Time domain analysis:

Time Response:

The time response of the system is the output of the closed loop system as a function of time. denoted by c(t).

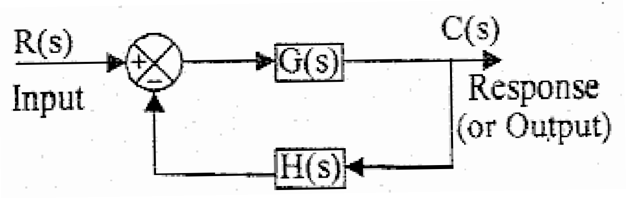
The time response can be obtained by solving the differential equation governing the system

Alternatively, the response c(t) can be obtained from the transfer function of the system and the input(excitation) to the system.

The closed loop transfer function $$\frac{C(s)}{R(s)} = \frac{G(s)}{1+G(s)\,H(s)} = M(s)$$

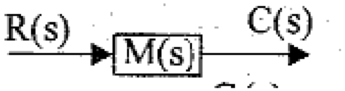
In any practical system, output of the system takes some finite time to reach to its final value. This time varies from system to system and is dependent on different factors. Similarly final value achieved by the output also depends on the different factors like friction, mass or inertia of moving elements, some nonlinearities present etc.

For example consider a simple ammeter as a system. It is connected in a system so as to measure current of magnitude 5A. Ammeter pointer hence must deflect to show us 5A reading on it. So 5A is its ideal value that it must show. Now pointer will take some finite time to stabilise to indicate some reading and after stabilising also, it depends on various factors like friction, pointer inertia etc. whether it will show us accurate 5 A or not.



The closed loop transfer function $\frac{C(s)}{R(s)} = \frac{G(s)}{1 + G(s)\,H(s)} = M(s)$

The Output or Response in s-domain, C(s) is given by the product of the transfer function and the input, R(s).



On taking inverse Laplace transform of this product the time domain response, C(t) can be obtained

Response in s-domain, C(s) = R(s) M(s)

where, $M(s) = \frac{G(s)}{1+G(s)H(s)}$

Response in time domain $c(t) = \mathcal{L}^{-1}\{C(s)\} = \mathcal{L}^{-1}\{R(s) \times M(s)\}$

The time response of a control system consists of two parts : **the transient** and the **steady state response.**

Transient Response

The transient response is the response of the system when the input changes from one state to another

The output variation during the time, it takes to achieve its final value is called transient response. The time required to achieve the final value is called transient period.

The transient response may be exponential or oscillatory. It is represented as $C_t(t)$.

To get the desired output, system must pass satisfactorily through transient period. Transient response must vanish after some time to get the final value closer to the desired value. Such systems in which transient response dies out after some time are called Stable Systems

From transient response can get following information about the system,

- When the system has started showing its response to the applied excitation
- What is the rate of rise of output ? From this, parameters of system can be designed which can withstand such rate of rise. It also gives indication about speed of the system
- Whether output is increasing exponentially or it is oscillating
- If output is oscillating, whether it is over shootting its final value.
- When it is settling down to its final value ?

All this information matters much at the time of designing the systems

Steady state response

It is that part of the time response which remains after complete transient response vanishes from the system output.

The **steady state response** is the response as time, t approaches infinity.

Steady state response is the final value achieved by the system output.

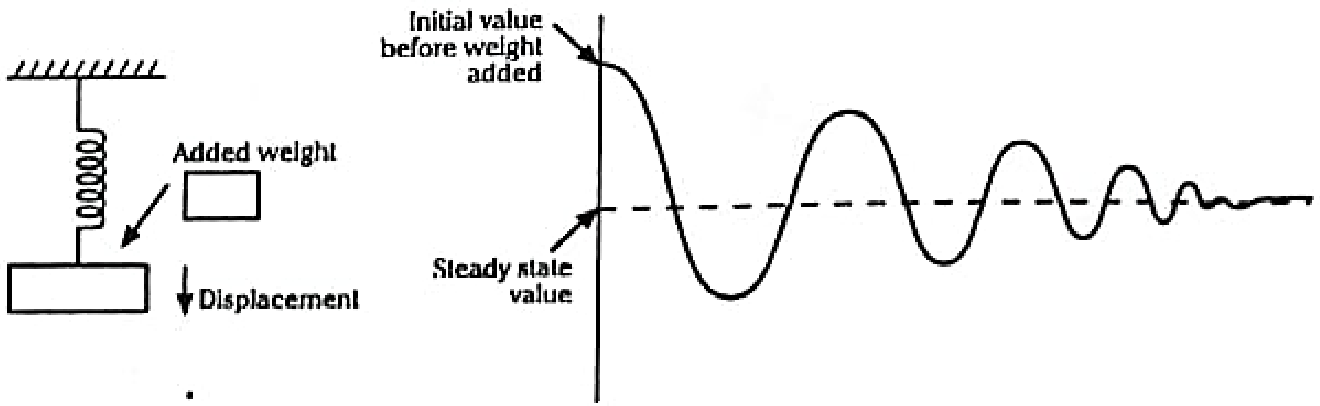
The steady state response indicates the accuracy of the system and it is denoted as C_{SS}.

From steady state response we get;

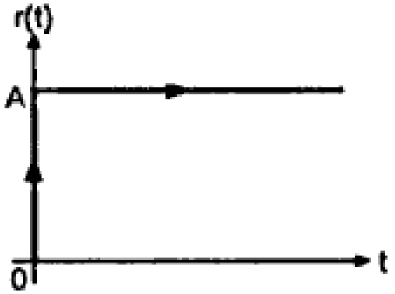
1. **How much away the system output is from its desired value which indicates error.**
2. **The error is constant or varying with the time**

Consider a vertically suspended spring and when the weight is added as shown in

The deflection of spring abruptly increases and oscillates for some time and then settles down to a steady value. The steady value is steady state response of the spring. The oscillation that occurs prior to this steady state is the transient response

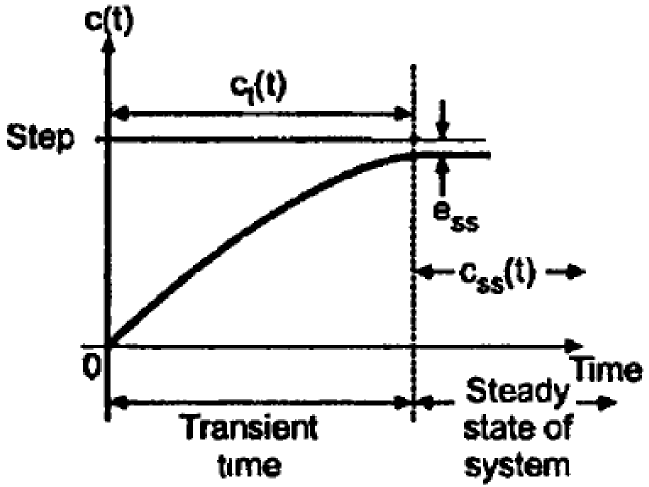


Transient and steady-state response of a spring system

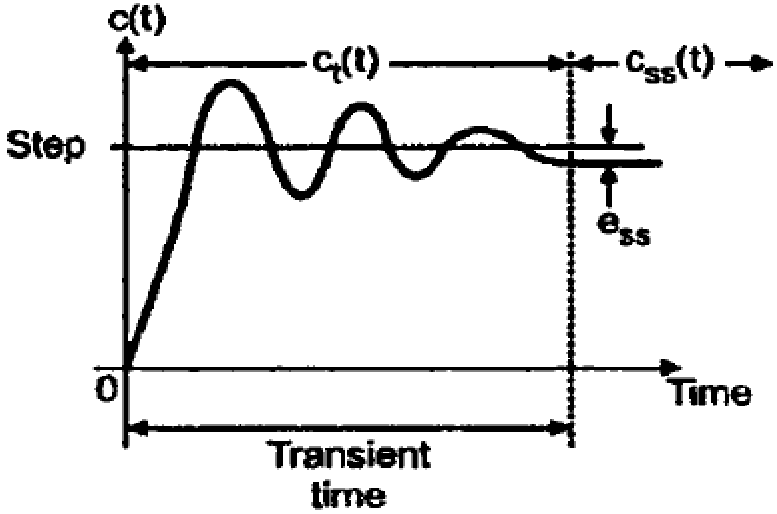


The step signal is a signal whose value changes from 0 to A and remains constant at A for t > 0.

Step Input



Response for 1st Order System for step Input



Response for 2nd Order System (Under Damped Case) for step Input

In Time response analysis of control system, the test input signals are applied and the response produced by various system to these input signals are compared and then the performance index is specified. Once the control system is designed based on the test input signals, the performance of the system to the actual input signals are generally found to be satisfactory.

The input signals to control systems are not known fully ahead of time, the characteristics of control system which suddenly strain a control system are:

a) Sudden shock

b) Sudden change

c) Constant velocity and acceleration

System dynamic behavior for analysis and design is therefore judged and compared under standard test signals.

The test signals can be easily generated in test laboratories and the characteristics of test signals resembles, the characteristics of actual input signals..

The test signals are used to predetermine the performance of the system. If the response of a system is satisfactory for a test signal, then the system will be suitable for practical applications

The commonly used test input signals in control system are Impulse, Step, Ramp, Acceleration and Sinusoidal signals.

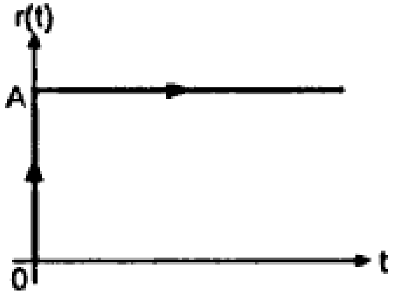
For analyzing transient response mainly step input signal is used and also other signals mainly ramp and parabolic are not used for this analysis but they are used for steady state analysis.

Standard Test Signals

1.Step Input (Position Function): Sudden application of input at a specified time.

The step signal is a signal whose value changes from 0 to A and remains constant at A for t > 0.

The step signal resembles an actual steady input to a system. A special case of step signal is unit step in which A is unity.



Mathematically it can be described as,

$$\boxed{\begin{aligned} r(t) &= A \quad && \text{for } t \geq 0 \\ &= 0 && \text{for } t < 0 \end{aligned}}$$

If A = 1, then it is called unit step function and denoted by u(t).

Laplace transform of such input is $\frac{A}{s}$.

If a system is subjected to sudden disturbance then the step input can be used as a test input signal.

Example.

If the input is the angular position of shaft, step input represents sudden rotation of the shaft

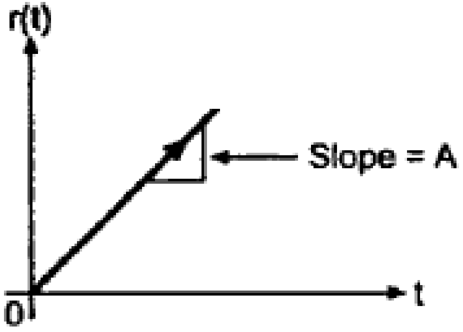
Change in fluid flow made available by the sudden opening of a value in a line from a pump

Voltage impressed upon an electric network when the network is suddenly connected to a battery by closing the switch

Ramp Input

The ramp signal is a signal whose value increases linearly with time from an initial value of zero at t= 0. The ramp signal resembles a constant velocity input to the system. A special case of ramp signal is unit ramp signal in which the value of A is unity

The signal will have constant change in value with respect to time i.e., it starts at zero and increases linearly with time



Magnitude of Ramp input is nothing but its slope. Mathematically it is defined as,

$$\boxed{\begin{aligned} r(t) &= At && \text{for } t \geq 0 \\ &= 0 && \text{for } t < 0 \end{aligned}}$$

If A = 1, it is called Unit Ramp input. It is denoted as r(t). Its Laplace transform is $\frac{A}{s^2}$.

Since the input to the system is unbounded, the output will also be unbounded and the system's response is said to be unstable. Ramp inputs are useful in determining the performance of certain system (eg : Machine tools, missiles).

If the input to a control systems are a gradually changing with respect to time, ramp input is a suitable test signal.

Example

If the input is of the form of angular displacement of the shaft, the ramp input represents constant speed rotation of the shaft. Similarly if a temperature is increasing at a constant rate of say 5°C per second then the quantity is said to have a velocity of this value.

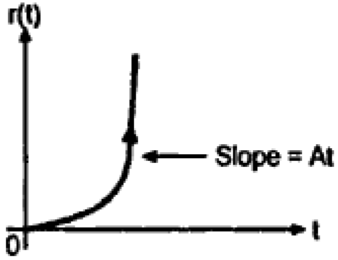
Unit ramp input employed as mathematical model of the input to a radar fire control system that is suddenly required to track a target moving with constant velocity.

Ramp signal denotes constant velocity and also basic definition states that its value increases linearly with time.

Parabolic Input

In parabolic signal, the instantaneous value varies as square of the time from an initial value of zero at t = 0. The sketch of the signal with respect to time resembles a parabola.

The parabolic signal resembles a constant acceleration input to the system. A special case of parabolic signal is unit parabolic signal in which A is unity



Mathematically this function is described as,

$$\boxed{\begin{aligned} r(t) &= \frac{A}{2}t^2, & \text{for } t \geq 0 \\ &= 0, & \text{for } t < 0 \end{aligned}}$$

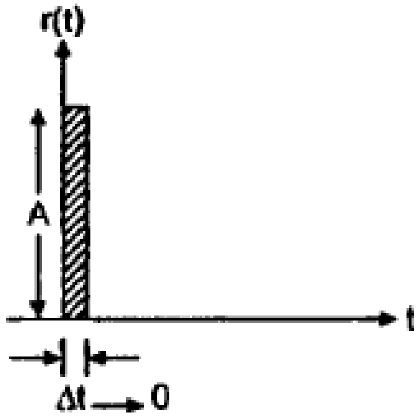
where A is called magnitude of the parabolic input.

If A = 1, i.e. $r(t) = \frac{t^2}{2}$ it is called unit parabolic input. Its Laplace transform is $\frac{A}{s^3}$.

Note : Integral of step signal is ramp signal. Integral of ramp signal is parabolic signal.

Impulse Input

A signal of very large magnitude which is available for very short duration is called impulse signal. Ideal impulse signal is a signal with infinite magnitude and zero duration but with an area of A. The unit impulse signal is a special case, in which A is unity.



Area of the impulse is nothing but its magnitude. If its area is unity it is called **Unit Impulse Input**, denoted as $\delta(t)$.

Mathematically it can be expressed as,

$$\therefore \quad \boxed{\begin{aligned} r(t) &= A, \quad \text{for } t = 0 \\ &= 0, \quad \text{for } t \neq 0 \end{aligned}}$$

It is a signal which has zero value everywhere except at t = 0, where its magnitude is infinity.

Since perfect impulse cannot be achieved in practice, it is approximated by a pulse of unit area having small width.

If the system is subjected to shock inputs, the impulse input can be used.

Pulse inputs are useful in dealing with an operating system because there is no prolonged disturbance of the output variable and also because pulses are readily imposed on the input variable.

For analyzing transient response mainly step input signal is used and also other signals mainly ramp and parabolic signal are not used for this analysis but they are used for steady state analysis.

Name of the signal	Time domain equation of signal, r(t)	Laplace transform of the signal, R(s)
Step	A	$\frac{A}{s}$
Unit step	1	$\frac{1}{s}$
Ramp	At	$\frac{A}{s^2}$
Unit ramp	t	$\frac{1}{s^2}$
Parabolic	$\frac{At^2}{2}$	$\frac{A}{s^3}$
Unit parabolic	$\frac{t^2}{2}$	$\frac{1}{s^3}$
Impulse	$\delta(t)$	1

ORDER OF A SYSTEM

The input and output relationship of a control system can be expressed by **nth** order differential equation shown in equation

$$a_0 \frac{d^n}{dt^n} p(t) + a_1 \frac{d^{n-1}}{dt^{n-1}} p(t) + a_2 \frac{d^{n-2}}{dt^{n-2}} p(t) + \ldots\ldots + a_{n-1} \frac{d}{dt} p(t) + a_n p(t) = b_0 \frac{d^m}{dt^m} q(t)$$
$$+ b_1 \frac{d^{m-1}}{dt^{m-1}} q(t) + b_2 \frac{d^{m-2}}{dt^{m-2}} q(t) + \ldots\ldots + b_{m-1} \frac{d}{dt} q(t) + b_m q(t)$$

Where p(t) is the Output/Response, q(t) is the Input/Excitation

The order of the system is given by the order of the differential equation governing the system. If the system is governed by **nth** order differential equation, then the system is called **nth** order system

Alternatively, the order can be determined from the transfer function of the system. The transfer function of the system can be obtained by taking Laplace transform of the differential equation governing the system and rearranging them as a ratio of two polynomials in s, as shown in equation

$$\text{Transfer function, } T(s) = \frac{C(s)}{R(s)} = \frac{b_0 s^m + b_1 s^{m-1} + b_2 s^{m-2} + \dots\dots + b_{m-1} s + b_m}{a_0 s^n + a_1 s^{n-1} + a_2 s^{n-2} + \dots\dots + a_{n-1} s + a_n}$$

Where C(s) is the Numerator polynomial , R(s) is the Denominator polynomial

The order of the system is given by the maximum power of s in the denominator polynomial

$$R(s) = a_0 s^n + a_1 s^{n-1} + a_2 s^{n-2} + \dots\dots\dots + a_{n-1} s + a_n.$$

n is the order of the system

Note : The order can be specified for both open loop system and closed loop system

The numerator and denominator polynomial of equations can be expressed in the factorized form as shown in below equation

$$T(s)=\frac{C(s)}{R(s)}=\frac{(s+z_1)(s+z_2).......(s+z_m)}{(s+p_1)(s+p_2).......(s+p_n)}$$

Z1, Z2,…… Zm are the Zeros of the system

P1, P2, ...Pn are poles of the system

Order of the system is equal to the number of poles of the transfer function

Denominator of the Transfer function represents Characteristic Equation. Consequently if all the roots of the Characteristic equation have negative real part, the system is stable

Poles and Zeros. The roots of the denominator polynomial R(s) are the poles of G(s).

The roots of the numerator polynomial C(s) are zeros of G(s).

$$G(s)=\frac{s+2}{s^2+2s+2}=\frac{s+2}{(s+1)^2+1}=\frac{s+2}{(s+1-j)(s+1+j)}$$

The zeroes can be calculated by equating the numerator to zero:

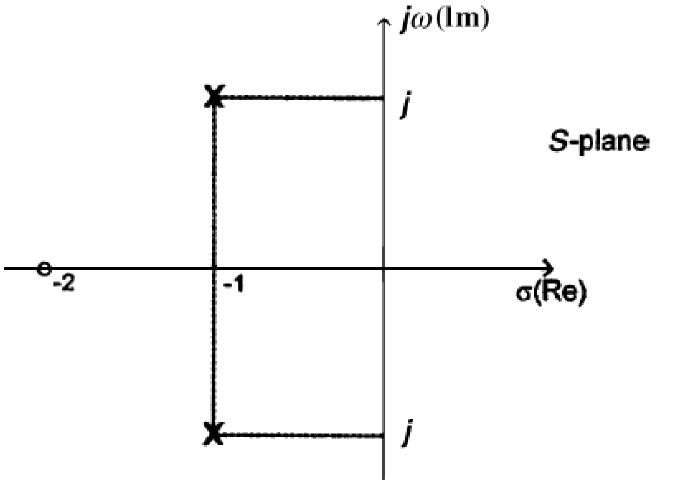
The poles can be calculated by equating the denominator to zero

G(s) has one zero at −2 and two complex conjugate poles at $-1 \pm j$

In general, a transfer function with m zeros and n poles can be written as

$$G(s)=k\frac{(s-z_1)\quad(s-z_m)}{(s-p_1)\quad(s-p_n)}$$

where k is the gain.



Complex conjugate poles of $G(s)$.

Example

Let, $T(s) = \dfrac{2}{s(s+1)(s+2)}$

By partial fraction expansion, T(s) can be expressed as,

$$T(s) = \frac{2}{s(s+1)(s+2)} = \frac{A}{s} + \frac{B}{s+1} + \frac{C}{s+2}$$

A is obtained by multiplying T(s) by s and letting s = 0.

$$A = T(s) \times s\Big|_{s=0} = \frac{2}{s(s+1)(s+2)} \times s\Big|_{s=0} = \frac{2}{(s+1)(s+2)}\Bigg|_{s=0} = \frac{2}{1\times 2} = 1$$

B is obtained by multiplying T(s) by (s +1) and letting s = –1.

$$B = T(s) \times (s+1)\Big|_{s=-1} = \frac{2}{s(s+1)(s+2)} \times (s+1)\Big|_{s=-1} = \frac{2}{s(s+2)}\Bigg|_{s=-1} = \frac{2}{-1(-1+2)} = -2$$

C is obtained by multiplying T(s) by (s +2) and letting s = –2.

$$C = T(s) \times (s+2)\Big|_{s=-2} = \frac{2}{s(s+1)(s+2)} \times (s+2)\Big|_{s=-2} = \frac{2}{s(s+1)}\Bigg|_{s=-2} = \frac{2}{-2(-2+1)} = +1$$

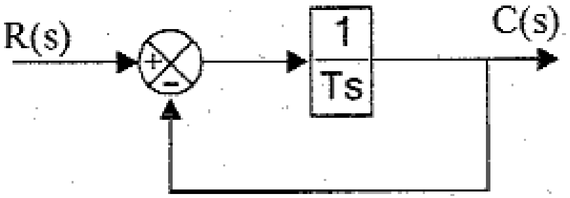
$$\therefore T(s) = \frac{2}{s(s+1)(s+2)} = \frac{1}{s} - \frac{2}{s+1} + \frac{1}{s+2}$$

Time Response of First Order Control Systems

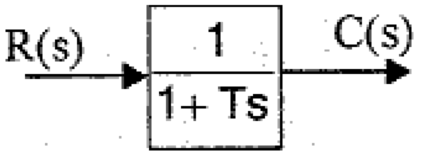
Only one energy storage element, and thus could be modeled by a first-order differential equation

When the maximum power of s in the denominator of a transfer function is one, the transfer function represents a first order control system.

