The Root Locus

The relative stability and transient performance of a closed loop control system are directly related to the location of the closed loop poles.

The closed loop poles are the roots of the characteristic equation

The response of a closed loop control system can be adjusted by judicious selection of system parameter or appropriate gain value to achieve desired performance as the system parameter or gain value chosen determine the location of the closed loop poles

The root locus technique is a graphical method of plotting the locus of roots of the characteristic equation in the s-plane as the system parameter or gain is varied

Such a plot clearly depicts the contribution of each open loop pole or zero to the location of closed loop poles

Further the roots corresponding to a particular value of the system gain can be located on the root locus or the value of the system gain for a desired location can be determined from the root locus

The gain of the system has a crucial effect on the stability of the system.

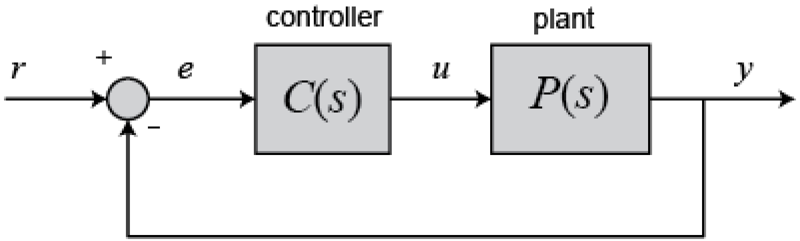
Root loci provide very convenient means of analysing the system where this parameter occurs in the characteristic equation. Thus root locus method is a powerful tool in analysing and designing a feedback control system

The Root Locus Concept

The root locus method involves plotting the roots of the characteristic equation of closed loop system as the gain is varied from zero to infinity. Consider the system shown Fig

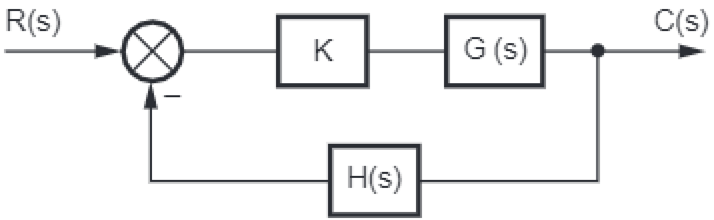
- In general, the characteristic equation of a closed loop system is given as,

$$1 + G(s)H(s) = 0$$

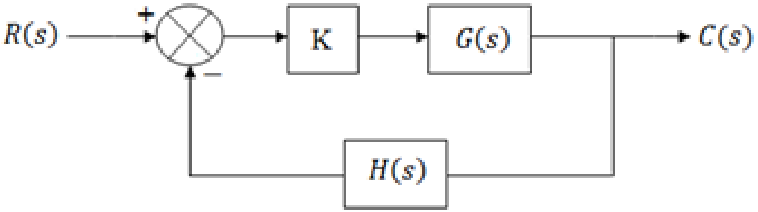


- For root locus, the gain `K' is assumed to be a variable parameter and is a part of forward path of the closed loop system. Consider the system shown in the

$$G(s) = KG'(s)$$



The Root Locus Concept



$$\frac{C(s)}{R(s)} = \frac{KG(s)}{1 + KG(s)H(s)}$$

- where K = Gain of the amplifier in forward path or also called **System Gain.** The characteristic equation becomes,

Key Point The closed loop poles i.e. the roots of the above equation are now an on the values of `K'.

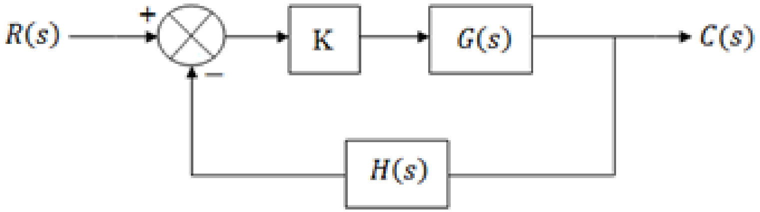
The characteristic equation will be the denominator of transfer function equated to 0 i.e 1+KG(s)H(s)=0.

Now, for different values of K, different roots of the characteristic equation are obtained. In other words, the location of roots of characteristic equation on the s-plane will vary with parameter ‘K’.

