pi}{2}\right) = \frac{\sqrt{2}z}{z^2+2}$
- $y_n = \sqrt{2}^n \cos\frac{n\pi}{2} + \sqrt{2}^n \sin\frac{n\pi}{2}$

If $\overline{u(z)} = \frac{3z^2-5z+7}{(z-2)^3}$, *find* u_0, u_1, u_2, u_3.

Solution:

$$u_0 = \lim_{z\to\infty} \bar{u}(z) = \lim_{z\to\infty} \frac{3z^2 - 5z + 7}{(z-2)^3} = \lim_{z\to\infty} \frac{z^2(3 - \frac{5}{z} + \frac{7}{z^2})}{z^3(1 - \frac{2}{z})^3} = 0.$$

$$u_1 = \lim_{z\to\infty}\{z[\bar{u}(z) - u_0]\} = \lim_{z\to\infty}\left\{z\left[\overline{\frac{3z^2 - 5z + 7}{(z-2)^3}} - 0\right]\right\}$$

$$= \lim_{z\to\infty} \frac{z^3(3 - \frac{5}{z} + \frac{7}{z^2})}{z^3(1 - \frac{2}{z})^3} = 3$$

$$u_2 = \lim_{z\to\infty}\left\{z^2\left[\bar{u}(z) - u_0 - \frac{u_1}{z}\right]\right\} = \lim_{z\to\infty}\left\{z^2\left[\frac{3z^2 - 5z + 7}{(z-2)^3} - 0 - \frac{3}{z}\right]\right\}$$

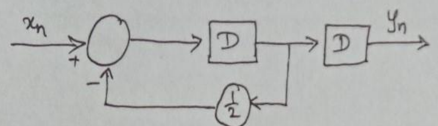
$$= \lim_{z\to\infty}\left\{z^2\left[\frac{3z^3 - 5z^2 + 7z - 3(z-2)^3}{z(z-2)^3}\right]\right\}$$

$$= \lim_{z\to\infty}\left\{z\left[\frac{\cancel{3z^3} - 5z^2 + 7z - 3\cancel{z^3} - 18z^2 - 36z + 24}{(z-2)^3}\right]\right\}$$

$$= \lim_{z\to\infty}\left\{z^3\left[\frac{13 - \frac{29}{z} + \frac{24}{z^2}}{z^3(1 - \frac{2}{z})^3}\right]\right\} = 13$$

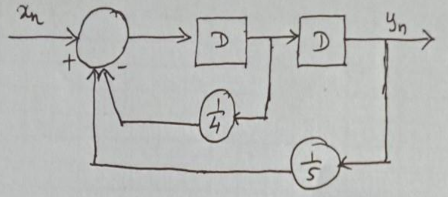
- Obtain the inverse Z- transform of $\frac{8z-z^3}{(4-z)^3}$
- Solution:
- $\frac{8z-z^3}{(4-z)^3} = \frac{z^3-8z}{(z-4)^3}$
- $\frac{z^3-8z}{(z-4)^3} = \frac{Az}{(z-4)}+\frac{B4z}{(z-4)^2}+\frac{C(4z^2+16z)}{(z-4)^3}$----(1)
- Dividing z, we get
- $\frac{z^2-8}{(z-4)^3} = \frac{A}{(z-4)}+\frac{B4}{(z-4)^2}+\frac{C(4z+16)}{(z-4)^3}$
- $\frac{z^2-8}{(z-4)^3} = \frac{A(z-4)^2+4B(z-4)+C(4z+16)}{(z-4)^3}$
- $z^2 - 8 = A(z-4)^2 + 4B(z-4) + C(4z+16)$
- Equating z^2, we get 1=A,
- Put z= 0 , we get -8=16A-16B+16C
- -8-16=-16B+16C
- -3=-2B+2C ------------(2)
- Equating z, we get
- 0=-8A+4B+4C
- 2=B+C -----------------(3)
- Solving 2 and 3 we get $C = \frac{1}{4}\ and\ B = \frac{7}{4}$
- Substituting
- $\frac{z^3-8z}{(z-4)^3} = \frac{z}{(z-4)}+\frac{7}{4}\frac{4z}{(z-4)^2}+\frac{1}{4}\frac{(4z^2+16z)}{(z-4)^3}$
- Apply inverse Z –transform, we get
- $z^{-1}\left(\frac{z^3-8z}{(z-4)^3}\right)=z^{-1}\left(\frac{z}{(z-4)}+\frac{7}{4}\frac{4z}{(z-4)^2}\right. +$

Find difference equations representing the discrete-time systems



Soln:

$y_{n+2} + \frac{1}{2} y_{n+1} = x_n$



Soln: $y_{n+2} = -\frac{1}{4} y_{n+1} + \frac{1}{5} y_n + x_n$

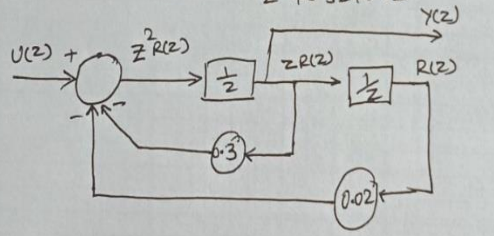
$y_{n+2} + \frac{1}{4} y_{n+1} - \frac{1}{5} y_k = x_n$

A System is specified by its transfer function

$G(Z) = \frac{Z}{Z^2 + 0.3Z + 0.02}$, Draw a block diagram to illustrate a time-domain realization of the system. Find a second structure that also implement the system.

Soln.

$G(z) = \frac{Y(z)}{U(z)} = \frac{z}{z^2+0.3z+0.02}$, $(z^2+0.3z+0.02)\,zR(z) = zU(z)$



$\frac{z}{(z+0.2)(z+0.1)} = \frac{A}{z+0.2} + \frac{B}{z+0.1} = \frac{2}{z+0.2} - \frac{1}{z+0.1}$