The closed loop order system with unity feedback is shown in fig



$\Rightarrow$



Closed loop first order system. The closed loop transfer function of first order system, $\frac{C(s)}{R(s)} = \frac{1}{1+Ts}$

If the input is unit step then, $r(t) = 1$ and $R(s) = 1/S$

T is called Time constant of the system

$$\frac{C(s)}{R(s)} = \frac{1}{1+Ts}$$

R(s) =1/S

The response in s-domain, $C(s) = R(s)\frac{1}{(1+Ts)} = \frac{1}{s}\frac{1}{(1+Ts)}$

By partial fraction expansion

$$\frac{1}{s(Ts+1)} = \frac{A}{s} + \frac{B}{Ts+1}$$

$$1 = ATs + A + Bs$$

$$A = 1$$

$$AT + B = 0 \quad \text{or} \quad B = -AT = -T$$

$$C(s) = \frac{1}{S} - \frac{T}{Ts+1} = \frac{1}{S} - \frac{1}{(s+1/T)}$$

The response in time domain

$$C(s) = \frac{1}{S} - \frac{1}{(s+1/T)}$$

The response in time domain

$$\mathcal{L}\{e^{-at}\} = \frac{1}{s+a}$$

$$c(t) = \mathcal{L}^{-1}\{C(s)\} = \mathcal{L}^{-1}\left\{\frac{1}{s} - \frac{1}{s+\frac{1}{T}}\right\} = 1 - e^{-\frac{t}{T}}$$ ....Equation (1)

The response of the closed loop first order system for unit step input.

For step input of step value, A, the equation (1) is multiplied by A

For closed loop first order system, Unit step response $= 1 - e^{-\frac{t}{T}}$

For closed loop first order system Step response $= A\left(1 - e^{-\frac{t}{T}}\right)$

For closed loop first order system, Unit step response $= 1 - e^{-\frac{t}{T}}$

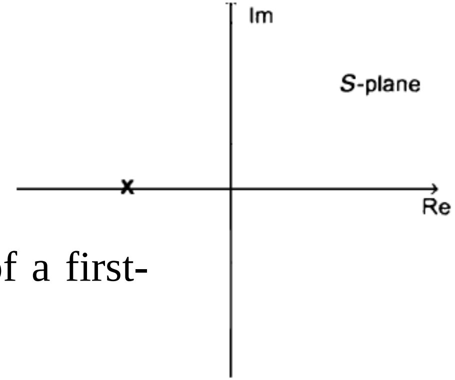
When

$$
\begin{aligned}
t &= 0, & c(t) &= 1 - e^{0} = 0 \\
t &= 1T, & c(t) &= 1 - e^{-1} = 0.632 \\
t &= 2T, & c(t) &= 1 - e^{-2} = 0.865 \\
t &= 3T, & c(t) &= 1 - e^{-3} = 0.95 \\
t &= 4T, & c(t) &= 1 - e^{-4} = 0.9817 \\
t &= 5T, & c(t) &= 1 - e^{-5} = 0.993 \\
t &= \infty, & c(t) &= 1 - e^{-\infty} = 1
\end{aligned}
$$

Here T is called Time constant of the system. In a time of 5T, the system is assumed to have attained steady state

The input and output signal of the first order system is shown in fig

Response of First order control Systems to unit step input



Pole location of a first-order system

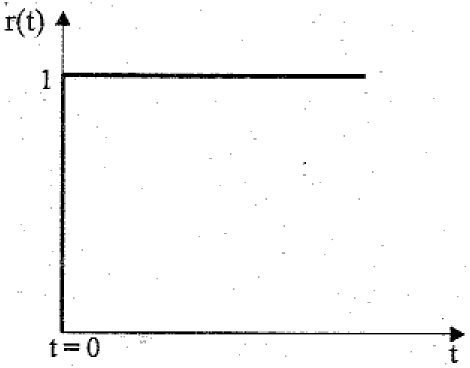


Fig a : Unit step input.

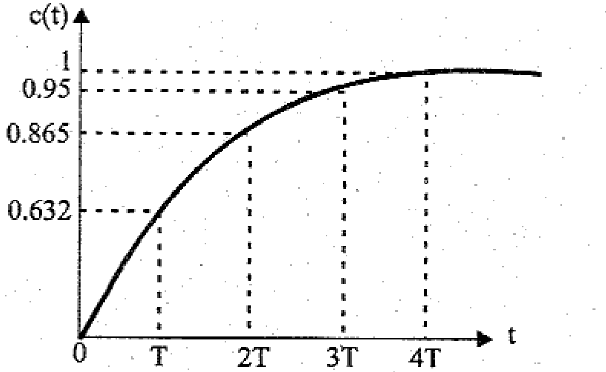


Fig b : Response for Unit step input.

Example: Partial fraction

$$T(s) = \frac{K}{s(s+p_1)(s+p_2)} = \frac{A}{s} + \frac{B}{s+p_1} + \frac{C}{s+p_2}$$

The residues A, B and C are given by,

$$A = T(s) \times s\Big|_{s=0} \qquad B = T(s) \times (s+p_1)\Big|_{s=-p_1} \qquad C = T(s) \times (s+p_2)\Big|_{s=-p_2}$$

The First order control system is represented by a TF. C(s)/R(s)= 1/s+5. Determine the time constant and response for a unit step input.

$$\left[\because\ R(s)=\frac{1}{s}\ \text{for unit step } I/P\right]$$

$$\frac{C(s)}{R(s)}=\frac{1}{s+5}$$

$$C(s)=\frac{R(s)}{s+5}=\frac{1}{s(s+5)}$$

$$C(s)=\frac{1}{s(s+5)}=\frac{A}{s}+\frac{B}{s+5}=\frac{A(s+5)+Bs}{s(s+5)}$$

$$C(s)=\frac{0.2}{s}-\frac{0.2}{s+5}$$

$$=0.2\left(\frac{1}{s}-\frac{1}{s+5}\right)$$

$$As+5A+Bs=1$$

$$A+B=0$$

$$5A=1$$

$$A=\frac{1}{5}\ \text{and}\ B=-\frac{1}{5}$$

$$C(t)=0.2\left(1-e^{-5t}\right)$$

$$=0.2\left(1-e^{-t/0.2}\right)\quad\left[1-e^{-t/T}\right]$$

$$T=0.2\ sec$$

Second order system

First Order Control Systems had only one energy storage element and thus could be modeled by a first-order differential equation

Second order system are models with two energy storage elements. The simple step of adding an additional energy storage element to First Order Control Systems allows much greater variation in the types of responses, The largest difference is that systems can now exhibit oscillations in time in their natural response

In this case of the mechanical systems, energy is stored in a spring or an Mass/inertia.

In the case of electrical systems, energy can be stored either in a capacitance or an inductance.

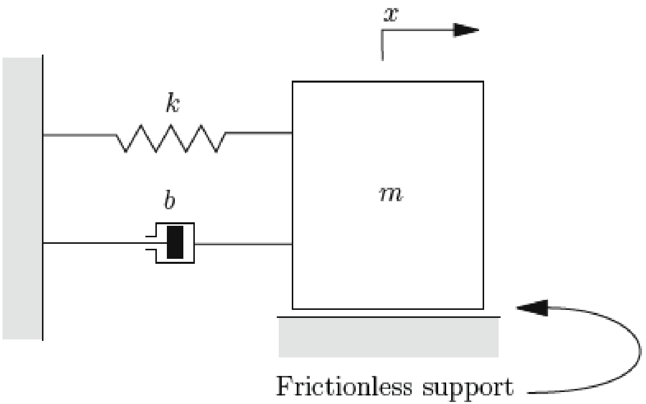
Fluid systems store energy via pressure in fluid capacitances, and via flow rate in fluid inertia (inductance)

Second order system

When the highest power of s in the denominator of the closed loop transfer function of a system is 2, then it represents a second order system.

Second order systems are very important as it characterizes the dynamics of many control system application, such as servomechanisms, air craft and space craft control systems.

Example of Second order system are models with two energy storage elements



The system consists of a spring and damper attached to a mass which moves laterally on a frictionless surface

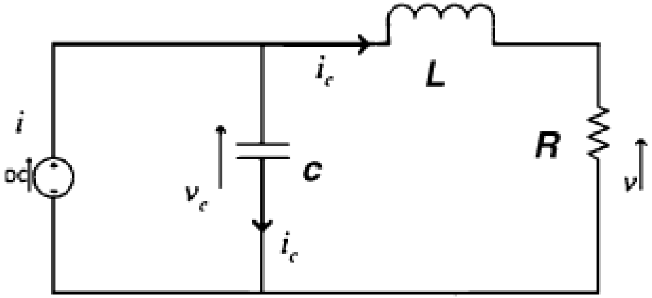


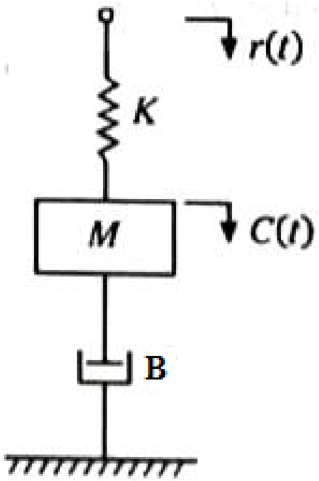
FIGURE . RLC circuit.

Electric RLC circuit with $i(t)$ the input current of a current source, and $v(t)$ the output voltage across a load resistance R.

The example of second order system is servomechanisms or spring-mass-dash pot system

As an example consider a simple mechanical system, a spring/mass/damper.

It consists of a weight of mass 'M' Kg, on a spring with spring constant 'K' N/m, its motion damped by friction with coefficient 'B' N-sec/m



$$C(s)/R(s) = K / Ms^2 + Bs + K$$

$$C(s)/R(s) = (K/M) / (s^2 + (B/M)\ s + (K/M)$$

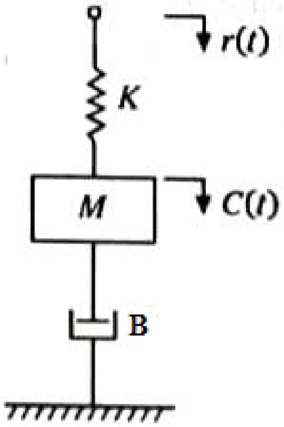
In transient response analysis

$$\omega_n^2 = K / M \quad \text{and} \quad 2\ \xi\ \omega_n = B/M$$

$$\omega_n = \sqrt{\ } K / M \qquad \xi = (B/M) / (2\ \omega_n) = (B/M) / (2\sqrt{K/M})$$

$$\xi = B / 2\sqrt{KM} = B/Bc$$

The damping ratio ξ is defined as the ratio of actual damping 'B' to the critical damping 'Bc'



C(s)/R(s) = K / Ms^2 + Bs + K

The characteristic equation **Ms^2 + Bs + K** = 0, This second-order polynomial has two Roots, which are the pole locations (natural frequencies) of the system

$$s_1 = -\frac{b}{2m} + \frac{\sqrt{b^2 - 4mk}}{2m} \qquad s_2 = -\frac{b}{2m} - \frac{\sqrt{b^2 - 4mk}}{2m}$$

Before further analysis, it is helpful to introduce some standard terms. The pole locations are conveniently parameterized in terms of the damping ratio ξ, and natural frequency $\boldsymbol{\omega_n}$

Where $$\omega_n = \sqrt{\frac{k}{m}} \qquad \zeta = \frac{b}{2\sqrt{km}}.$$

ωn is the (undamped) natural frequency of the system in rad/sec, i.e., the frequency of oscillations when the damping ‘B’is zero.

Lets make the physically reasonable assumption that the values of m, and k are greater than zero (to maintain system order) and that b is non-negative (to keep things stable). With these assumptions, there are four classes of pole locations:

First, if b = 0, the poles are Purely Imaginary lying on the imaginary axis at $s1 = +j\sqrt{k/m}$ and $s2 = -j\sqrt{k/m}$. This corresponds to $\xi = 0$, and is **referred to as the undamped case**

If $\mathbf{b^2 - 4mk} < 0$ then the poles are complex conjugates lying in the left half of the s-plane. This corresponds to the range $0 < \xi < 1$, and is **referred to as the underdamped case**

If $\mathbf{b^2 - 4mk = 0}$ then the poles coincide on the real axis at $s1 = s2 = -b/2m$. This corresponds to $\xi = 1$, and is **referred to as the critically damped case**.[Two identical (repeated) real roots]

Finally, if $\mathbf{b^2 - 4mk > 0}$ then the poles are at distinct locations on the real axis in the left half of the s-plane. This corresponds to $\xi > 1$, and is **referred to as the overdamped case.**

$$\frac{Y(s)}{X(s)} = K\,\omega_n^2/(s|2 + 2\omega_n \delta s + \omega_n^2)$$

Damping Ratio

The **damping ratio** of a second-order system, denoted with the Greek letter zeta (ζ), is a real number that defines the damping properties of the system. More damping has the effect of less percent overshoot, and slower settling time. Damping is the inherent ability of the system to oppose the oscillatory nature of the system's transient response. Larger values of damping coefficient or damping factor produces transient responses with lesser oscillatory nature.

Undamped natural frequency of a second order system has the following influence on the response due to various excitations:
a) Increase in speed of response and decrease sensitivity
b) Decrease in speed of response and increase sensitivity
c) Has no influence in the dynamic response
d) Increase oscillatory behavior
Answer: a
Explanation: Undamped natural frequency is the frequency that has suffered damping and gets affected by the increase in the speed of response and decrease in sensitivity.

Second order system

- The closed loop transfer function of Standard second order system is given by $\frac{C(s)}{R(s)} = \frac{\omega_n^2}{s^2 + 2\xi\omega_n s + \omega_n^2}$

  Hence the value of ξ and ω_n describe the dynamic behaviour of the 2nd order system

- Every system has a tendency to oppose the oscillator behaviour of the system which is called damping.
- The damping is measured by a ratio called damping ratio of the system.(ξ)

  The second-order system is parameterized by the two parameters zeta ξ and $\boldsymbol{\omega_n}$, Different choices for zeta and $\boldsymbol{\omega_n}$ lead to different pole locations and to different behavior of the (modes of) the system

  $\xi < 1$ the system is said to be underdamped, if $\xi > 1$, it is overdamped and $\xi = 1$ the system is said to be critically damped

- If the $\xi = 0$, the system will oscillate with maximum frequency. This frequency is called natural frequency of oscillation (ω_n) in rad/sec

The damping ratio gives us an idea about the nature of the transient response detailing the amount of overshoot & oscillation that the system will undergo. This is completely regardless of time scaling.

The closed loop transfer function of Standard second order system is given by $\frac{C(s)}{R(s)} = \frac{\omega_n^2}{s^2 + 2\xi\omega_n s + \omega_n^2}$

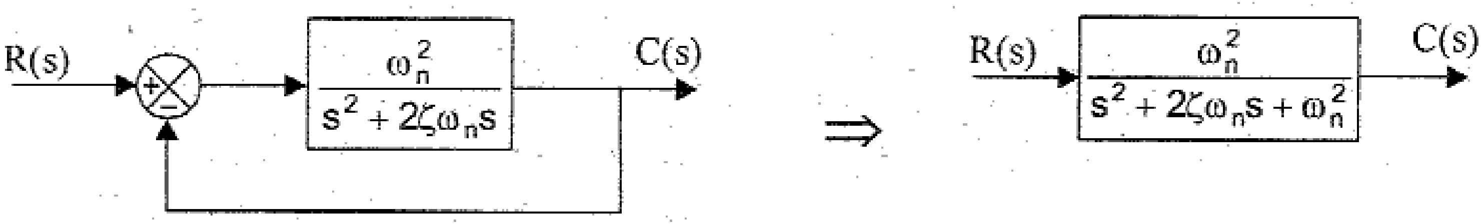
- For a second order system the denominator of closed loop T.F. is quadratic and the coefficients of this equation are directly related to ξ and ω_n
- Where characteristic equation is, $s^2 + 2\xi\omega_n s + \omega_n^2 = 0$
- The standard second order system is that where in C.L.T.F. numerator is ω_n^2.

$$\frac{C(s)}{R(s)} = \frac{\omega_n^2}{s^2 + 2\xi\,\omega_n\, s + \omega_n^2}$$

Key Point : In practice it is not necessary that numerator must be always ω_n^2. It may be other constant or polynomial of `s' but denominator ***middle term coefficient*** *and* ***last term coefficient*** *always reflect* `$2\,\xi\,\omega_n$' *and* `ω_n^2' *of the system respectively.*

- Hence always denominator of a T.F. must be compared with the standard form $s^2 + 2\xi\,\omega_n\, s + \omega_n^2 = 0$ to decide the values of ξ and ω_n of the system. The numerator should not be used for comparison to obtain the values of ξ and ω_n.

Second order system and Time domain specifications



The closed loop transfer function of second order system is given by

$$\frac{C(s)}{R(s)} = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

$\boldsymbol{\omega_n}$ is the (undamped) natural frequency of the system, i.e., the frequency of oscillations when the damping ξ is zero.

ζ is the damping ratio of the system

Undamped system,	$\zeta = 0$
Under damped system,	$0 < \zeta < 1$
Critically damped system,	$\zeta = 1$
Over damped system,	$\zeta > 1$

http://www.controlsystemsacademy.com/0024/0024.html

Second order system and Time domain specifications

$$\frac{C(s)}{R(s)} = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

where, ω_n = Undamped natural frequency, rad/sec.

ζ = Damping ratio.

Undamped system,	$\zeta = 0$
Under damped system,	$0 < \zeta < 1$
Critically damped system,	$\zeta = 1$
Over damped system,	$\zeta > 1$

Thus, for

- **undamped** (ξ=0) systems there are
  - two purely imaginary roots (poles), $s_{1,2} = \pm j\omega_n$
- **critically damped** (ξ=1)systems, there are
  - two identical (repeated) real roots, $s_{1,2} = -\omega_n$
- **overdamped** (ξ>1) systems, there are
  - two distinct negative real roots, and $s_1 = -\omega_{n1}$, $s_2 = -\omega_{n2}$
- **underdamped** (ξ<1) systems, there are
  - two complex conjugate roots. $s_{1,2} = -\xi\omega_n \pm j\omega_n\sqrt{1-\xi^2}$

Response of Undamped second order system for unit step input

$$\frac{C(s)}{R(s)} = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

Undamped system, $\zeta = 0$

When $\zeta = 0$, $s_1, s_2 = \pm j\omega_n$; $\left\{\begin{array}{l}\text{roots are purely imaginary} \\ \text{and the system is undamped}\end{array}\right.$

Response is completely oscillatory (sustained oscillations)

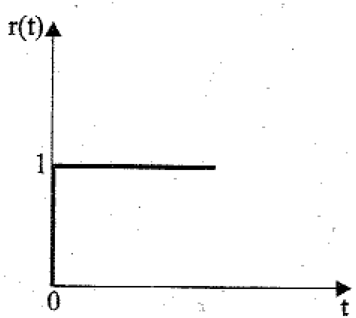


Fig 2.9.a : Input.

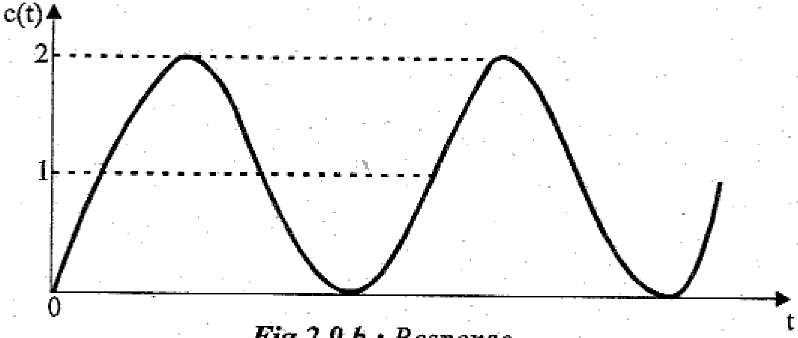
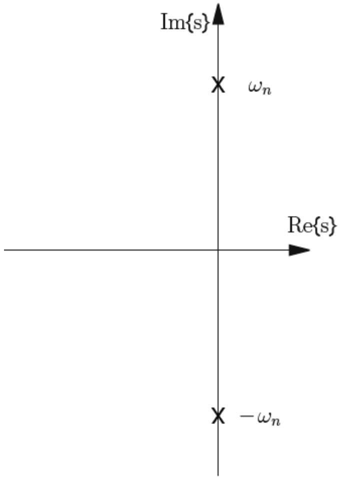


Fig 2.9.b : Response.



Pole locations in the s-plane for second-order mechanical system in the undamped case ($\xi = 0$).

Response of Underdamped second order system for unit step input

$$\frac{C(s)}{R(s)} = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

Under damped system, $0 < \zeta < 1$

When $0<\zeta<1$, $s_1, s_2 = -\zeta\omega_n \pm j\omega_d$; $\begin{cases} \text{roots are complex conjugate} \\ \text{the system is underdamped} \end{cases}$

where, $\omega_d = \omega_n\sqrt{1-\zeta^2}$

The poles lie at $s = -\sigma \pm j\omega_d$

$$\sigma = \zeta\omega_n$$

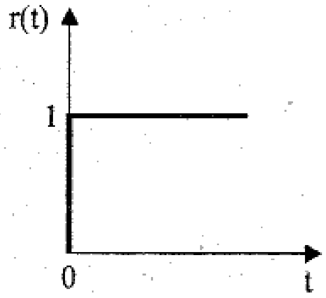


Fig a : Input.

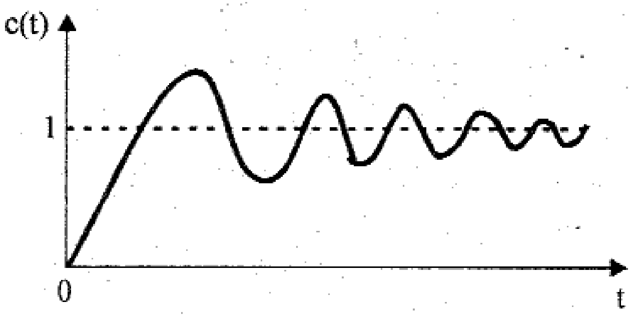
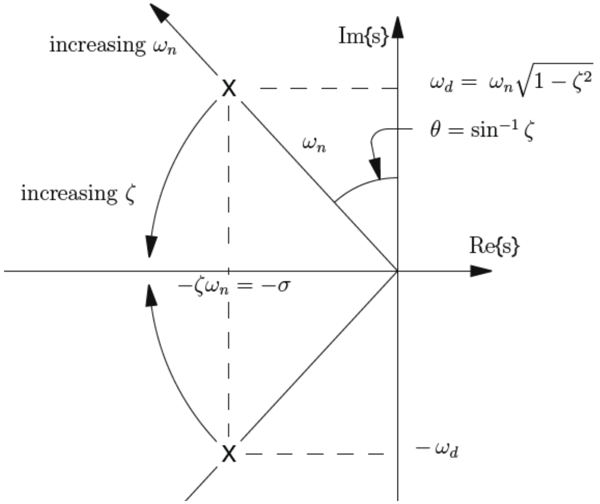


Fig .b : Response.