If the roots are plotted on the s-plane for different values of K, then the collection of all those points i.e the locus of roots — is called root locus. When K varies from 0 to positive infinity, it is termed as direct root locus and when K varies from negative infinity to 0, it is called complementary root locus.

The direct and complementary root locus together (K varying from positive infinity to negative infinity) form the complete root locus.

The Root Locus Concept



$$\frac{C(s)}{R(s)} = \frac{KG(s)}{1 + KG(s)H(s)}$$

$G(s) = KG'(s)$

- where K = Gain of the amplifier in forward path or also called **System Gain.** The characteristic equation becomes,

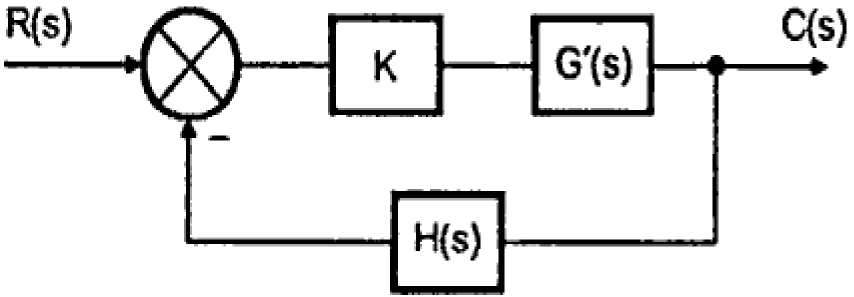
$1 + G(s)H(s) = 0$ i.e. $1 + KG'(s)H(s) = 0$

- which contains `K' as a variable parameter.

Key Point The closed loop poles i.e. the roots of the above equation are now an on the values of `K'.

- If now gain `K' is varied from $-\infty$ to $+\infty$ then for each separate value of `K' we will get separate set of locations of the roots of the characteristic equation. If all such locations are joined, the resulting locus is called **Root Locus.** So we can define root locus as, **the locus of the closed loop poles obtained when system gain `K' is varied from $-\infty$ to $+\infty$ is called Root Locus.**

Key Point When `K' is varied from 0 to $+\infty$, the plot is called ***direct root locus*** *while when `K' is varied from $-\infty$ to 0, the plot obtained is called* ***inverse root locus.***



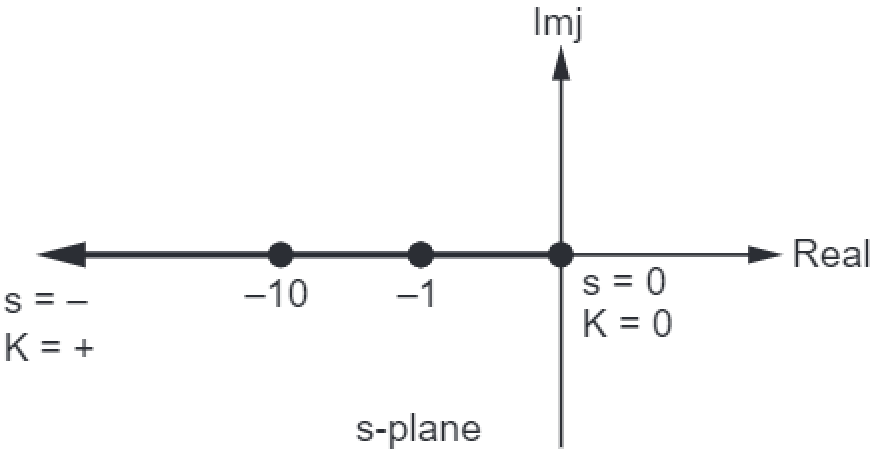
When K is varied from 0 to +infinity, the plot is called **direct root locus**.

When K is varied from –infinity to 0, the plot is called **inverse root locus**.

- But generally the term root locus is used in the sense of direct root locus. Unless otherwise stated, the variation in gain K is assumed to be 0 to $+\infty$ and plot is called root locus.

- The locus obtained by joining all such locations when K is varied from 0 to $+\infty$ is called root locus.

K	$s = -K$ Root location
0	0
1	-1
10	-10
:	:
$+\infty$	$-\infty$



In order to satisfy the performance specifications such as time domain specifications, frequency domain specifications a compensator is introduced in open loop transfer function

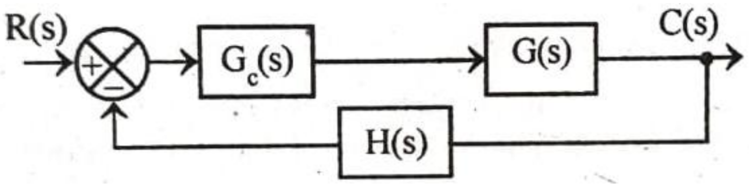
Two methods of designing a control system are design using root locus and design using bode plot.

In design using root locus, the system is designed to satisfy the specified time domain specifications.

In design using bode plot, the system is designed to satisfy the specified frequency domain specifications.

What is series compensation?

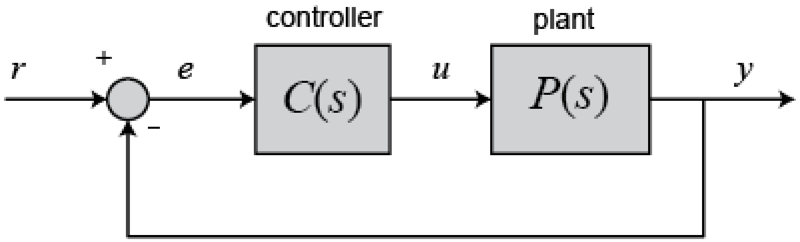
The series compensation is a design procedure in which a compensator is introduced in series with plant to alter the system behaviour and to provide satisfactory performance (i.e., to meet the desired specifications). The block diagram of series compensation scheme is shown in fig



Gc(s) = Transfer function of series compensator

G(s) = Open loop transfer function of the plant.

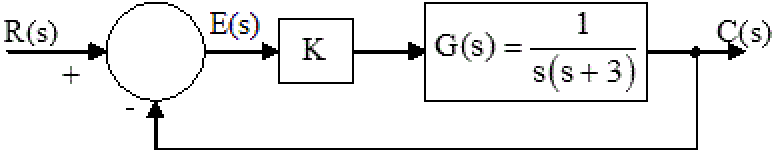
H(s) = Feedback path transfer function.



To understand what root locus plots are, and why they are important, let's examine the behavior of a system when it is in a control system. Assume that the system is defined by the transfer function:

$$G(s) = \frac{1}{s(s+3)}$$

We'll control this system with a very simple proportional controller in which the input to the system to be controlled is proportional (with gain, K) to the difference between the input, R(s), and the output, C(s).



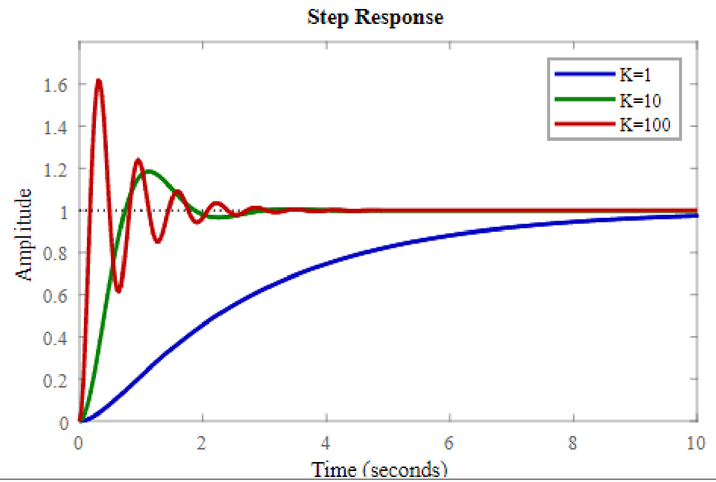
The loop gain is K·G(s), so the closed loop gain is given by $H(s) = \frac{C(s)}{R(s)} = \frac{K \cdot G(s)}{1 + K \cdot G(s)} = \frac{K \cdot \frac{1}{s(s+3)}}{1 + K \cdot \frac{1}{s(s+3)}}$

$$= \frac{K}{s(s+3)+K} = \frac{K}{s^2+3s+K}$$

This expression for H(s) is easily derived: E(s)=R(s)-C(s) and C(s)=K·E(s)=K·(C(s)-R(s)). Collect terms and solve for H(s)=C(s)/R(s).