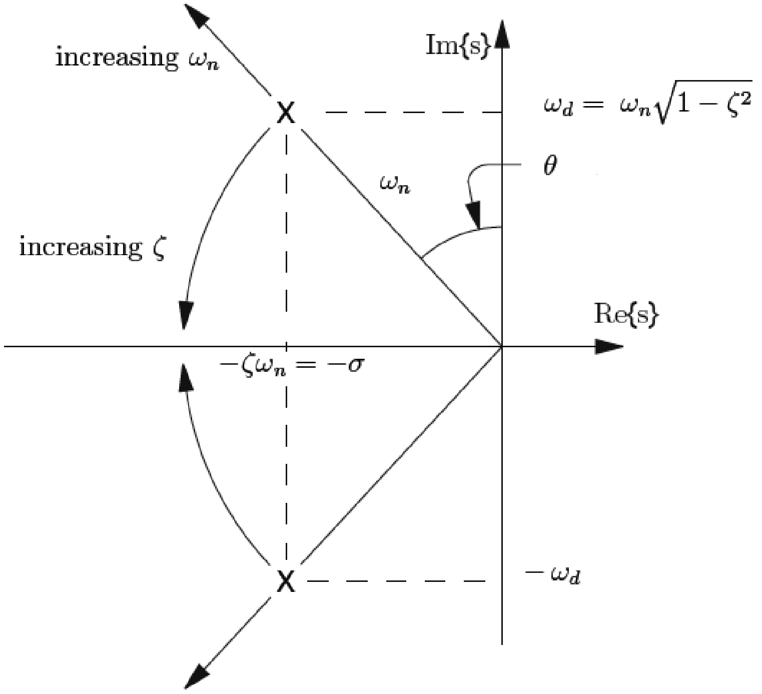


Response is oscillates before settling final value. The oscillations depends on the value of Damping ratio

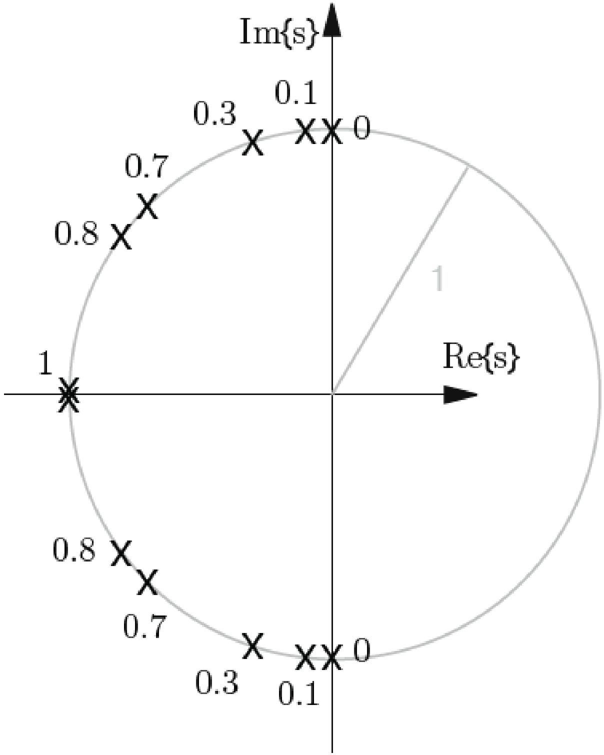
$$\theta = \cos^{-1}\zeta = \tan^{-1}\left(\sqrt{1-\zeta^2}/\zeta\right), \omega_d = \omega_n\sqrt{1-\zeta^2}$$

Pole locations in the s-plane for second-order mechanical system in the underdamped case ($0 < \xi < 1$). Arrows show the effect of increasing **ωn** and ξ , respectively.

The poles are at a radius from the origin of **ωn** and at an angle from the imaginary axis of $\boldsymbol{\Theta} = \mathbf{Cos^{-1}}\ \xi$. The figure also shows the effect of increasing ξ and **ωn**. As ξ increases from 0 to 1, the poles move along an arc of radius **ωn** from $\boldsymbol{\Theta} = 0$ to $\boldsymbol{\Theta} = \pi/2$. As **ωn** increases, the poles move radially away from the origin, maintaining constant angle $\boldsymbol{\Theta} = \mathbf{Cos^{-1}}\ \xi$ and thus constant damping ratio



Pole locations for **ωn** = 1 and ξ =0, 0.1, 0.3, 0.7, 0.8, and 1.



Response of Critically Damped second order system for unit Step input

$$\frac{C(s)}{R(s)} = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

Critically damped system, $\zeta = 1$

When $\zeta = 1$, $s_1, s_2 = -\omega_n$; $\begin{cases} \text{roots are real and equal and} \\ \text{the system is critically damped} \end{cases}$

Poles are real, repeated, and located at − **ωn**

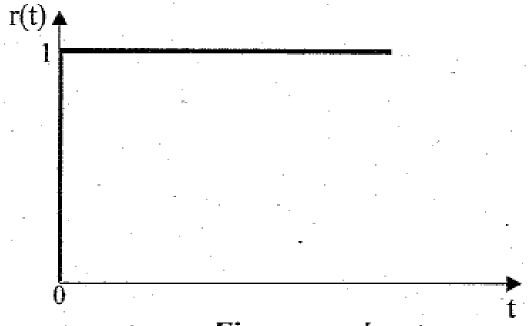


Fig a : Input.

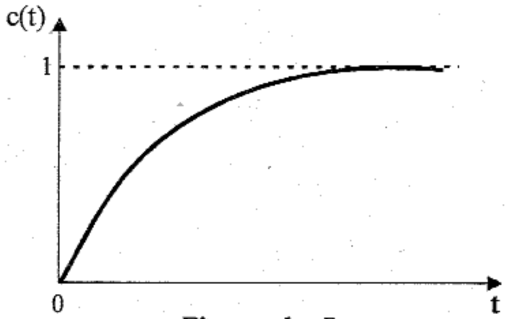
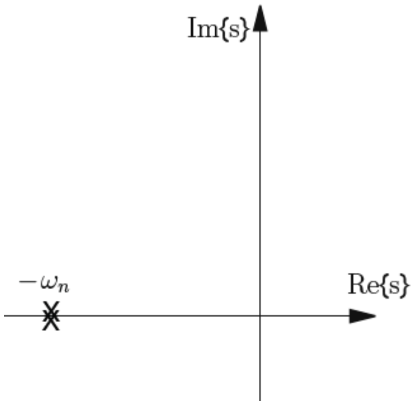


Fig .b : Response.



Response have no oscillations

Pole locations in the *s*-plane for second-order mechanical system in the critically-damped case (ξ = 1).

Response of Over damped second order system for unit Step input

$$\frac{C(s)}{R(s)} = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

Over damped system, $\zeta > 1$

When $\zeta > 1$, $s_1, s_2 = -\zeta\omega_n \pm \omega_n\sqrt{\zeta^2 - 1}$; $\left\{\begin{array}{l}\text{roots are real and unequal and}\\ \text{the system is overdamped}\end{array}\right.$

When ξ > 1 the poles are real and distinct and the response approaches its steady state value of 1 without overshoot. In this case the system is overdamped

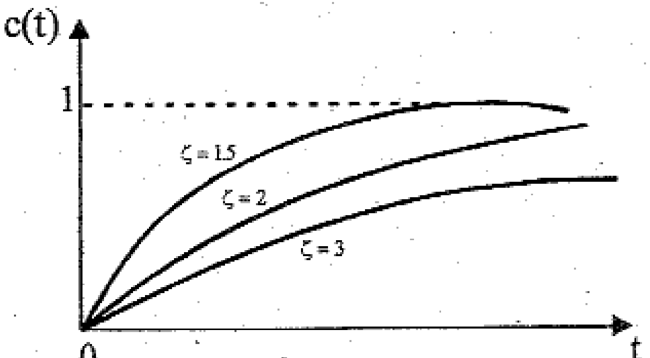
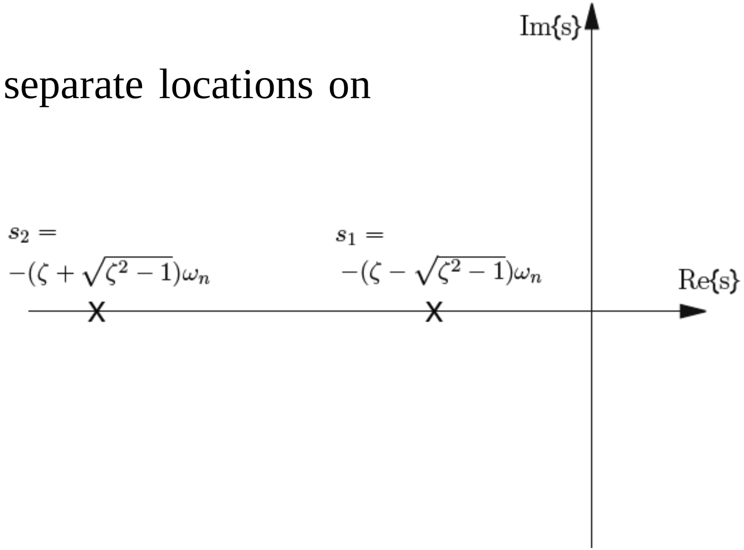


Fig b : Response.

The two poles are at separate locations on the real axis



Response have no oscillations, but it takes longer time for the response to reach the final steady value

Sr. No.	Range of ξ	Type of closed loop poles	Nature of response	System classification
1.	$\xi = 0$	Purely imaginary	Oscillations with constant frequency and amplitude	Undamped
2.	$0 < \xi < 1$	Complex conjugates with negative real part	Damped oscillations	Underdamped
3.	$\xi = 1$	Real, equal and negative	Critical and pure exponential	Critically damped
4.	$1 < \xi < \infty$	Real, unequal and negative	Purely exponential slow an sluggish	Overdamped

Sr. No.	Range of ξ	Type of closed loop poles	Nature of response	System classification
1.	$\xi = 0$	Purely imaginary	Oscillations with constant frequency and amplitude	Undamped
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3.	$\xi = 1$	Real, equal and negative	Critical and pure exponential	Critically damped
4.	$1 < \xi < \infty$	Real, unequal and negative	Purely exponential slow an sluggish	Overdamped

If $\xi = 0$, then such a second-order system is marginally stable in that the natural response is of constant amplitude in time

If $0 < \xi < 1$, then such a second-order system is underdamped, the poles have imaginary components, and the natural response contains some amount of oscillatory component. Lower values of ξ correspond with relatively more oscillatory responses, i.e., are more lightly damped.

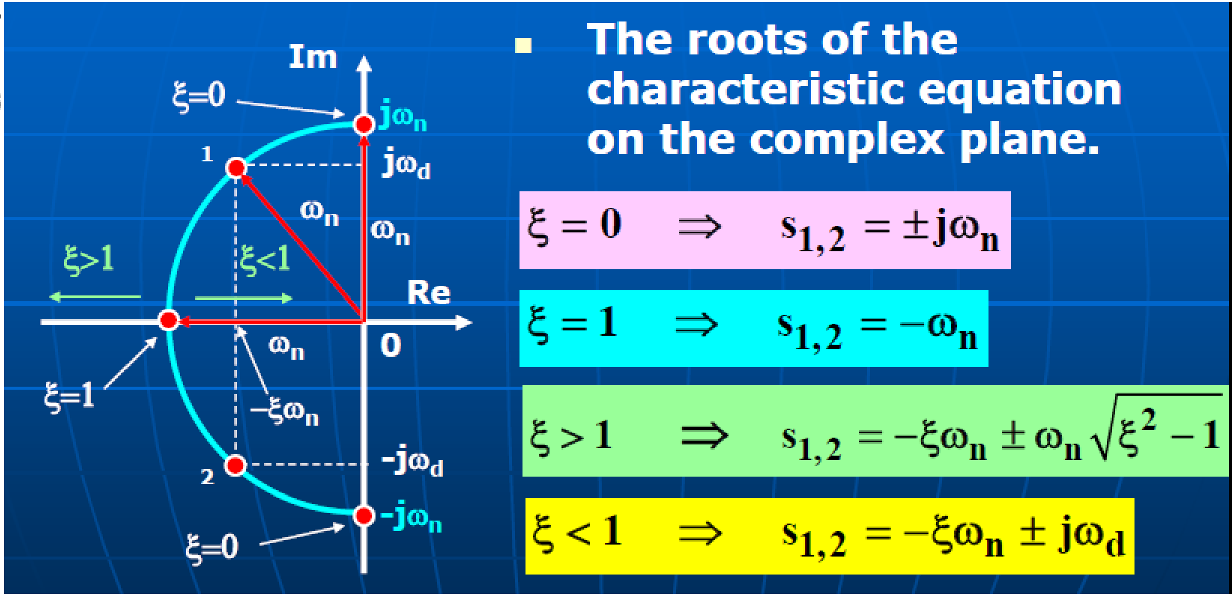
If $\xi = 1$, then such a second-order system is critically damped, and the poles are coincident on the negative real axis at a location $\boldsymbol{\omega_n}$

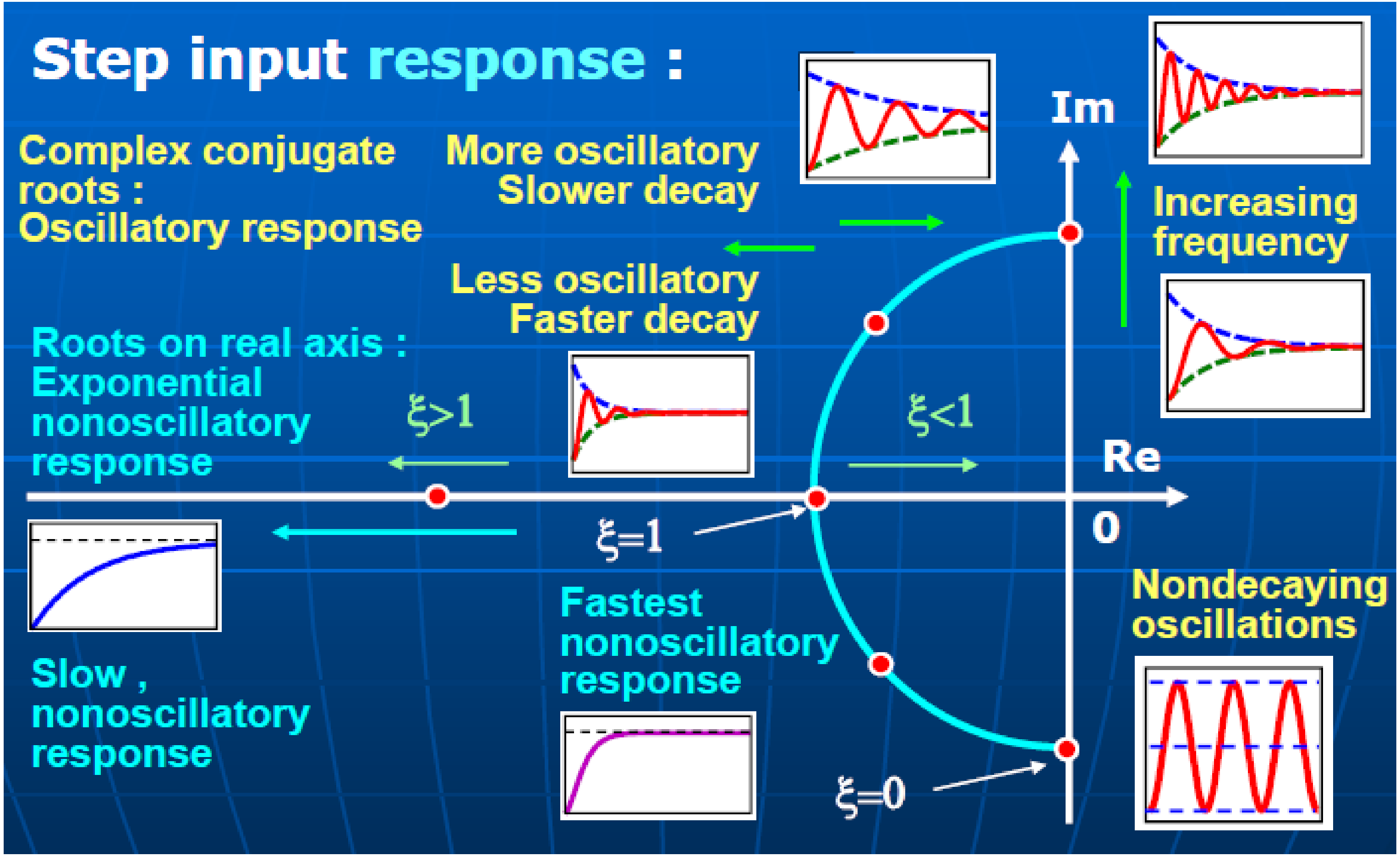
If $\xi > 1$, then such a second-order system is overdamped, and the poles are at distinct locations on the negative real axis. This case can also be thought of as two independent first-order systems

The second-order system is parameterized by the two parameters zeta ξ and $\boldsymbol{\omega_n}$, Different choices for zeta and $\boldsymbol{\omega_n}$ lead to different pole locations and to different behavior of the (modes of) the system

- Locations of these roots of characteristic equation i.e. closed loop poles of second order system can be shown in s-plane as shown in the Fig.

Effect of ξ on locations of closed loop poles




Step input response :
Complex conjugate roots : Oscillatory response
More oscillatory Slower decay
Less oscillatory Faster decay
Im
Increasing frequency
Roots on real axis : Exponential nonoscillatory response
ξ>1
ξ<1
Re
0
ξ=1
Nondecaying oscillations
Slow , nonoscillatory response
Fastest nonoscillatory response
ξ=0

Transfer function stability is solely determined by its denominator.

The roots of a denominator are called **poles**.

Poles located in the left half-plane are stable while poles located in the right half-plane are not stable.

The reasoning is very simple: the Laplace operator **"s"**, which is location in the Laplace domain,

can be also written $s = \sigma + j\omega$

Left half-plane has negative *sigma*

The plane below shows the damping frequency and damping coefficient "zeta" graphically.

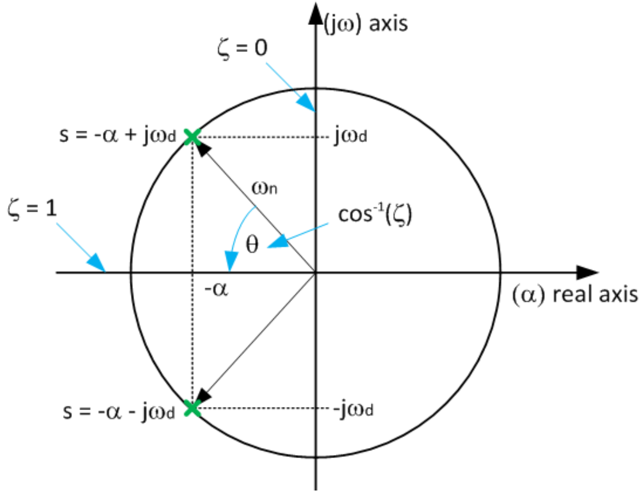
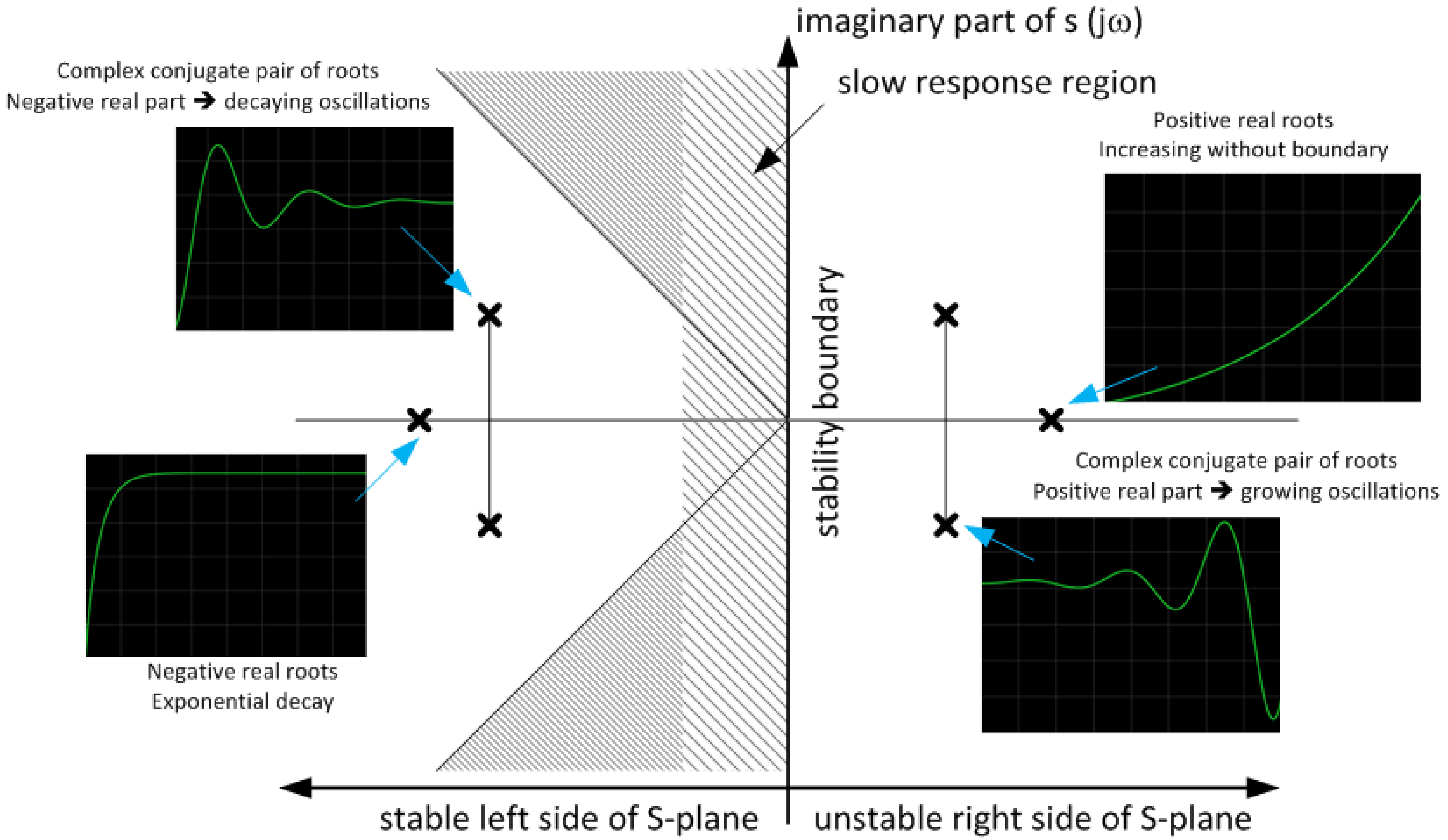


figure shows the time-domain response based on pole location in the Laplace domain.



Response of 2nd Order System to Step Inputs

Overdamped	Sluggish, no oscillations
Critically damped	Faster than overdamped, no oscillation
Underdamped	Fast, oscillations occur

Ways to describe underdamped responses:

- Rise time
- Time to first peak
- Settling time
- Overshoot
- Decay ratio
- Period of oscillation

Speed of response is the speed at which the response takes the final value and this is determined by damping factor which reduces the oscillations and peak overshoot, as the peak is less then the speed of response will be more.

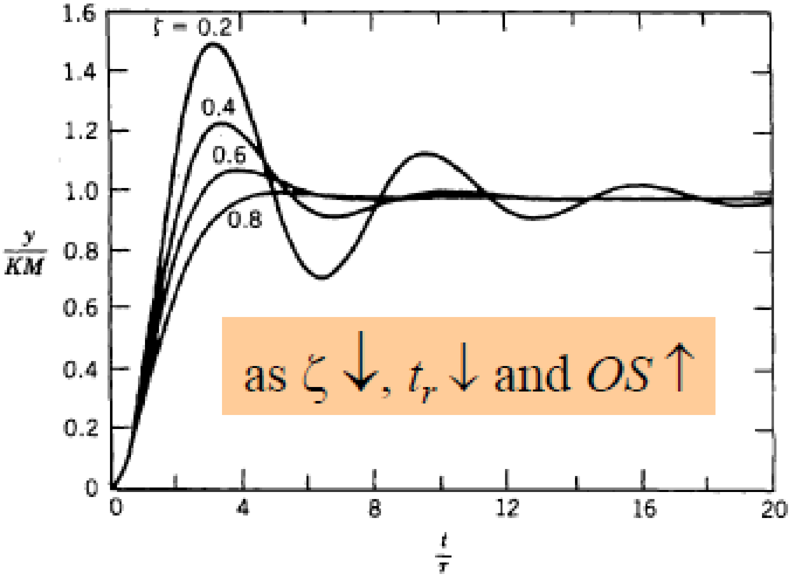


Figure 5.8. Step response of underdamped second-order processes.

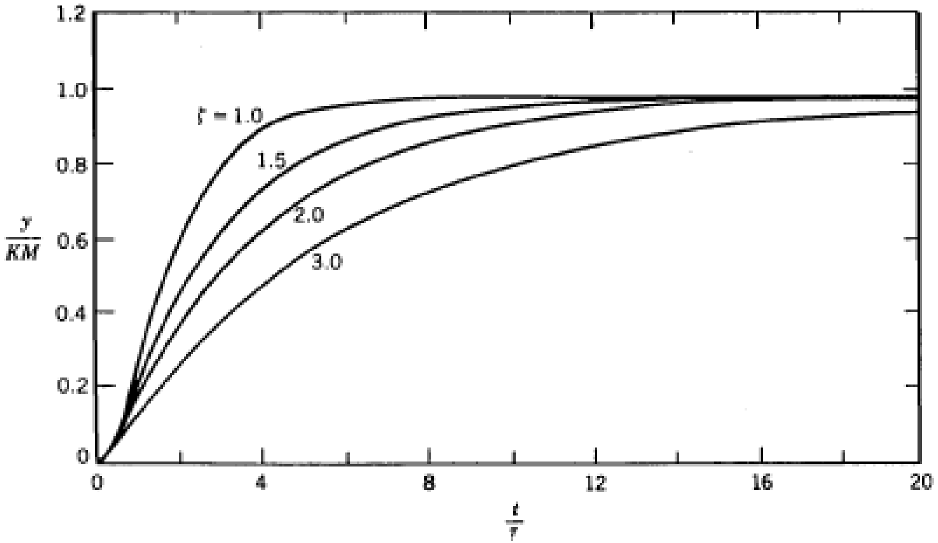


Figure 5.9. Step response of critically-damped and overdamped second-order processes.

$$0 < \zeta < 1$$

$$\zeta \geq 1$$

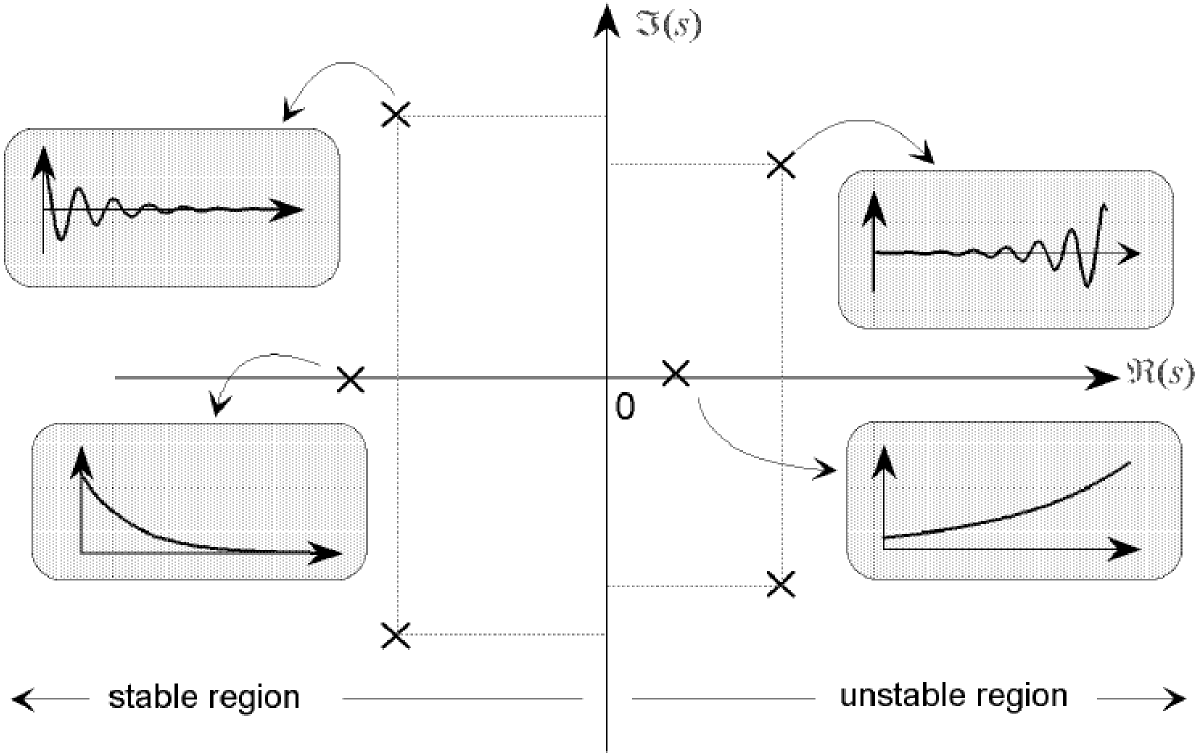
Note that $\zeta < 0$ gives an unstable solution

Effect of pole location on Stability:

The stability of a feedback system is directly related to the location of the roots of the characteristic equation of the system transfer function.

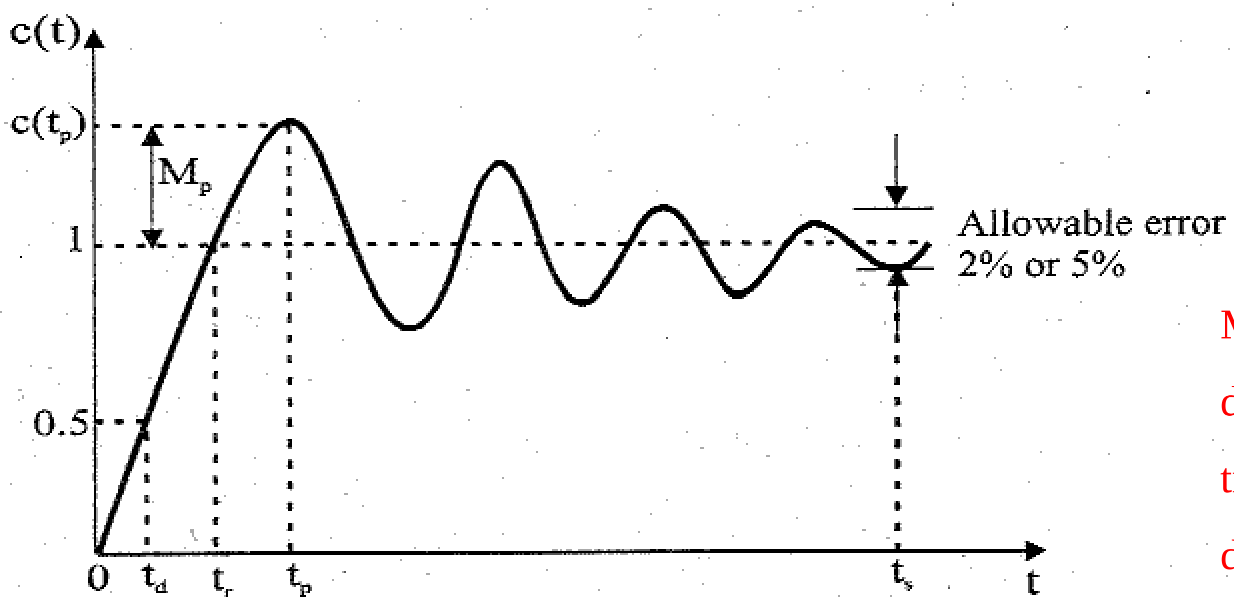
For BIBO (Bounded input bounded output) Stability the integral of impulse response should be finite, which implies that the impulse response should be finite as time 't' tends to infinity

"A linear system will be stable if and only if all the poles of the transfer function are located on the left half of the 'S' plane".



Pole locations on the pole-zero plot

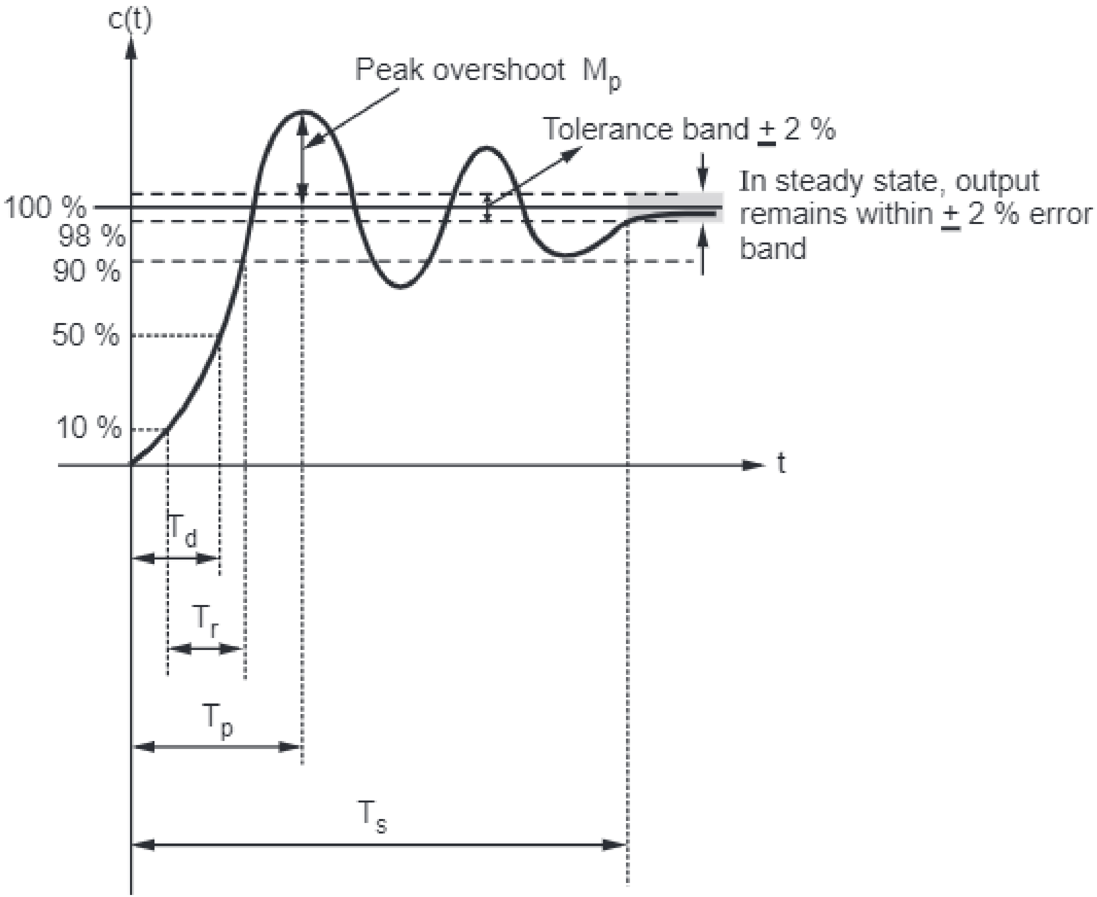
Damped oscillatory response of a second order system for unit step input



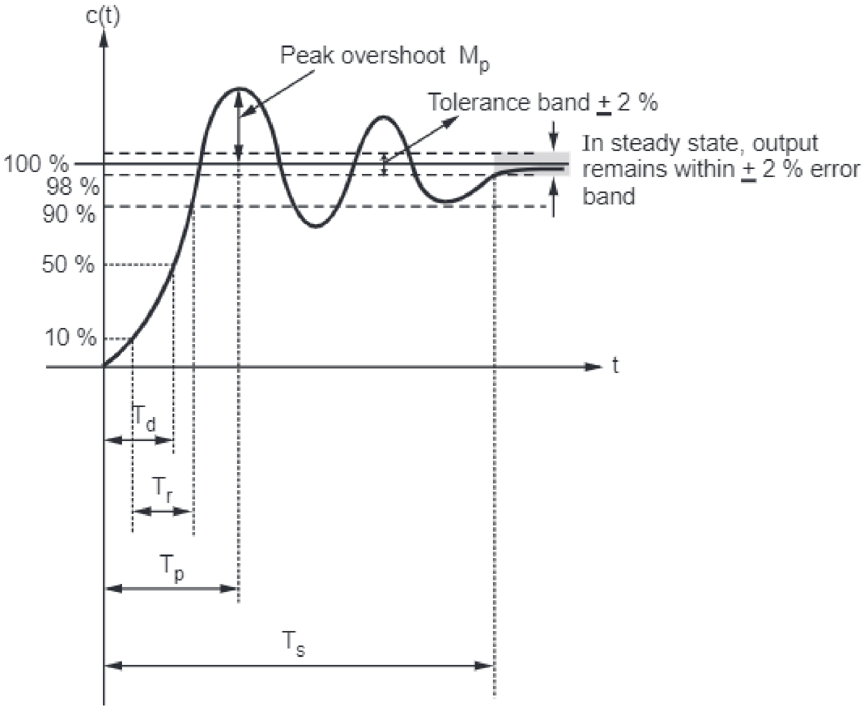
Maximum overshoot, rise time and delay time are the major factor of the transient behaviour of the system and determines the transient characteristics

Damping factor is minimum hence the system will have the maximum overshoot

Transient Response Speci



Transient Response Specifications of Second Order System



Delay Time (Td): **It is the time required for the response to reach 50% of the final value in the first attempt.**

$$T_d = \frac{1 + 0.7\,\xi}{\omega_n}$$

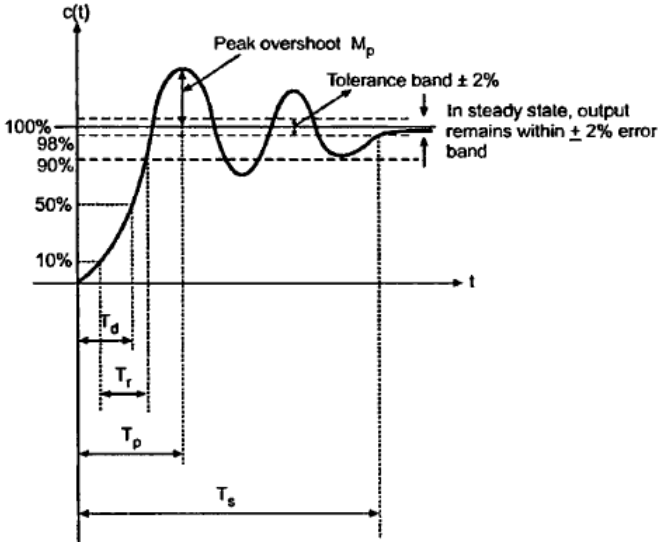
Rise Time (Tr): It is the time required for the response to rise from 10% to 90% of the final value for over damped systems and 0 to 90% of the final value for under damped systems.

$$T_r = \frac{\pi - \theta}{\omega_d} \text{ sec where } \theta \text{ must be in radians.}$$

$$\theta = \tan^{-1}\frac{\sqrt{1-\xi^2}}{\xi} \text{ radians}$$

3. **Peak Time (Tp):** It is the time required for the response to reach its peak value.

$$T_p = \frac{\pi}{\omega_d} = \frac{\pi}{\omega_n\sqrt{1-\xi^2}} \text{ sec}$$

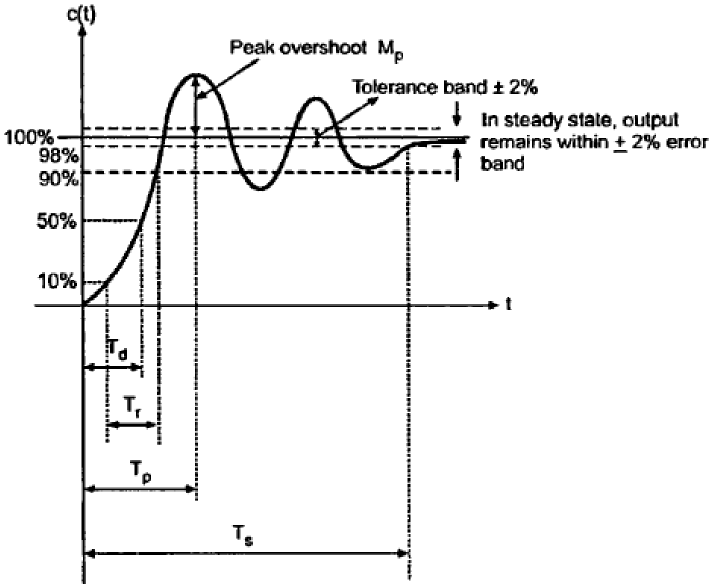


4. **Peak Overshoot (Mp):** The amount by which output overshoots its reference steady state value during the first overshoot.

$$\%M_p = e^{-\frac{\zeta\pi}{\sqrt{1-\zeta^2}}} \times 100$$

5. Settling Tine (Ts): It is the time required for the response to decrease and stay within specified percentage of its final value.

$$t_s = \frac{4}{\zeta \omega_n}$$ ..... for a tolerance band of ± 2% of steady state



Translent response specifications

$$c(t) = 1 - \frac{e^{-\xi\omega_n t}}{\sqrt{1-\xi^2}} \sin(\omega_d + \theta)$$

for underdamped system, unit step input

where $\omega_d = \omega_n \sqrt{1-\xi^2}$

and $\theta = \tan^{-1}\left[\frac{\sqrt{1-\xi^2}}{\xi}\right]$ radians.

For a step of A units,

$$c(t) = A\left[1 - \frac{e^{-\xi\omega_n t}}{\sqrt{1-\xi^2}} \sin(\omega_d t + \theta)\right]$$

$$T_d = \frac{1+0.7\,\xi}{\omega_n} \text{ sec}, \quad T_p = \frac{\pi}{\omega_d} \text{ sec}, \quad T_r = \frac{\pi-\theta}{\omega_d} \text{ sec}$$

$$\%\,M_p = \left[e^{-\pi\xi/\sqrt{1-\xi^2}}\right] \times 100$$

$$T_s = \frac{4}{\xi\,\omega_n} \text{ sec}, \quad \text{for} \pm 2\,\% \text{ tolerance band.}$$

Consider an example for the second order system with an open loop transfer function as $G(S) = \frac{5}{S(S+1)}$ and unity feedback system $H(s) = 1$, Find the rise time, peak time, settling time and peak overshoot for unit step input.

The closed loop transfer function is determined by using the equation

$$TF = \frac{C(s)}{R(s)} = \frac{G(s)}{1 + G(s)H(s)}$$

The closed loop transfer function is determined by using the equation

$$TF = \frac{C(s)}{R(s)} = \frac{G(s)}{1 + G(s)H(s)}$$

By substituting the values of G(s) and H(s) in above equation we get, $TF = \frac{C(s)}{R(s)} = \frac{5}{s^2+s+5}$

Standard form of transfer function of a second order system is

$$\frac{C(s)}{R(s)} = \frac{\omega_n^2}{s^2 + 2\delta\omega_n s + \omega_n^2}$$

Comparing Equations, we get ω_n = 2.236 and ξ= 0.224. Since ξ < 1, the given system is under damped

Solved Example

1. **A unity feedback system has an open loop transfer function $G(s) = \frac{5}{s(s+1)}$ Find the rise time, peak time, settling time and peak overshoot for unit step input.**

$$\frac{C(s)}{R(s)} = \frac{G(s)}{1+G(s)} = \frac{\frac{5}{s(s+1)}}{1+\frac{5}{s(s+1)}} = \frac{5}{s^2+s+5}$$

$$\frac{C(s)}{R(s)} = \frac{\omega_n^2}{s^2+2\xi\omega_n s+\omega_n^2} = \frac{5}{s^2+s+5}$$

$$\omega_n^2 = 5$$

$$\omega_n = \sqrt{5} = 2.236 \text{ rad/s}$$

$$2\xi\omega_n = 1$$

$$\xi = \frac{1}{2\omega_n} = \frac{1}{2\times 2.236} = 0.223$$

$$\omega_d = \omega_n\sqrt{1-\xi^2} = 2.236\sqrt{1-0.223^2} = 2.124 \text{ rad/s}$$

$$\theta = \tan^{-1}\frac{\sqrt{1-\xi^2}}{\xi} = \tan^{-1}\frac{\sqrt{1-0.223^2}}{0.223} = 1.346 \text{ rad/s}$$

Rise time $$t_r = \frac{\pi-\theta}{\omega_d} = \frac{3.141-1.346}{2.124} = 0.845\text{s}$$

Peak time $$t_p = \frac{\pi}{\omega_d} = \frac{\pi}{2.124} = 1.479\text{s}$$

Solved Example

The time constant

$$T = \frac{1}{\xi\omega_n} = \frac{1}{0.223 \times 2.236} = 2\,s$$

For 5% error, the settling time

$$t_s = 3T = 3 \times 2 = 6s$$

For 2% error, the settling time

$$t_s = 4T = 4 \times 2 = 8s$$

$$\% M_P = e^{-\pi\xi\sqrt{1-\xi^2}} * 100$$ **= 47.8%**

2. A unity feedback system has an Closed loop transfer function is $\frac{C(s)}{R(s)} = \frac{25}{s^2 + 6s + 25}$.

Find the damping ratio, the rise time, peak time, settling time and peak overshoot for unit step input and expression of output response

Standard form of transfer function of a second order system is

$$\frac{C(s)}{R(s)} = \frac{\omega_n^2}{s^2 + 2\delta\omega_n s + \omega_n^2}$$

$$\omega_n^2 = 25 \quad \text{and} \quad 2\xi\,\omega_n = 6$$

$$\omega_n = 5 \qquad \therefore \ \xi = 0.6$$

Comparing Equations, we get $\omega_n = 5$ and $\xi = 0.6$, Since $\xi < 1$, the given system is under damped

$$\theta = \tan^{-1}\left[\frac{\sqrt{1-\xi^2}}{\xi}\right] = 0.9272 \text{ radians}$$

$$\omega_d = \omega_n\sqrt{1-\xi^2} = 5\sqrt{1-(0.6)^2} = 4 \text{ rad/sec.}$$

$$T_r = \frac{\pi-\theta}{\omega_d} = \frac{\pi-0.9272}{4} = \mathbf{0.5535 \text{ sec.}}$$

$$T_p = \frac{\pi}{\omega_d} = \frac{\pi}{4} = \mathbf{0.785 \text{ sec.}}$$

$$\% M_p = e^{-\pi\xi/\sqrt{1-\xi^2}} \times 100 = \mathbf{9.48\ \%}$$

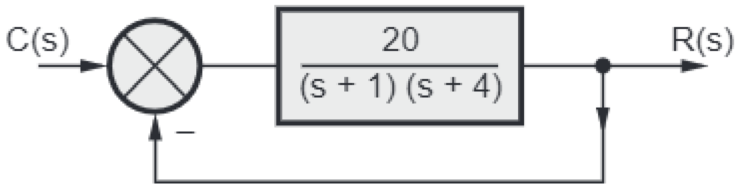
$$T_s = \frac{4}{\xi\,\omega_n} = \mathbf{1.33 \text{ sec.}}$$

$$c(t) = 1 - \frac{e^{-\xi\omega_n t}}{\sqrt{1-\xi^2}}\sin(\omega_d t + \theta)$$

$$= 1 - \frac{e^{-3t}}{\sqrt{1-(0.6)^2}}\sin(4t + 0.9272)$$

$$\boxed{c(t) = 1 - 1.5625\, e^{-3t} \sin(4t + 0.9272)}$$

3. For the system Shown, find the closed loop transfer function, damping ratio, the rise time, peak time, settling time and peak overshoot for unit step input, and expression of output response.



The closed loop transfer function is determined by using the equation

$$TF = \frac{C(s)}{R(s)} = \frac{G(s)}{1 + G(s)H(s)}$$

Standard form of transfer function of a second order system is

$$\frac{C(s)}{R(s)} = \frac{\omega_n^2}{s^2 + 2\delta\omega_n s + \omega_n^2}$$

$$\frac{C(s)}{R(s)} = \frac{\frac{20}{(s+1)(s+4)}}{1+\frac{20}{(s+1)(s+4)}} = \frac{20}{s^2+5s+24}$$

Comparing, $s^2 + 5s + 24$ with $s^2 + 2\xi\,\omega_n\,s + \omega_n^2$

$$\omega_n^2 = 24 \quad \therefore \quad \omega_n = \mathbf{4.8989\ rad/sec.}$$

$$2\,\xi\,\omega_n = 5 \quad \therefore \quad \xi = \mathbf{0.51031}$$

$$\omega_d = \omega_n\sqrt{1-\xi^2} = \mathbf{4.2129\ rad/sec.}$$

$$\theta = \tan^{-1}\frac{\sqrt{1-\xi^2}}{\xi}\ \text{radians}$$

$$= 1.03\ \text{radians}$$

$$c(t) = \left[1 - \frac{e^{-\xi\omega_n t}}{\sqrt{1-\xi^2}}\sin(\omega_d t + \theta)\right]$$

4. A system is given by differential equations $\frac{d^2 y}{dt^2} + 4\frac{dy}{dt} + 8y = 8x \quad y = output \quad and \quad x = input.$

Find the damping ratio, the rise time, peak time, settling time and peak overshoot for unit step input and expression of output response

$$\frac{d^2 y}{dt^2} + 4\frac{dy}{dt} + 8y = 8x$$

Take Laplace Transform of both side

$$s^2\, Y(s) + 4s\, Y(s) + 8\, Y(s) = 8\, X(s)$$

$$\text{i.e.} \quad Y(s)\left[s^2 + 4s + 8\right] = 8\, X(s)$$

$$\therefore \quad \text{T.F.} \quad \frac{Y(s)}{X(s)} = \frac{8}{s^2 + 4s + 8}$$

Standard form of transfer function of a second order system is

$$\frac{C(s)}{R(s)} = \frac{\omega_n^2}{s^2 + 2\delta\omega_n s + \omega_n^2}$$

T.F. $\frac{Y(s)}{X(s)} = \frac{8}{s^2 + 4s + 8}$

$\omega_n^2 = 8$ i.e. $\omega_n = \mathbf{2.83\ rad/sec}$

$2\,\xi\,\omega_n = 4$ $\therefore$ $\xi = \mathbf{0.7067}$

$$\omega_d = \omega_n\sqrt{1-\xi^2}$$

$$= 2.83\sqrt{1-(0.7067)^2} = \mathbf{2.002\ rad/sec}$$

T_p = Time for peak overshoot

$$= \frac{\pi}{\omega_d} = \frac{\pi}{2.002} = \mathbf{1.57\ sec}$$

$$\%\ M_p = e^{-\pi\,\xi/\sqrt{1-\xi^2}} \times 100$$

$$= e^{-\pi \times 0.706/\sqrt{1-(0.706)^2}} \times 100 = \mathbf{4.33\ \%}$$

T_s = Settling time $= \frac{4}{\xi\,\omega_n}$

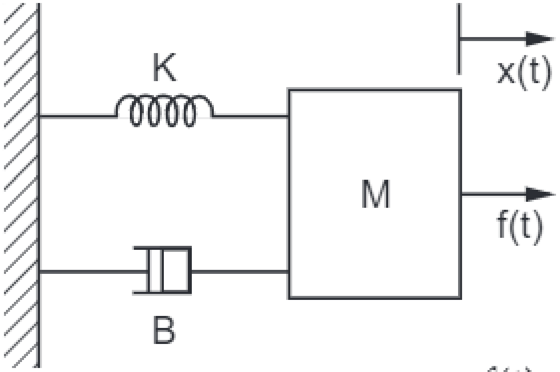
$$= \frac{4}{0.7067 \times 2.83} = \mathbf{2\ sec}$$

$$c(t) = 1 - \frac{e^{-\xi \omega_n t}}{\sqrt{1-\xi^2}} \sin(\omega_d t + \theta) \quad \text{where } \theta = \tan^{-1}\left(\frac{\sqrt{1-\xi^2}}{\xi}\right) = 45^\circ = \frac{\pi}{4} \text{ rad}$$

$$c(t) = 1 - \frac{e^{-0.7067 \times 2.87\, t}}{\sqrt{1-(0.7067)^2}} \sin\left(2t + \frac{\pi}{4}\right)$$

$$= 1 - 1.41\, e^{-2t} \sin\left(2t + \frac{\pi}{4}\right)$$

4. For the system Shown, find the closed loop transfer function, damping ratio, the rise time, peak time, settling time and peak overshoot for unit step input



i) Transfer function $\frac{X(s)}{F(s)}$ *and ii)* ξ , ω_n, % M_p , T_s *and* T_p

Take : $K = 33$ N/m , $B = 15$ N – S/m , $M = 3$ kg.

$$f(t) = M\frac{d^2x(t)}{dt^2} + B\frac{dx(t)}{dt} + K\,x(t)$$ Take Laplace Transform of both side

$$F(s) = Ms^2X(s) + Bs\,X(s) + K\,X(s)$$

$$\frac{X(s)}{F(s)} = \frac{1}{Ms^2 + Bs + K} = \frac{1/M}{s^2 + \frac{B}{M}s + \frac{K}{M}}$$

For $K = 33, \quad B = 15, \quad M = 3$

$$\frac{X(s)}{F(s)} = \frac{1}{Ms^2 + Bs + K} = \frac{1/M}{s^2 + \frac{B}{M}s + \frac{K}{M}}$$

$$\frac{X(s)}{F(s)} = \frac{1/3}{s^2 + 5s + 11} = \frac{0.333}{s^2 + 5s + 11}$$

Comparing denominator with $s^2 + 2\xi\omega_n s + \omega_n^2$,

$\omega_n^2 = 11$ hence $\omega_n =$ **3.3166 rad/sec**

$2\xi\omega_n = 5$ hence $\xi =$ **0.7537**

$$\% M_p = e^{-\pi\xi/\sqrt{1-\xi^2}} \times 100$$

$$\% M_p = e^{-\pi\xi/\sqrt{1-\xi^2}} \times 100 = \mathbf{2.7234\ \%}$$

$$\omega_d = \omega_n \sqrt{1-\xi^2} = \mathbf{2.1797\ rad/sec}$$

$$T_p = \frac{\pi}{\omega_d} = \mathbf{1.4412\ sec}$$

$$T_s = \frac{4}{\xi\omega_n} = \mathbf{1.6\ sec}$$

TYPE NUMBER OF CONTROL SYSTEMS

The Type number is specified for open loop transfer function G(S)*H(s). **The number of poles of the open loop transfer function lying at the origin decides the type number of the system**

if N is the number of poles at the origin then the type number is N.

Type Number is increased, accuracy is improved, however it aggravates the stability problem of the system

The Open loop transfer function can be expressed as a ratio of two polynomials in s.

$$G(s)\,H(s) = K\frac{P(s)}{Q(s)} = K\frac{(s+z_1)\,(s+z_2)\,(s+z_3)\,\ldots\ldots\ldots}{s^N\,(s+p_1)\,(s+p_2)\,(s+p_3)\,\ldots\ldots}$$

If N = 0, then the system is type – 0 system

If N = 1, then the system is type – 1 system

If N = 2, then the system is type – 2 system

Z1, Z2, Z3 are the Zeros of the system

P1, P2, P3 are Poles of the system

K = System Gain

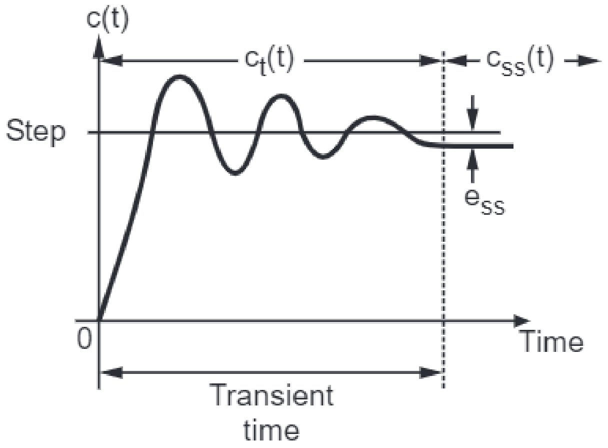
N = Number of poles at the origin

STEADY STATE ERROR

The steady state error is the difference between the actual output and the desired output

The steady state error is the value of error signal e(t), when t tends to infinity.

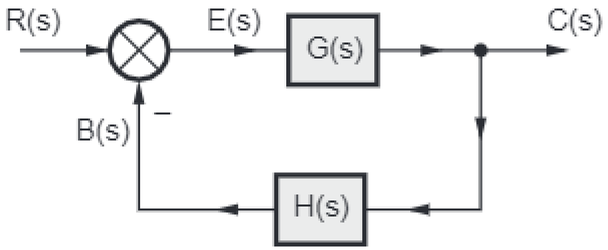
The steady state error is a measure of system accuracy.



These errors arise from the nature of inputs, type of system and from non linearity of system components

The steady state performance of a stable control system is generally judged by its steady state error to step, ramp and parabolic inputs

Consider a closed loop system shown in fig



where $E(s)$ = Error signal, and

$B(s)$ = Feedback signal

Now, $E(s) = R(s) - B(s)$

But $B(s) = C(s)H(s)$

$\therefore$ $E(s) = R(s) - C(s)H(s)$

and $C(s) = E(s)G(s)$

$\therefore$ $E(s) = R(s) - E(s)\,G(s)H(s)$

$\therefore E(s) + E(s)G(s)H(s) = R(s)$

$\therefore$ $E(s) = \dfrac{R(s)}{1+G(s)H(s)}$ for nonunity feedback

$E(s) = \dfrac{R(s)}{1+G(s)}$ for unity feedback

E(s) = error signal in Laplace domain

Let e(t) = error signal in time domain

$$\therefore e(t) = \mathcal{L}^{-1}\{E(s)\} = \mathcal{L}^{-1}\left\{\frac{R(s)}{1+G(s)\,H(s)}\right\}$$

Let, ess = steady state error..

The steady state error is defined as the value of e(t) when t tends to infinity

Steady state error, $e_{ss} = \lim_{t \to \infty} e(t)$

- Now we can relate this in Laplace domain by using **final value theorem** which states that,

$$\lim_{t \to \infty} F(t) = \lim_{s \to 0} sF(s)$$

where $F(s) = L\{ F(t) \}$

Therefore,

$$\boxed{e_{ss} = \lim_{t \to \infty} e(t) = \lim_{s \to 0} sE(s)}$$

where E(s) is L{ e(t) }.

Steady state error depends on

1.Input, its type and magnitude

2.G(s)H(S) Open loop Transfer function (Type of the system)

3. Dominant Nonlinearity present if any

- Substituting E(s) from the expression derived, we can write

$$\boxed{e_{ss} = \lim_{s \to 0} \frac{sR(s)}{1 + G(s)H(s)}}$$

Static Error Constants

When a control system is excited with standard input signal, the steady state error may be zero, constant or infinity

The value of steady state error depends on the type number and the input signal

Type-0 system will have a constant steady state error when the input is step signal.

Type-1 system will have a constant steady state error when the input is ramp signal or velocity signal.

Type-2 system will have a constant steady state error when the input is parabolic signal or acceleration signal.

For the three cases mentioned, the steady state error is associated with one of the constants defined as follows,

Positional error constant $K_p = \lim_{s \to 0} G(s)H(s)$

Velocity error constant $K_v = \lim_{s \to 0} s\,G(s)H(s)$

Acceleration error constant $K_a = \lim_{s \to 0} s^2\,G(s)H(s)$

Kp, Kv , Ka are in general called static error constant

Steady State Error when The Input Is Step Signal

Positional Error coefficient

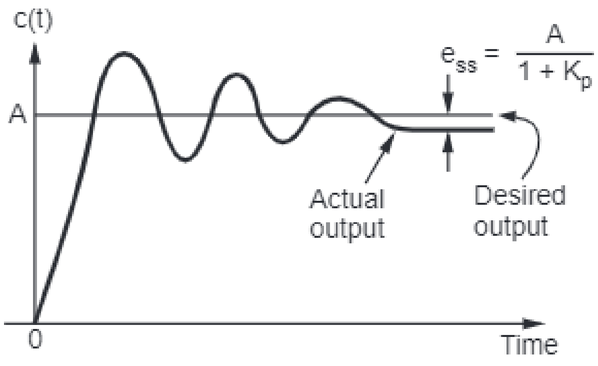


Fig. 7.6.1

$$R(s) = \frac{A}{s}$$

$$e_{ss} = \lim_{s \to 0} \frac{s\,R(s)}{1+G(s)H(s)}$$

$$= \lim_{s \to 0} \frac{s \cdot A/s}{1+G(s)H(s)} = \lim_{s \to 0} \frac{A}{1+G(s)H(s)}$$

$$e_{ss} = \frac{A}{1+\lim\limits_{s \to 0} G(s)H(s)}$$

$$\therefore \quad e_{ss} = \frac{A}{1+\lim\limits_{s \to 0} G(s)H(s)}$$

- For a system selected, $\lim\limits_{s \to 0} G(s)H(s)$ is constant and called **positional error coefficient** of the system denoted as K_p.

$$\boxed{K_p = \lim_{s \to 0} G(s)H(s) = \text{Positional error coefficient}}$$

- And corresponding error is,

$$\boxed{e_{ss} = \frac{A}{1+K_p}}$$

Steady State Error When The Input Is Ramp Signal

Velocity Error coefficient

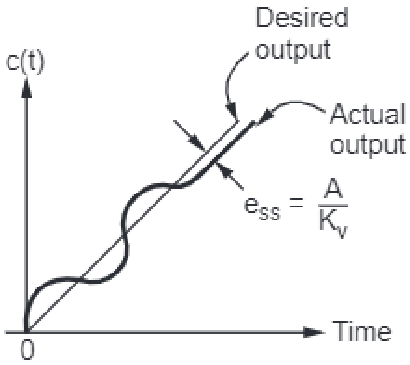


Fig. 7.6.2

$$e_{ss} = \lim_{s \to 0} \frac{sR(s)}{1 + G(s)H(s)}$$

$$= \lim_{s \to 0} \frac{s \cdot A/s^2}{1 + G(s)H(s)}$$

$$= \lim_{s \to 0} \frac{A}{s\,[1 + G(s)H(s)]}$$

$$= \lim_{s \to 0} \frac{A}{s + s\,G(s)H(s)}$$

$$\therefore \qquad e_{ss} = \frac{A}{\lim\limits_{s \to 0} s\,G(s)H(s)}$$

- For a selected system $\lim\limits_{s \to 0} sG(s)H(s)$ is constant and called **velocity error coefficient** as K_v.

$$K_v = \lim_{s \to 0} s\,G(s)H(s)$$
$$= \text{Velocity error coefficient}$$

- And corresponding error is,

$$e_{ss} = \frac{A}{K_v}$$

Steady State Error When The Input Is Parabolic

Acceleration Error coefficient

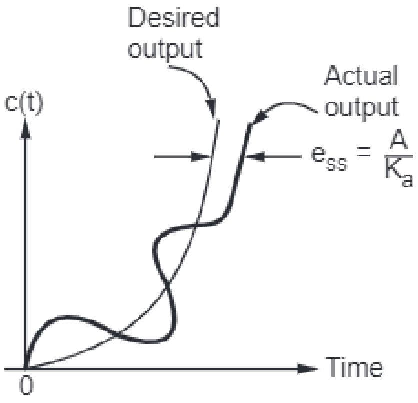


Fig. 7.6.3

$$e_{ss} = \lim_{s \to 0} \frac{sR(s)}{1+G(s)H(s)}$$

$$= \lim_{s \to 0} \frac{s \cdot A/s^3}{1+G(s)H(s)}$$

$$= \lim_{s \to 0} \frac{A}{s^2 [1+G(s)\, H(s)]}$$

$$= \lim_{s \to 0} \frac{A}{s^2 [1+G(s)\, H(s)]}$$

$$= \lim_{s \to 0} \frac{A}{s^2 + s^2\, G(s)H(s)}$$

$$\therefore \qquad e_{ss} = \lim_{s \to 0} \frac{A}{s^2\, G(s)H(s)}$$

- So for a selected system $\lim_{s \to 0} s^2\, G(s)H(s)$ is constant and called **acceleration error coefficient** as K_a.

$$\therefore \qquad \boxed{K_a = \lim_{s \to 0} s^2\, G(s)H(s) = \text{Acceleration error coefficient}}$$

- And corresponding error is,

$$\boxed{e_{ss} = \frac{A}{K_a}}$$

Static Error Constants for various Type number of system

Error Constant	Type number of system			
	0	1	2	3
K_p	constant	∞	∞	∞
K_v	0	constant	∞	∞
K_a	0	0	constant	∞

Steady State Error for various Type of Inputs

Input Signal	Type number of system			
	0	1	2	3
Unit Step	$\frac{1}{1+K_p}$	0	0	0
Unit Ramp	∞	$\frac{1}{K_v}$	0	0
Unit Parabolic	∞	∞	$\frac{1}{K_a}$	0

TABLE 7.2 Relationships between input, system type, static error constants, and steady-state errors

		Type 0		Type 1		Type 2	
Input	Steady-state error formula	Static error constant	Error	Static error constant	Error	Static error constant	Error
Step, $u(t)$	$\frac{1}{1+K_p}$	$K_p = \text{Constant}$	$\frac{1}{1+K_p}$	$K_p = \infty$	0	$K_p = \infty$	0
Ramp, $tu(t)$	$\frac{1}{K_v}$	$K_v = 0$	∞	$K_v = \text{Constant}$	$\frac{1}{K_v}$	$K_v = \infty$	0
Parabola, $\frac{1}{2}t^2u(t)$	$\frac{1}{K_a}$	$K_a = 0$	∞	$K_a = 0$	∞	$K_a = \text{Constant}$	$\frac{1}{K_a}$

Analysis of Static Error Constants and its Steady State Error for various input for different Type number of system

when The Input Is Step Signal

$$K_p = \lim_{s \to 0} G(s)H(s)$$

$$= \text{Positional error coefficient}$$

- And corresponding error is,

$$e_{ss} = \frac{A}{1 + K_p}$$

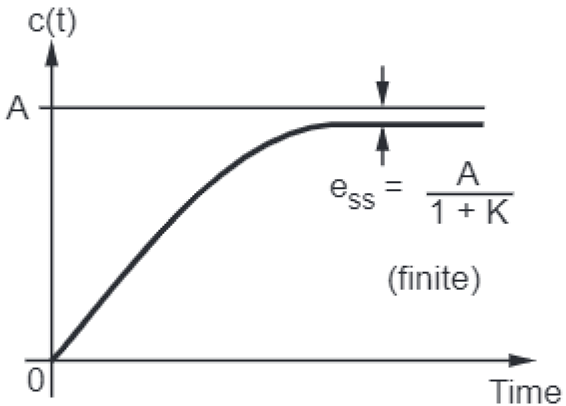
Type-0 system

$$K_p = \underset{s\to 0}{\mathrm{Lt}}\, G(s)\,H(s) = \underset{s\to 0}{\mathrm{Lt}}\, K\frac{(s+z_1)\,(s+z_2)\,(s+z_3)\ldots\ldots}{(s+p_1)\,(s+p_2)\,(s+p_3)\ldots\ldots} = K\frac{z_1.z_2.z_3\ldots\ldots}{p_1.p_2.p_3\ldots\ldots} = \text{constant}$$

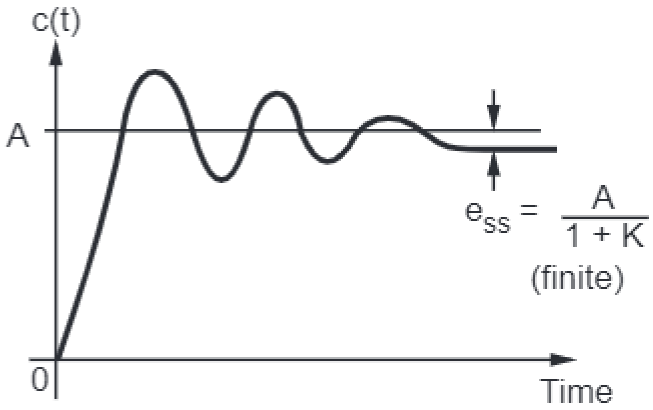
$$e_{ss} = \frac{A}{1 + K_p} = \frac{A}{1 + K}$$

$$\therefore e_{ss} = \frac{1}{1+K_p} = \text{constant}$$

Hence in type-0 systems when the input is unit step there will be a constant steady state error.



(a)



(b)

Response for 1st Order System for step Input

Response for 2nd Order System (Under Damped Case) for step Input

Finite error can be reduced by change in 'A' or 'K' or both as per requirement

Type-1 system

$$K_p = \underset{s \to 0}{Lt}\, G(s)H(s) = \underset{s \to 0}{Lt}\, K \frac{(s+z_1)(s+z_2)(s+z_3)......}{s(s+p_1)(s+p_2)(s+p_3)......} = \infty$$

$$K_p = \lim_{s \to 0} G(s)H(s) = \infty$$

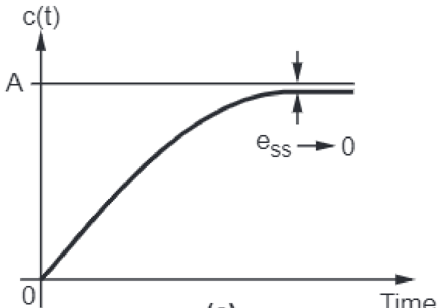
In system with type number 1, for unit step input the value the Value of Kp is infinite and so steady state error is zero

$$e_{ss} = \frac{A}{1+K_p} = \frac{A}{\infty} = 0$$

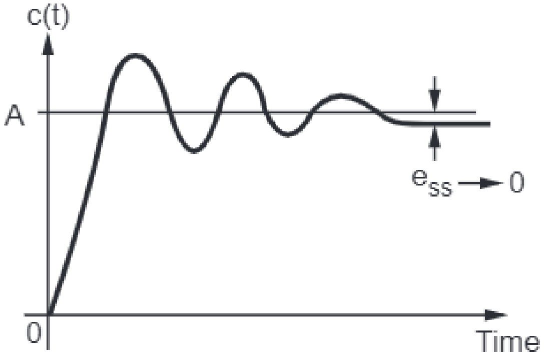
$$\therefore e_{ss} = \frac{1}{1+K_p} = \frac{1}{1+\infty} = 0$$

Mathematically answer for error is zero, practically small error will be present but it will be negligibly small.

Such type of responses may take one of the forms shown in the Fig



(a)



when The Input Is Ramp Signal

$$K_v = \lim_{s \to 0} s\,G(s)H(s)$$

$$= \text{Velocity error coefficient}$$

- And corresponding error is,

$$e_{ss} = \frac{A}{K_v}$$

Type-0 system

$$K_v = \underset{s\to 0}{\mathrm{Lt}}\ sG(s)\,H(s) = \underset{s\to 0}{\mathrm{Lt}}\ sK\frac{(s+z_1)\,(s+z_2)\,(s+z_3)......}{(s+p_1)\,(s+p_2)\,(s+p_3)......} = 0$$

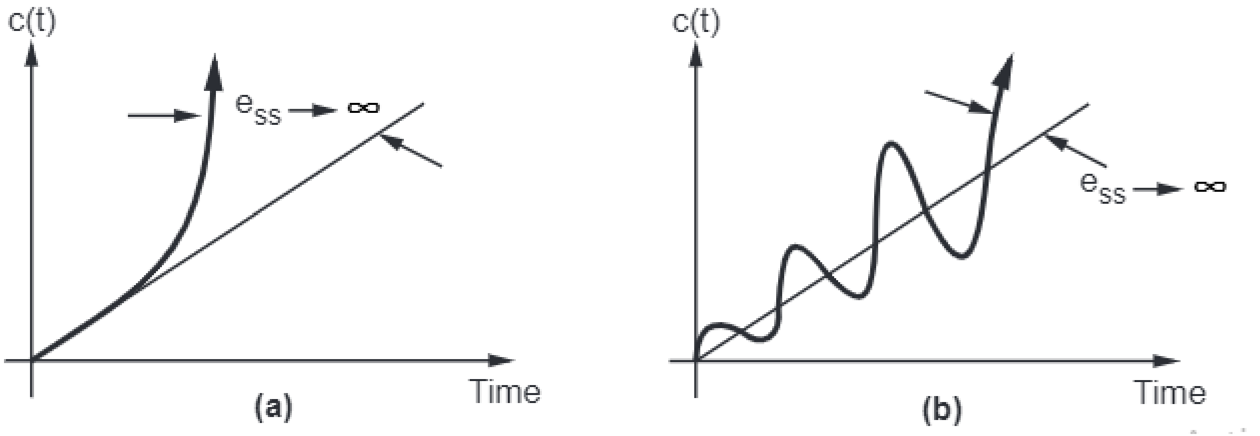
$$K_v = \lim_{s \to 0} s\,G(s)H(s) = 0$$

$$e_{ss} = \frac{A}{K_v} = \frac{A}{0} = \infty$$

In system with type number 0, for unit ramp input the value the Value of Kv is zero and so steady state error is infinite

$$\therefore e_{ss} = 1/K_v = 1/0 = \infty$$

TYPE 0 systems will not follow ramp input of any magnitude and will give large error in the output which may damage the parameters of system or may cause the saturation in parameters. Hence ramp input should not be applied to TYPE 0 systems. The output may take the form as shown in the Fig



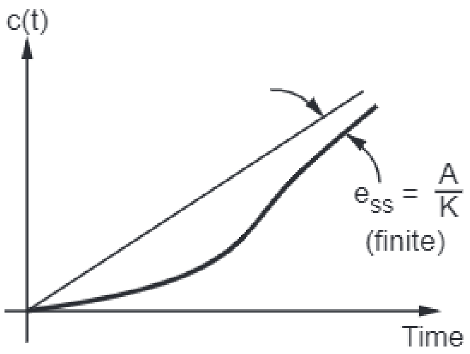
Type-1 system

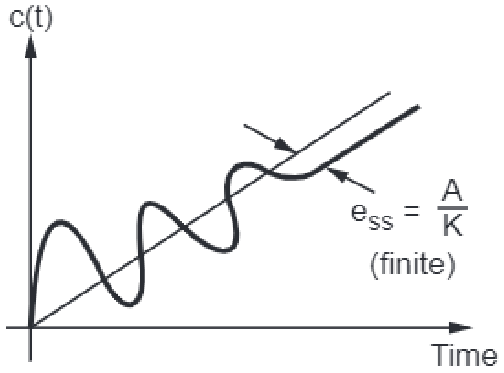
$$K_v = \underset{s\to 0}{Lt}\, sG(s)\,H(s) = \underset{s\to 0}{Lt}\, sK\frac{(s+z_1)(s+z_2)(s+z_3)......}{s(s+p_1)(s+p_2)(s+p_3)......} = K\frac{z_1.z_2.z_3......}{p_1.p_2.p_3......} = \text{constant}$$

$$K_v = \lim_{s\to 0} s\,G(s)H(s) = K$$

$$e_{ss} = \frac{A}{K_v} = \frac{A}{K}\ \text{finite}$$

In system with type number-1, for unit ramp input the value of Kv is Constant and so steady state error is constant(finite)





Type-2 system

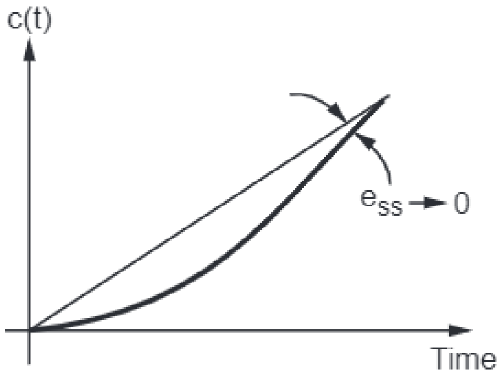
$$K_v = \underset{s\to 0}{Lt}\, sG(s)\,H(s) = \underset{s\to 0}{Lt}\; sK\frac{(s+z_1)\,(s+z_2)\,(s+z_3)......}{s^2\,(s+p_1)\,(s+p_2)\,(s+p_3)......} = \infty$$

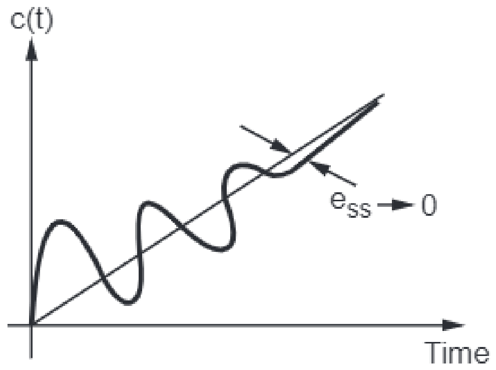
$$K_v = \lim_{s \to 0} s\,G(s)H(s) = \infty$$

$$e_{ss} = \frac{A}{K_v} = \frac{A}{\infty} = 0$$

In system with type number-2, for unit ramp input the value of Kv is Infinite and so steady state error is Zero

$$\therefore e_{ss} = 1/K_v = 1/\infty = 0$$





when The Input Is Parabolic Signal

$$K_a = \lim_{s \to 0} s^2 G(s)H(s)$$

$$= \text{Acceleration error coefficient}$$

- And corresponding error is,

$$e_{ss} = \frac{A}{K_a}$$

Type-0 system

$$K_a = \underset{s\to 0}{\mathrm{Lt}}\ s^2 G(s)\,H(s) = \underset{s\to 0}{\mathrm{Lt}}\ s^2 K \frac{(s+z_1)\,(s+z_2)\,(s+z_3)\ldots\ldots}{(s+p_1)\,(s+p_2)\,(s+p_3)\ldots\ldots} = 0$$

$$K_a = \lim_{s \to 0} s^2 G(s)H(s) = 0$$

$$e_{ss} = \frac{A}{K_a} = \frac{A}{0} = \infty$$

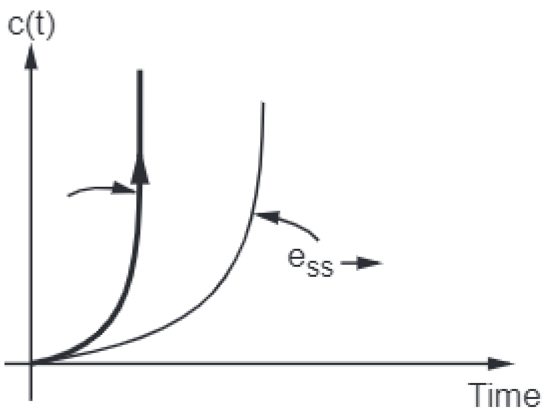
$$\therefore e_{ss} = \frac{1}{K_a} = \frac{1}{0} = \infty$$

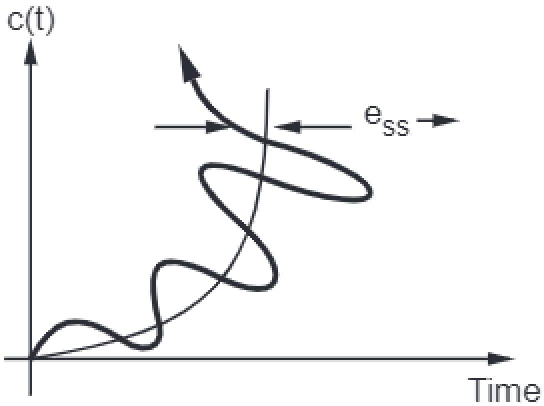
In system with type number-0, for unit parabolic input the value of Ka is Zero and so steady state error is Infinity

Type-1 system

$$K_a = \underset{s\to 0}{\mathrm{Lt}}\ s^2 G(s)\,H(s) = \underset{s\to 0}{\mathrm{Lt}}\ s^2 K \frac{(s+z_1)\,(s+z_2)\,(s+z_3)\ldots\ldots}{s\,(s+p_1)\,(s+p_2)\,(s+p_3)\ldots\ldots} = 0$$

$$e_{ss} = \frac{A}{K_a} = \frac{A}{0} = \infty$$


c(t)
e_{ss}
Time

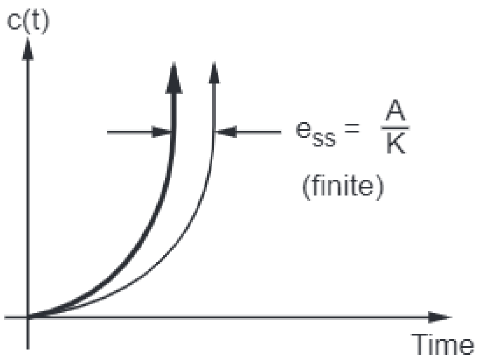

c(t)
e_{ss}
Time

Type-2 system

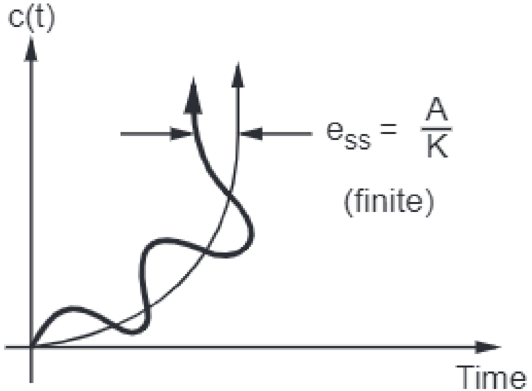
$$K_a = \underset{s \to 0}{Lt}\, s^2 G(s)\, H(s) = \underset{s \to 0}{Lt}\, s^2 K \frac{(s+z_1)\,(s+z_2)\,(s+z_3)......}{s^2(s+p_1)\,(s+p_2)\,(s+p_3)......} = K \frac{z_1.z_2.z_3......}{p_1.p_2.p_3.....} = \text{constant}$$

$$K_a = \lim_{s \to 0} s^2\, G(s)\, H(s) = K$$

$$e_{ss} = \frac{A}{K_a} = \frac{A}{K} \text{ finite}$$



(a)



Static Error Constants for various Type number of system

Error Constant	Type number of system			
	0	1	2	3
K_p	constant	∞	∞	∞
K_v	0	constant	∞	∞
K_a	0	0	constant	∞

Steady State Error for various Type of Inputs

Input Signal	Type number of system			
	0	1	2	3
Unit Step	$\frac{1}{1+K_p}$	0	0	0
Unit Ramp	∞	$\frac{1}{K_v}$	0	0
Unit Parabolic	∞	∞	$\frac{1}{K_a}$	0

1. A unity feedback system is characterised by the open loop transfer $G(s) = \dfrac{10}{(s+1)(s+2)}$. function. Determine the steady state errors for unit step

For unity feedback system H(s) =1

$$K_p = \lim_{s \to 0} G(s)H(s)$$

= Positional error coefficient

- And corresponding error is,

$$e_{ss} = \frac{A}{1+K_p}$$

Steady state errors for unit step input

$$\therefore e_{ss} = \frac{1}{1+K_p}$$

$$\therefore K_p = \underset{s\to 0}{Lt}\ G(s) = \underset{s\to 0}{Lt}\ \frac{10}{(s+1)(s+2)} = 5$$

$$e_{ss} = \frac{1}{1+K_p} = \frac{1}{1+5} = \frac{1}{6}$$

A unity feedback system is characterised by the open loop transfer function. $G(s) = \dfrac{25\,(s+4)}{s(s+0.5)\,(s+2)}$

Determine the steady state errors for unit ramp input

For unity feedback system H(s) =1

$$K_v = \lim_{s \to 0} s\,G(s)H(s)$$
$$= \text{Velocity error coefficient}$$

- And corresponding error is,

$$\boxed{e_{ss} = \frac{A}{K_v}}$$

Steady state errors for unit Ramp input

$$e_{ss} = \frac{1}{K_v}$$

$$\therefore K_v = \underset{s\to 0}{\mathrm{Lt}}\; sG(s) = \underset{s\to 0}{\mathrm{Lt}}\; s\left[\frac{25\,(s+4)}{s(s+0.5)\,(s+2)}\right] = \frac{25\times 4}{0.5\times 2} = 100$$

$$e_{ss} = \frac{1}{K_v} = \frac{1}{100} = 0.01$$

A unity feedback system is characterised by the open loop transfer function. $G(s) = \dfrac{20(s+5)}{s(s+0.1)(s+3)}$

Determine the steady state errors for unit acceleration input

For unity feedback system H(s) =1

$$\therefore \quad \boxed{\begin{aligned} K_a &= \lim_{s \to 0} s^2 G(s)H(s) \\ &= \text{Acceleration error coefficient} \end{aligned}}$$

- And corresponding error is,

$$\boxed{e_{ss} = \frac{A}{K_a}}$$

Steady state errors for unit acceleration input $e_{ss} = \dfrac{1}{K_a}$

$$\therefore K_a = \underset{s \to 0}{Lt}\ s^2 \left[\frac{20(s+5)}{s^2(s+0.1)(s+3)}\right] = \frac{20 \times 5}{0.1 \times 3} = \frac{100}{0.3} = 333.33$$

$$e_{ss} = \frac{1}{K_a} = \frac{1}{333.33} = 0.003$$

A unity feedback system is characterised by the open loop transfer function. $G(s)H(s) = \dfrac{20(1+s)}{s^2(2+s)(4+s)}$

Determine the Static error coefficient and steady state errors for applied input $r(t) = 40 + 2t + 5t^2$.

$$G(s)H(s) = \frac{20(1+s)}{s^2(2+s)(4+s)} = \frac{20\,(1+s)}{s^2 \times 2\left(1+\frac{s}{2}\right)\times 4\left(1+\frac{s}{4}\right)}$$

$$G(s)H(s) = \frac{2.5(1+s)}{s^2(1+0.5\,s)(1+0.25\,s)}$$

Type-2 system

$$K_p = \lim_{s\to 0} G(s)H(s)$$

$$= \lim_{s\to 0} \frac{2.5\,(1+s)}{s^2\,(1+0.5\,s)\,(1+0.25\,s)} = \infty$$

$$K_v = \lim_{s\to 0} sG(s)H(s)$$

$$= \lim_{s\to 0} \frac{s\times 2.5\,(1+s)}{s^2\,(1+0.5\,s)\,(1+0.25\,s)} = \infty$$

$$K_a = \lim_{s\to 0} s^2G(s)H(s)$$

$$= \lim_{s\to 0} \frac{s^2\times 2.5\,(1+s)}{s^2\,(1+0.5\,s)\,(1+0.25\,s)} = 2.5$$

Name of the signal	Time domain equation of signal, r(t)	Laplace transform of the signal, R(s)
Step	A	$\frac{A}{s}$
Unit step	1	$\frac{1}{s}$
Ramp	At	$\frac{A}{s^2}$
Unit ramp	t	$\frac{1}{s^2}$
Parabolic	$\frac{At^2}{2}$	$\frac{A}{s^3}$
Unit parabolic	$\frac{t^2}{2}$	$\frac{1}{s^3}$
Impulse	$\delta(t)$	1

$$r(t) = 40 + 2t + 5t^2 = 40 + 2t + \frac{10}{2}t^2$$

$$= A_1 + A_2 t + \frac{A_3}{2}t^2 \qquad \text{Hence } A_1 = 40 \text{ step, } A_2 = 2 \text{ ramp, } A_3 = 10 \text{ parabolic}$$

$$e_{ss} = e_{ss1} + e_{ss2} + e_{ss3} = \frac{A_1}{1+K_p} + \frac{A_2}{K_v} + \frac{A_3}{K_a}$$

Total steady state errors $e_{ss} = \frac{40}{1+\infty} + \frac{2}{\infty} + \frac{10}{2.5}$

$$= 0 + 0 + 4 = 4$$

A unity feedback system is characterised by the open loop transfer function. $G(s) = \frac{10(s+2)}{s^2(s+1)}$.

Determine the Static error coefficient and steady state errors for applied input $R(s) = \frac{3}{s} + \frac{2}{s^2} + \frac{1}{3s^3}$

Type-2 system

$$K_p = \lim_{s \to 0} G(s)H(s) = \lim_{s \to 0} \frac{10(s+2)}{s^2(s+1)} = \infty$$

$$R(s) = \frac{3}{s} + \frac{2}{s^2} + \frac{1}{3s^3} \quad \text{i.e.} \quad r(t) = 3 + 2t + \frac{1}{6}t^2$$

$$r(t) = A_1 + A_2 t + \frac{A_3}{2}t^2$$

$$K_v = \lim_{s \to 0} sG(s)H(s) = \lim_{s \to 0} \frac{s \times 10(s+2)}{s^2(s+1)} = \infty$$

$$\text{where } A_1 = 3,\ A_2 = 2,\ A_3 = \frac{1}{3}$$

$$K_a = \lim_{s \to 0} s^2G(s)H(s) = \lim_{s \to 0} \frac{s^2 \times 10(s+2)}{s^2(s+1)} = 20$$

$$e_{ss} = \frac{A_1}{1+K_p} + \frac{A_2}{K_v} + \frac{A_3}{K_a} = 0 + 0 + \frac{\left(\frac{1}{3}\right)}{20} = \mathbf{0.01667}$$

Concept of stability

The term stability refers to the stable working condition of a control system. Every working system is designed to be stable. In a stable system, the response or output is predictable, finite and stable for a given input (or for any changes in input or for any changes in system parameters).

The different definitions of the stability are the following

1. A system is stable, if its output is bounded (finite) for any bounded (finite) input.

2. A system is asymptotically stable, if in the absence of the input, the output tends towards zero (or to the equilibrium state) irrespective of initial conditions

3. A system is stable if for a bounded disturbing input signal , the output vanishes ultimately as t approaches infinity

4. A system is unstable if for a bounded disturbing input signal, the output is of infinite amplitude or oscillatory.

5. For a bounded input signal, if the output has constant amplitude oscillations and constant frequency of oscillations Such a system is called Marginally stable system/Limited stable

6. If a system output is stable for all variations of its parameters, then the system is called absolutely stable system.

7.If a system output is stable for a limited range of variations of its system parameters, then the system is called conditionally stable system.

In **summary**, the following three points may be stated regarding the stability of the system depending on the location of roots of characteristic equation

(i) If all the roots of characteristic equation has negative real parts, then the system is stable

(ii) If any root of the characteristic equation has a positive real part or if there is a repeated root on the imaginary axis then the system is unstable

(iii) If the condition (i) is satisfied except for the presence of one or more non repeated roots on the imaginary axis, then the system is limitedly or marginally stable.

Consider the characteristic equation of the order 'n' is

$$a_0 s^n + a_1 s^{n-1} + a_2 s^{n-2} + \ldots + a_{n-1} s^1 + a_n s^0 = 0$$

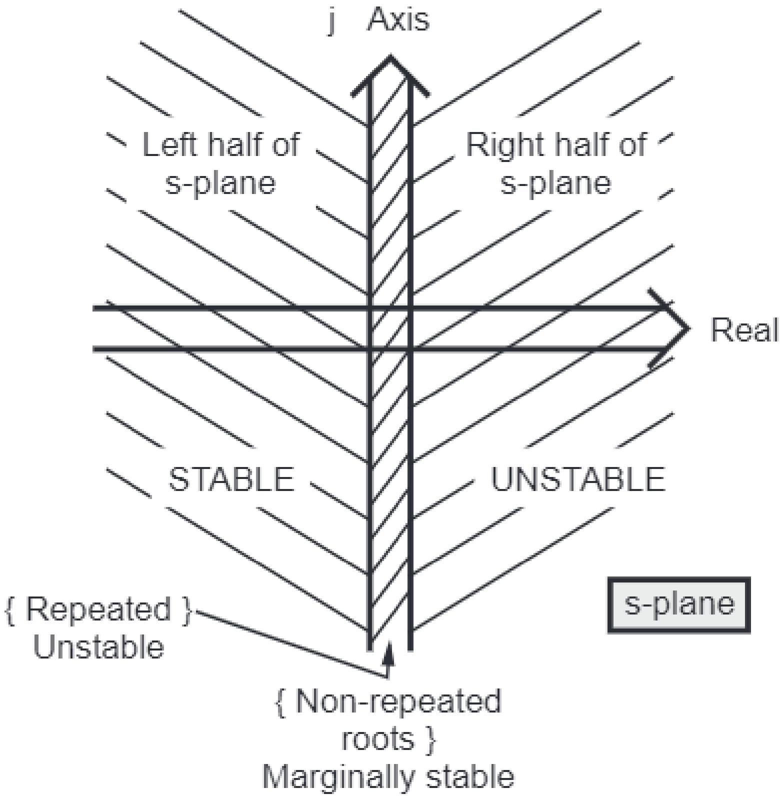
The necessary condition is that the coefficients of the characteristic polynomial should be positive.

This implies that all the roots of the characteristic equation should have negative real parts

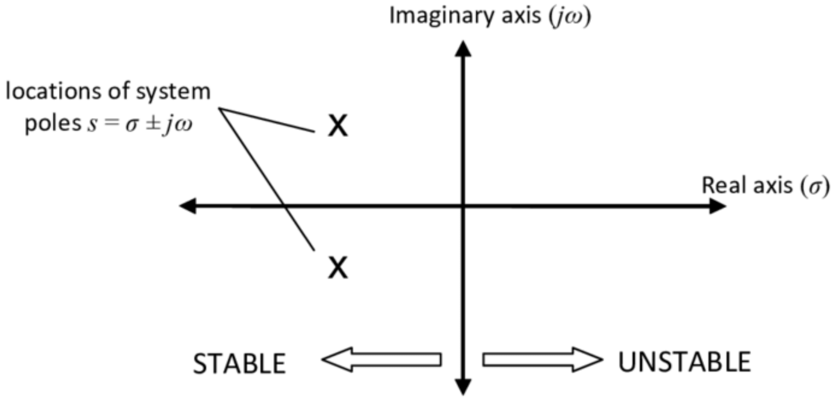
If any coefficient is zero/Negative, the system is unstable

Effect of pole location on Stability:

The ‘S’ plane” is divided into 3 distinct zone from stability point of view.



The stability of a feedback system is directly related to the location of the roots of the characteristic equation of the system transfer function.



“A linear system will be stable if and only if all the poles of the transfer function are located on the left half of the ‘S’ plane”.

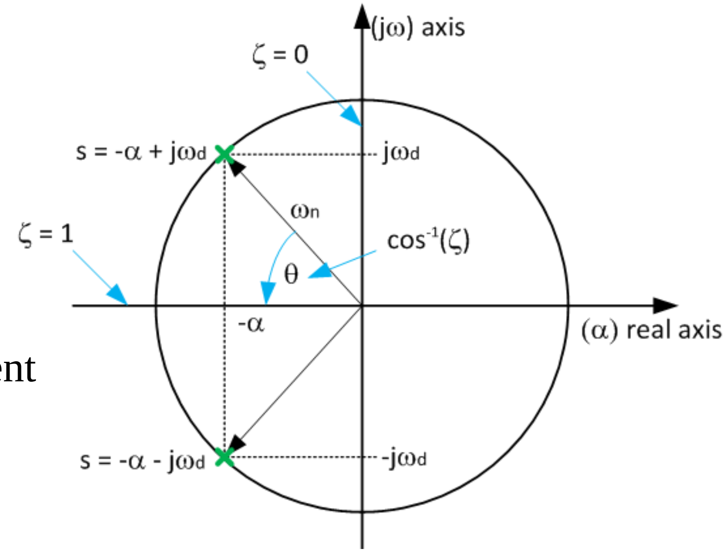
Transfer function stability is solely determined by its denominator.

The roots of a denominator are called **poles**.

Poles located in the left half-plane are stable while poles located in the right half-plane are not stable.

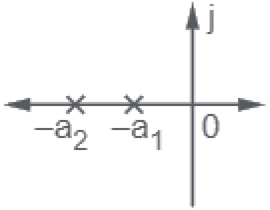
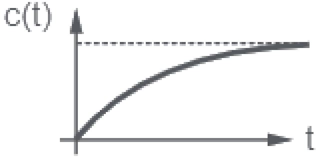
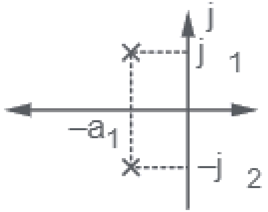
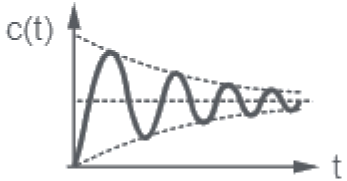
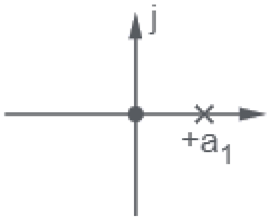
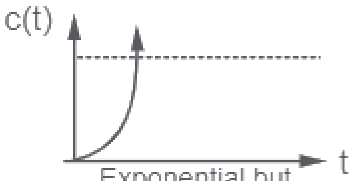
The reasoning is very simple: the Laplace operator **"s"**, which is location in the Laplace domain, can be also written $s = \sigma + j\omega$

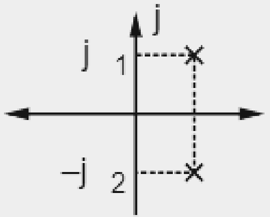
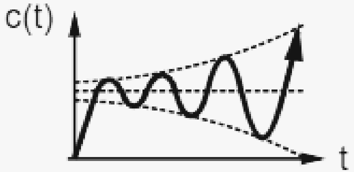
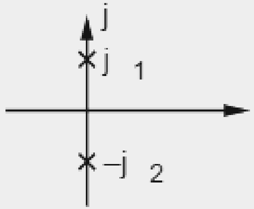
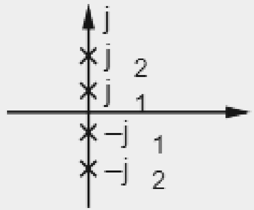
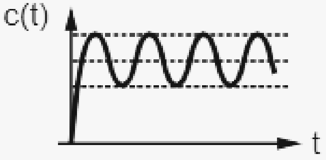
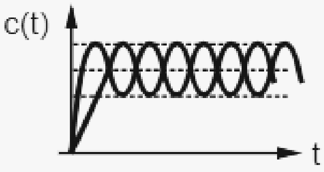
Left half-plane has negative *sigma*

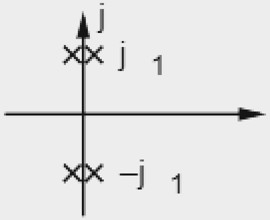
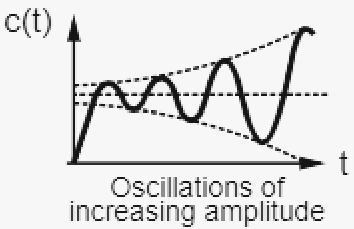


The plane shows the damping frequency and damping coefficient "zeta" graphically.

Effect of pole location on Stability:

Sr. No.	Nature of closed loop poles	Locations of closed loop poles in s-plane	Step response	Stability condition
1.	Real, negative i.e. in L.H.S. of s-plane		 Pure exponential	Absolutely stable
2.	Complex conjugate with negative real part i.e. in L.H.S. of s-plane		 Damped oscillations	Absolutely stable
3.	Real, positive i.e. in R.H.S. of s-plane (Any one closed loop pole in right half irrespective of number of poles in left half of s-plane)		 Exponential but increasing towards	Unstable

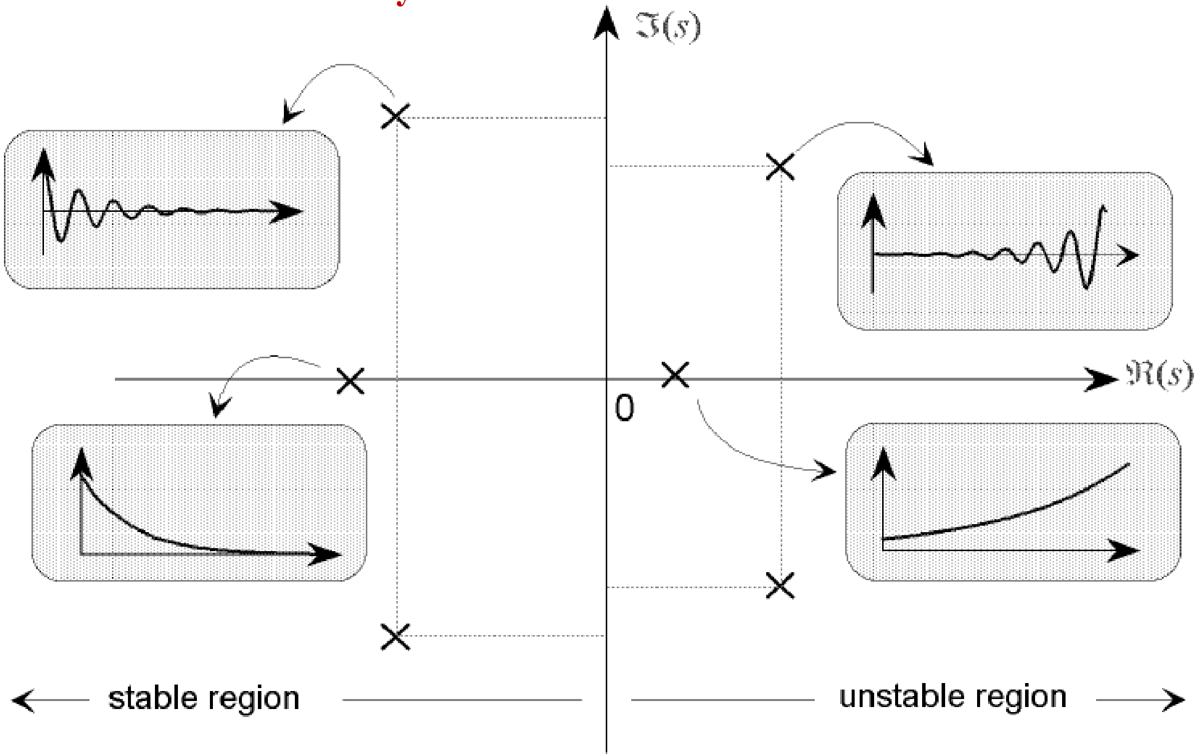
4.	Complex conjugate with positive real part i.e. in R.H.S. of s-plane		Oscillations with increasing amplitude	Unstable
5.	Non repeated pair on imaginary axis without any pole in R.H.S. of s-plane	ortwo non repeated pairs on imaginary axis.	Frequency of oscillations = ω_1Sustained oscillations with two frequency components ω_1 and ω_2	Marginally or critically stable Marginally or critically stable.

6.	Repeated pair on imaginary axis without any pole in R.H.S. of s-plane			Unstable

IMPULSE RESPONSE AND STABILITY

For BIBO (Bounded input bounded output) Stability the integral of impulse response should be finite, which implies that the impulse response should be finite as time 't' tends to infinity

"A linear system will be stable if and only if all the poles of the transfer function are located on the left half of the 'S' plane".



Pole locations on the pole-zero plot

RH Criterion

The Rouths Stability Criterion is a quick and easy method of establishing system stability. The stability of system can be ascertained without actually having to determine its roots

The Routh-Hurwitz stability criterion is an analytical procedure for determining whether all the roots of a polynomial characteristic Equation have negative real part or not.

Necessary Condition for Routh-Hurwitz Stability

Consider the Linear time invariant system characteristic equation of the order 'n' is -

$$a_0 s^n + a_1 s^{n-1} + a_2 s^{n-2} + \ldots + a_{n-1} s^1 + a_n s^0 = 0$$

The first step in analysing the stability of a system is to examine its characteristic equation. The necessary condition for stability is that all the coefficients of the polynomial characteristic be real and have same sign. There should be no missing term, If some of the coefficients missing (are zero) or are negative, it can be concluded that the system is not stable

$$a_0 s^n + a_1 s^{n-1} + a_2 s^{n-2} + \ldots + a_{n-1} s^1 + a_n s^0 = 0$$

Note that, there should not be any term missing in the **n^th^** order characteristic equation. This means that the **n^th^** order characteristic equation should not have any coefficient that is of zero value.

When all the coefficients (a0, a1, a2…an) are Real & have same sign and there are no missing terms (none is zero), No guarantee that the system will be stable. For this, we use Routh Hurwitz Criterion to check the stability of the system. Even though the coefficient are same sign, some of the roots may lie on the right half of s-plane or on the imaginary axis

$$a_0 s^n + a_1 s^{n-1} + a_2 s^{n-2} + \ldots + a_{n-1} s^1 + a_n s^0 = 0$$

One should proceed further to examine the sufficient conditions of stability by use of Routh Hurwitz Criterion to check the stability of the system..

R-H Criteria is based on ordering the coefficients of characteristic Polynomial in the form of array called the Routh's array

"The necessary and sufficient condition for stability that all of the elements in the first column of the Routh's array Table must be positive. If this condition is not met, the system is unstable and the number of sign changes in the elements of the first column of the Routh array corresponds to the number of roots of the characteristic equation in the right half of the s-plane". i.e. equals to the number of roots with positive real parts.

Advantages of Routh- Hurwitz Criterion

We can find the stability of the system without solving the equation.

We can easily determine the relative stability of the system.

By this method, we can determine the range of K(Gain) for stability

By this method, we can also determine the point of intersection for root locus with an imaginary axis

Limitations of Routh- Hurwitz Criterion

This criterion is applicable only for a linear system.

It does not provide the exact location of poles on the right and left half of the S plane.

In case of the characteristic equation, it is valid only for real coefficients.

The Routh stability criterion is based on ordering the coefficients of the characteristic equation, into a schedule, called the Routh array.

The T.F of Linear time invariant system can be represented as $\frac{C(s)}{R(s)} = \frac{b_0 s^m + b_1 s^{m-1} + ... + b_m}{a_0 s^n + a_1 s^{n-1} + ... + a_n}$

Consider the Linear time invariant system characteristic equation of the order 'n' is -

$$a_0 s^n + a_1 s^{n-1} + a_2 s^{n-2} + \ldots + a_{n-1} s^1 + a_n s^0 = 0$$

When the coefficients a0, a1, ...an are all of the same sign, and none is zero.

Then the Routh's array is given by the coefficients of the polynomial which is arranged in rows and column

Routh suggested a method of tabulating the coefficients of characteristic equation in a particular way. Tabulation of coefficients gives an array called Routh's array Table

$$a_0 s^n + a_1 s^{n-1} + a_2 s^{n-2} + \ldots + a_{n-1} s^1 + a_n s^0 = 0$$

Step 1: Arrange all the coefficients of the above equation in two rows:

Row 1	a_0	a_2	a_4	....
Row 2	a_1	a_3	a_5	....

Step 2: From these two rows we will form the third row:

Row 1	a_0	a_2	a_4	....
Row 2	a_1	a_3	a_5	....
Row 3	b_1	b_3	b_5	....

Where,

$$b_1 = -\frac{1}{a_1}\begin{vmatrix} a_0 & a_2 \\ a_1 & a_3 \end{vmatrix} = -\frac{a_0 a_3 - a_1 a_2}{a_1}$$

$$b_3 = -\frac{1}{a_1}\begin{vmatrix} a_0 & a_4 \\ a_1 & a_5 \end{vmatrix} = -\frac{a_0 a_5 - a_1 a_5}{a_1}$$

Step 3: Now, we shall form fourth row by using second and third row:

Row 1	a_0	a_2	a_4	....
Row 2	a_1	a_3	a_5	....
Row 3	b_1	b_3	b_5	....
Row 4	c_1	c_3	c_5	....

Where,

$$c_1 = -\frac{1}{b_1}\begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} = -\frac{a_1 b_3 - b_1 a_3}{b_1}$$

$$c_3 = -\frac{1}{b_1}\begin{vmatrix} a_1 & a_5 \\ b_1 & b_5 \end{vmatrix} = -\frac{a_1 b_5 - b_1 a_5}{b_1}$$

Step 4: Continue this procedure of forming a new rows:

$$a_0 s^n + a_1 s^{n-1} + a_2 s^{n-2} + \ldots + a_{n-1} s^1 + a_n s^0 = 0$$

Method of forming an array :

s^n	a_0	a_2	a_4	a_6
s^{n-1}	a_1	a_3	a_5	a_7
s^{n-2}	b_1	b_2	b_3	
s^{n-3}	c_1	c_2	c_3	
:	:	:	:	
s^0	a_n			

When n is even, the s^n row is formed by coefficients of even order terms (i.e., coefficients of even powers of s) and s^{n-1} row is formed by coefficients of odd order terms (i.e., coefficients of odd powers of s).

When n is odd, the s^n row is formed by coefficients of odd order terms (i.e., coefficients of odd powers of s) and s^{n-1} row is formed by coefficients of even order terms (i.e., coefficients of even powers of s).

Other row of Routh array upto s^0 is constructed by using the elements of previous two rows.

Consider the Linear time invariant system characteristic equation of the order 'n' is -

$$a_0 s^n + a_1 s^{n-1} + a_2 s^{n-2} + \ldots + a_{n-1} s^1 + a_n s^0 = 0$$

Coefficients for first two rows are written directly from characteristic equation.

Method of forming an array :

s^n	a_0	a_2	a_4	a_6
s^{n-1}	a_1	a_3	a_5	a_7
s^{n-2}	b_1	b_2	b_3	
s^{n-3}	c_1	c_2	c_3	
:	:	:	:	
s^0	a_n			

From these two rows, next rows can be obtained as follows.

$$b_1 = \frac{a_1 a_2 - a_0 a_3}{a_1}, \quad b_2 = \frac{a_1 a_4 - a_0 a_5}{a_1}, \quad b_3 = \frac{a_1 a_6 - a_0 a_7}{a_1}$$

From 2^{nd} and 3^{rd} row, 4^{th} row can be obtained as

$$c_1 = \frac{b_1 a_3 - a_1 b_2}{b_1}, \qquad c_2 = \frac{b_1 a_5 - a_1 b_3}{b_1}$$

This process is to be continued till the coefficient for s^0 is obtained which will be a_n. From this array stability of a system can be predicted.

RH Criterion: Array Formation

$$a_0s^n + a_1s^{n-1} + a_2s^{n-2} + \ldots + a_{n-1}s^1 + a_ns^0 = 0$$

S^n	a_0	a_2	a_4	a_6	-----
S^{n-1}	a_1	a_3	a_5	a_7	------
S^{n-2}	$b_1 = \frac{a_1a_2 - a_0a_3}{a_1}$	$b_2 = \frac{a_1a_4 - a_0a_5}{a_1}$	$b_3 = \frac{a_1a_6 - a_0a_7}{a_1}$		
S^{n-3}	$c_1 = \frac{b_1a_3 - a_1b_2}{b_1}$	$c_1 = \frac{b_1a_5 - a_1b_3}{b_1}$	---		
--	---	---			
S^1	---				
S^0	a_n				

S^n	a_0	a_2	a_4	a_6	-----
S^{n-1}	a_1	a_3	a_5	a_7	------
S^{n-2}	$b_1 = \frac{a_1 a_2 - a_0 a_3}{a_1}$	$b_2 = \frac{a_1 a_4 - a_0 a_5}{a_1}$	$b_3 = \frac{a_1 a_6 - a_0 a_7}{a_1}$		
S^{n-3}	$c_1 = \frac{b_1 a_3 - a_1 b_2}{b_1}$	$c_2 = \frac{b_1 a_5 - a_1 b_3}{b_1}$	---		
--	d1	---			
S^1					
S^0	a_n				

"The necessary and sufficient condition for stability that all of the elements in the first column of the Routh array must have same Sign. If this condition is not met, the system is unstable and the number of sign changes in the elements of the first column of the Routh array corresponds to the number of roots of the characteristic equation in the right half of the s-plane". i.e. equals to the number of roots with positive real parts.

Note: If the order of sign change of first column element is +, +, -, + and +. Then + to - is considered as one sign change and – to + as another sign change.

In the process of constructing Routh array the missing terms are considered as zeros. Also, all the elements of any row can be multiplied or divided by a positive constant to simplify the computational work

Determine the stability of the system using RH criteria whose characteristic equation is

$s^3 + 6s^2 + 11s + 6 = 0$

$$a_0 s^n + a_1 s^{n-1} + a_2 s^{n-2} + \ldots + a_{n-1} s^1 + a_n s^0 = 0$$

$a_0 = 1, \quad a_1 = 6, \quad a_2 = 11, \quad a_3 = 6, \quad n = 3$

S^3	1	11
S^2	6	6
S^1	$\frac{6*11-1*6}{6} = 10$	---
S^0	6	

There is no sign change in the first column hence the system is stable.

Determine the stability of the system using RH criteria whose characteristic equation is

$s^3 + 4s^2 + s + 16 = 0$

$a_0 = 1, \quad a_1 = 4, \quad a_2 = 1, \quad a_3 = 16$

S^3	1	1
S^2	4	16
S^1	$\frac{4*1-1*16}{4} = -3$	0
S^0	16	

There is two sign change in the first column hence the system is unstable. Two roots are located in the right half of the s-plane.

Determine the stability of the system using RH criteria whose characteristic equation is

$$s^4 + 2s^3 + 4s^2 + 6s + 8 = 0$$

S^4	1	4	8
S^3	2	6	0
S^2	+1	8	0
S^1	−10	0	
S^0	+8		

There is two sign change in the first column hence the system is unstable. Two Poles are located in the right half of the s-plane.

Determine the stability of the system using RH criteria whose characteristic equation is

$s^4 + 2s^3 + 10s^2 + 8s + 3 = 0$

S^4	1	10	3
S^3	2	8	--
S^2	$\frac{2*10-1*8}{2} = 6$	$\frac{2*3-1*0}{2} = 3$	---
S^1	$\frac{6*8-2*3}{6} = 7$	$\frac{6*0-2*0}{6} = 0$	---
S^0	$\frac{7*3-6*0}{7} = 3$		--

In construction of Routh array one may come across the following three cases

Case-I : Normal Routh array (Non-zero elements in the first column of Routh array).

Special Cases.

Case-II : A row of all zeros.

Case-III: First element of a row is zero but some or other elements are not zero.

Case-I : **Normal Routh array**

In this case, there is no difficulty in forming Routh array. The Routh array can be constructed explained above. The sign changes are noted to find the number of roots lying on the right half of S-Plane and the stability of the system can be estimated

In this case,

1.If there is no sign change in the first column of Routh array then all the roots are lying on left half of s-plane and the system is stable.

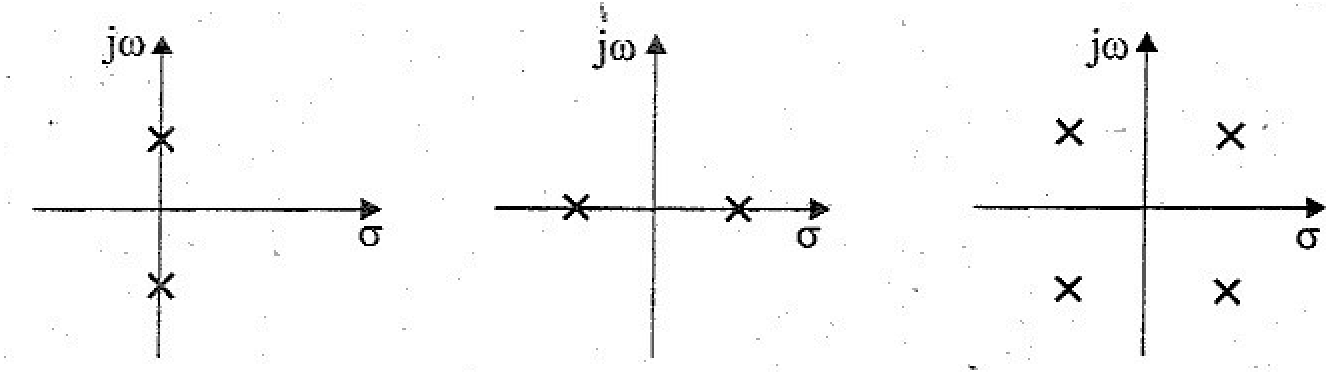
2. If there is sign change in the first column of routh array, then the system is unstable and the number of roots lying on the right half of s-plane is equal to number of sign changes. The remaining roots are lying on the left half of s-plane.

Case-II : A row of all zeros

An all zero row indicates the existence of an even polynomial as a factor of the given characteristic equation. In an even polynomial the exponents of s are even integers or zero only. This even polynomial factor is also called **auxiliary polynomial**. **The coefficients of the auxiliary polynomial will always be the elements of the row directly above the row of zeros in the array.**

The roots of an **auxiliary polynomial (**Even polynomial) occur in pairs that are equal in magnitude and opposite in sign. Hence, these roots can be purely imaginary, purely real or complex. The purely imaginary and purely real roots occur in pairs. The complex roots occur in groups of four and the complex roots have quadrantal symmetry, that is the roots are symmetrical with respect to both the real and imaginary axes

The roots of an **auxiliary polynomial (**Even polynomial) occur in pairs that are equal in magnitude and opposite in sign



The case-II polynomial can be analyzed by any one of the following two methods.

1. Determine the auxiliary polynomial, A(s)

2. Differentiate the auxiliary polynomial with respect to s, to get d A(s)/ds

3. The row of zeros is replaced with coefficients of dA(s)/ds

4. Continue the construction of the Routh array in the usual manner (as that of case-I) and the array is interpreted as follows.

A. If there are sign changes in the first column of routh array then the system is unstable. The number of roots lying on right half of s-plane is equal to number of sign changes. The number of roots on imaginary axis can be estimated from the roots of auxiliary polynomial. The remaining roots are lying on the left half of s-plane.

B. If there are no sign changes in the first column of routh array then the all zeros row indicate the existence of purely imaginary roots and so the system is limitedly or marginally stable. The roots of auxiliary equation lies on imaginary axis and the remaining roots lies on left half of s-plane.

Determine the stability of the system using RH criteria whose characteristic equation is

$s^6 + 2s^5 + 8s^4 + 12s^3 + 20s^2 + 16s + 16 = 0$

$$a_0 s^n + a_1 s^{n-1} + a_2 s^{n-2} + \ldots + a_{n-1} s^1 + a_n s^0 = 0$$

$a_0 = 1$, $a_1 = 2$, $a_2 = 8$, $a_3 = 12$, $a_4 = 20$, $a_5 = 16$, $a_6 = 16$

s^6	1	8	20	16
s^5	2	12	16	0
S^4	2	12	16	0
S^3	0	0	0	0
S^3				
S^2				
S^1				
S^0				

- **The 4th row (S^3) consists of zeros in all the elements. An all zero row**

Auxiliary equation: $A(s) = 2s^4 + 12s^2 + 16 = 0$

The coefficients of the auxiliary polynomial will always be the elements of the row directly above the row of zeros in the array

Differentiate Auxiliary equation w.r.t s

$dA(s)/ds = 8s^3 + 24s = 0$

The coefficients of **$dA(s)/ds$ are used to form S^3 row**

Auxiliary equation: $A(s) = 2s^4 + 12s^2 + 16 = 0$ **Differentiate w.r.t s = $dA(s)/ds = 8s^3 + 24s = 0$**

The coefficients of dA(s)/ds are used to form S^3 row, Complete the construction of array in the usual was (as that of Case-1)

s^6	1	8	20	16
s^5	2	12	16	0
S^4	2	12	16	0
S^3	8	24	0	0
S^2	6	16	0	
S^1	2.67	0		
S^0	16			

No sign change, hence the system is stable.

$2s^4 + 12s^2 + 16 = 0$

$s^4 + 6s^2 + 8 = 0$ **Put $s^2 = y$**

$y^2 + 6y + 8 = 0$

$y = -2$ & $y = -4$ $j = \sqrt{-1}$

$s^2 = -2$ & $s^2 = -4$ $s = \pm j\sqrt{2}$ & $s = \pm j2$

On examining the elements of 1st column of routh array it is observed that there is no sign change. The row with all zeros indicate the possibility of roots on imaginary axis. **Auxiliary equation** has **Non-repeated roots on imaginary axis. Hence the system is** limitedly or **marginally stable.**

Special Cases. Case-III: First element of a row is zero but some or other elements are not zero.

While constructing Routh array, **if a zero is encountered as first element of a row then all the elements of the next row will be infinite.** To overcome this problem let $0 \rightarrow \varepsilon$ and complete the construction of array in the usual way (as that of case-I)

Finally let $\varepsilon \rightarrow 0$ and determine the values of the elements of the array which are functions of ε. The resultant array is interpreted as follows.

- **Note**: If all the elements of a row are zeros then the solution is attempted by considering the polynomial as case-II polynomial. **Even if there is a single element zero on S^1 row, it is considered as a row of all zeros**

A. If there is no sign change in first column of routh array and if there is no row with all zeros, then all the roots are lying on left half of s-plane and the system is stable.

B. If there are sign changes in first column of routh array and there is no row with all zeros, then some of the roots are lying on the right half of s-plane and the system is unstable. The number of roots lying on the right half of s-plane is equal to number of sign changes and the remaining roots are lying on the left half of s-plane.

C. If there is a row of all zeros after letting $\varepsilon \rightarrow 0$, then there is a possibility of roots on imaginary axis. Determine the auxiliary polynomial as explained in method of case-II, The coefficients of dA(s)/ds are used to form row of all zeros, Complete the construction of array in the usual was (as that of Case-1)

Determine the stability of the system using RH criteria whose characteristic equation is $s^5+2s^4+3s^3+6s^2+2s+1=0$

Method 1: $a_0 = 1, a_1 = 2, a_2 = 3, a_3=6, a_4 = 2, a_5=1$

s^5	1	3	2
S^4	2	6	1
S^3	$\frac{2*3-1*6}{2}=0$	$\frac{2*2-1*1}{2}=1.5$	0
S^2			
S^1			
S^0			

s^5	1	3	2
S^4	2	6	1
S^3	ε	1.5	0
S^2	$\frac{6\varepsilon-3}{\varepsilon}$	1	0
S^1	$\frac{\frac{(6\varepsilon-3)1.5}{\varepsilon}-\varepsilon}{\frac{6\varepsilon-3}{\varepsilon}}$	0	
S^0	1		

- The 3rd row consists of zero in the first element.
- **Substitute a small positive number ‘ε’ in place of zero.**

S^5	1	3	2
S^4	2	6	1
S^3	ε	1.5	0
S^2	$\frac{6\varepsilon-3}{\varepsilon}$	1	0
S^1	$\frac{\frac{(6\varepsilon-3)1.5}{\varepsilon}-\varepsilon}{\frac{6\varepsilon-3}{\varepsilon}}$	0	
S^0	1		

Letting $\varepsilon \rightarrow 0$

S^5	1	3	2
S^4	2	6	1
S^3	$+\varepsilon$	1.5	0
S^2	- infinity	1	0
S^1	+ 1.5	0	
S^0	1		

$$\lim_{\varepsilon \rightarrow 0}\left(\frac{6\varepsilon-3}{\varepsilon}\right) = -\infty$$ sign is negative.

$$\lim_{\varepsilon \rightarrow 0} \frac{1.5(6\varepsilon-3)-\varepsilon^2}{6\varepsilon-3} = \lim_{\varepsilon \rightarrow 0} \frac{9\varepsilon-4.5-\varepsilon^2}{6\varepsilon-3}$$

$$= \frac{0-4.5-0}{0-3}$$

$= +1.5$ sign is positive.

Alternative Method for Case-III: $s^5+2s^4+3s^3+6s^2+2s+1=0$

- Replace S by (1/Z) in Char. Eq. and Transfer the Char Equation into Z plane and formulate the Routh Array Table

$$\frac{1}{z^5}+\frac{2}{z^4}+\frac{3}{z^3}+\frac{6}{z^2}+\frac{2}{z}+1=0$$

- Take LCM **Rearrange the** Char Equation **Terms in descending powers of Z**

$$1+2z+3z^2+6z^3+2z^4+z^5=0$$

$$z^5+2z^4+6z^3+3z^2+2z+1=0$$

Two sign changes, hence the system is unstable.

z^5	1	6	2
Z^4	2	3	1
Z^3	$\frac{2*6-1*3}{2}=4.5$	$\frac{2*2-1*1}{2}=1.5$	0
Z^2	$\frac{4.5*3-1.5*2}{4.5}=2.33$	1	0
Z^1	$\frac{2.33*1.5-4.5*1}{2.33}=-0.429$	0	0
Z^0	1		

$$s^5+s^4+2s^3+2s^2+3s+5=0$$

$$9s^5-20s^4+10s^3-s^2-9s-10=0$$

$$s^7+9s^6+24s^5+24s^4+24s^3+24s^2+23s+15=0.$$

Determine the range of K for stability of unity feedback system whose open loop transfer function is $G(s) = \frac{K}{s(s+1)(s+2)}$

$$\frac{C(s)}{R(s)} = \frac{G(s)}{1+G(s)} = \frac{\frac{K}{s(s+1)(s+2)}}{1+\frac{K}{s(s+1)(s+2)}} = \frac{K}{s(s+1)(s+2)+K}$$

Characteristic equation is $s(s+1)(s+2)+K=0$

$$\therefore s(s^2+3s+2)+K=0 \quad \Rightarrow \quad s^3+3s^2+2s+K=0$$

The highest power of s in the characteristic polynomial is odd number. Hence form the first row using the coefficients of odd powers of s and form the second row using the coefficients of even power s

s^3	:	1	2
s^2	:	3	K
s^1	:	$\frac{6-K}{3}$	
s^0	:	K	

Column-1

$$s^1 : \frac{3\times 2 - K\times 1}{3}$$

$$s^1 : \frac{6-K}{3}$$

$$s^0 : \frac{\frac{6-K}{3}\times K - 0\times 3}{(6-K)/3}$$

$$s^0 : K$$

s^3	:	1	2
s^2	:	3	K
s^1	:	$\frac{6-K}{3}$	
s^0	:	K	

Column-1

For the system to be stable there should not be any sign change in the elements of first column. Hence choose the value of K so that the first column elements are positive.

From s' row, for the system to be stable, $(6-K)/3 > 0$

From s^o row, for the system to be stable, $K > 0$

For $(6-K)/3 > 0$, the value of K should be less than 6.

:. The range of K for the system to be stable is $0 < K < 6$.

For the unity feedback system, $G(s) = \frac{K}{s(1+0.4s)(1+0.25s)}$, Find the range of values of Kfor stability of unity feedback system, Marginal value of K which causes sustained oscillation in the closed loop system and Corresponding frequency of sustained oscillation

Characteristic equation = 1+G(s)H(s) = 0 and H(s) = 1

$$1+\frac{K}{s(1+0.4s)(1+0.25s)} = 0$$

$$s[1+0.65s+0.1s^2]+K=0$$

$$0.1s^3+0.65s^2+s+K=0$$

S^3	0.1	1
S^2	0.65	K
S^1	$\frac{0.65*1 - 0.1K}{0.65}$	0
S^0	K	---

From s^1,

$0.65 - 0.1K > 0$

$\therefore\ 0.65 > 0.1K$

$\therefore\ 6.5 > K$

From s^0, $K > 0$

S^3	0.1	1
S^2	0.65	K
S^1	$\frac{0.65*1-0.1K}{K}$	0
S^0	K	---

The marginal value of K is value which makes any row other than s^0 as row of zeros.

$$0.65 - 0.1\ K_{mar} = 0$$

$$\boxed{K_{mar} = 6.5}$$

To find frequency, find out the auxiliary equation at K_{mar}

$$A(s) = 0.65s^2 + K = 0\ ;$$

$$0.65s^2 + 6.5 = 0 \qquad \because\ K_{mar} = 6.5$$

$$s^2 = -10$$

$$s = \pm j\ 3.162$$

$$s = \pm j\omega$$

ω = Frequency of oscillations

= **3.162 rad/sec.**

For the unity feedback system, $G(s) = \dfrac{K}{(s+2)(s+4)(s^2+6s+25)}$.

Find the range of values of K for stability of unity feedback system, Marginal value of K which causes sustained oscillation in the closed loop system and Corresponding frequency of sustained oscillation

The closed loop T.F $\dfrac{C(s)}{R(s)} = \dfrac{G(s)}{1+G(s)} = \dfrac{\dfrac{K}{(s+2)(s+4)(s^2+6s+25)}}{1+\dfrac{K}{(s+2)(s+4)(s^2+6s+25)}} = \dfrac{K}{(s+2)(s+4)(s^2+6s+25)+K}$

Characteristic equation = 1+G(s)H(s) = 0

$$\therefore (s^2+6s+8)(s^2+6s+25)+K=0 \quad \Rightarrow \quad s^4+12s^3+69s^2+198s+200+K=0$$

$$s^4 + 12s^3 + 69s^2 + 198s + 200 + K = 0$$

s^4	:	1	69	200+K	..... Row-1
s^3	:	12	198		..... Row-2

Divide s^3 row by 12 to simplify the calculations

s^4	:	1	69	200+K	..... Row-1
s^3	:	1	16.5		..... Row-2
s^2	:	52.5	200+K		..... Row-3
s^1	:	$\frac{666.25-K}{52.5}$			..... Row-4
s^0	:	200+K			..... Row-5
		Column-1			

$$s^2 : \frac{1\times 69 - 16.5\times 1}{1} \quad \frac{1\times(200+K)}{1}$$

$$s^2 : 52.5 \qquad 200+K$$

$$s^1 : \frac{52.5\times 16.5 - (200+K)\times 1}{52.5}$$

$$s^1 : \frac{666.25-K}{52.5}$$

$$s^0 : \frac{\frac{666.25-K}{52.5}\times(200+K)}{(666.25-K)/52.5}$$

$$s^0 : 200+K$$

For the system to be stable there should not be any sign change in the elements of first column. Hence choose the value of K so that the first column elements are positive.

$$s^4 + 12s^3 + 69s^2 + 198s + 200 + K = 0$$

s^4	:	1	69	200+K..... Row-1
s^3	:	12	198	..... Row-2

Divide s^3 row by 12 to simplify the calculations

s^4	:	1	69	200+K
s^3	:	1	16.5	
s^2	:	52.5	200+K	
s^1	:	$\frac{666.25 - K}{52.5}$		
s^0	:	200+K		

Column-1

From s' row, for the system to be stable [(666.25-K) / 52.5] > 0

Since (666.25-K) > 0, K should be less 666.25

From s° row, for the system to be stable (200+K) > 0

Since (200+K) > 0, K should be greater than- 200, but practical values of K starts from 0. Hence K should be greater than 0.

:. The range of K for the system to be stable is 0 < K < 666.25 .

$$s^4 + 12s^3 + 69s^2 + 198s + 200 + K = 0$$

s^4	:	1	69	200+K..... Row-1
s^3	:	12	198	..... Row-2

Divide s^3 row by 12 to simplify the calculations

		Column-1		
s^4	:	1	69	200+K
s^3	:	1	16.5	
s^2	:	52.5	200+K	
s^1	:	$\dfrac{666.25 - K}{52.5}$		
s^0	:	200+K		

:. The range of K for the system to be stable is $0 < K < 666.25$.

The marginal value of K is value which makes any row other than s^0 as row of zeros.

When K=666.25, the s' row becomes zero, which indicates the possibility of roots on imaginary . A system will oscillate if it has roots on imaginary axis and no roots on right half of s-plane the System is Marginally stable

When K=666.25, The coefficients of auxiliary equation are given by the s2 row.

The auxiliary equation is $52.5s^2+200+K=0$

$$52.5s^2 + 200 + 666.25 = 0$$

$$s^2 = \frac{-200 - 666.25}{52.5} = -16.5$$

$$s = \pm\sqrt{-16.5} = \pm j\sqrt{16.5} = \pm j4.06$$

When K=666.25, system has roots on imaginary axis and so it oscillate the System is Marginally stable

Corresponding frequency of sustained oscillation is given by the roots on imaginary axis ω=4.06rad/sec