Design: Trial and Error Solution

We want to examine how the behavior of the system varies as K changes, so let's try several values of K. Let's arbitrarily try K=1, 10 and 100 so that we have a wide range of K values.

<table>
<tr><th>K, Xfer Function</th><th>Step Response</th></tr>
<tr><td>$K=1, \quad \frac{C(s)}{R(s)}=\frac{1}{s^2+3s+1}$</td><td rowspan="3">
</td></tr>
<tr><td>$K=10, \quad \frac{C(s)}{R(s)}=\frac{10}{s^2+3s+10}$</td></tr>
<tr><td>$K=100, \quad \frac{C(s)}{R(s)}=\frac{100}{s^2+3s+100}$</td></tr>
</table>

The response with K=1 (blue) is very slow, the response with K=100 (maroon) is faster but very oscillatory. However the response with K=10 (green) is fast and has about 20% overshoot (we can reduce this overshoot and maintain the speed of the response). Clearly this method is rather "hit-or-miss" and it may take us a long time to find a suitable value for K, especially for more complicated systems.

For the very simple problem described above, it was possible to calculate the precise location of the roots, and choose a value of K that gave us a good response. For more complicated systems it is not so straightforward so we need a more general method for finding K.

This more general method is called the "root locus" method. With this technique we make a plot of the path of the roots as a parameter (usually the gain K) varies. We then choose pole locations, and find the value of K necessary

Root locus technique is a important tool in designing control systems with desired performance characteristics.

The desired performance of the system can be achieved by adjusting the location of its closed loop poles in S-Plane by varying one or more system parameters (Usually open loop gain, K)

$$G(s) = \frac{K}{s\,(s+p_1)\,(s+p_2)}$$

$$\frac{C(s)}{R(s)} = \frac{G(s)}{1+G(s)} = \frac{\dfrac{K}{s\,(s+p_1)\,(s+p_2)}}{1+\dfrac{K}{s\,(s+p_1)\,(s+p_2)}} = \frac{K}{s\,(s+p_1)\,(s+p_2)+K}$$

The denominator polynomial of C(S)/R(S) is the characteristic equation of the system.

The characteristic equation is given by $S(S+P1)(S+P2)+K=0$

The roots of characteristic equation depend on open loop gain K, when the gain K is varied from 0 to infinity, the roots of characteristic equation will take different values. When K=0, the roots are given by open loop poles. When K -> infinity, the roots will take the value of open loop zeros

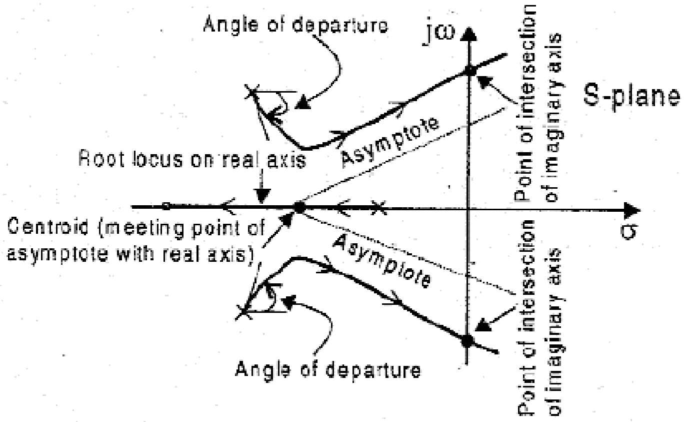
The path taken by the roots of characteristic equation when open loop gain K is varied from 0 to infinity are called root locus/root loci.

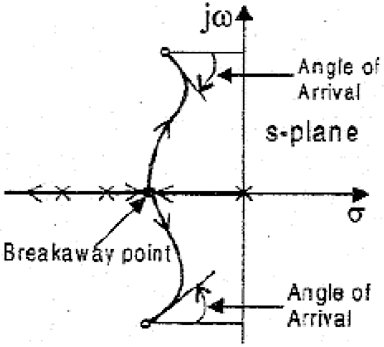
Root locus technique is a also used for stability analysis. Using root locus the range of value of K for a stable system can be determined

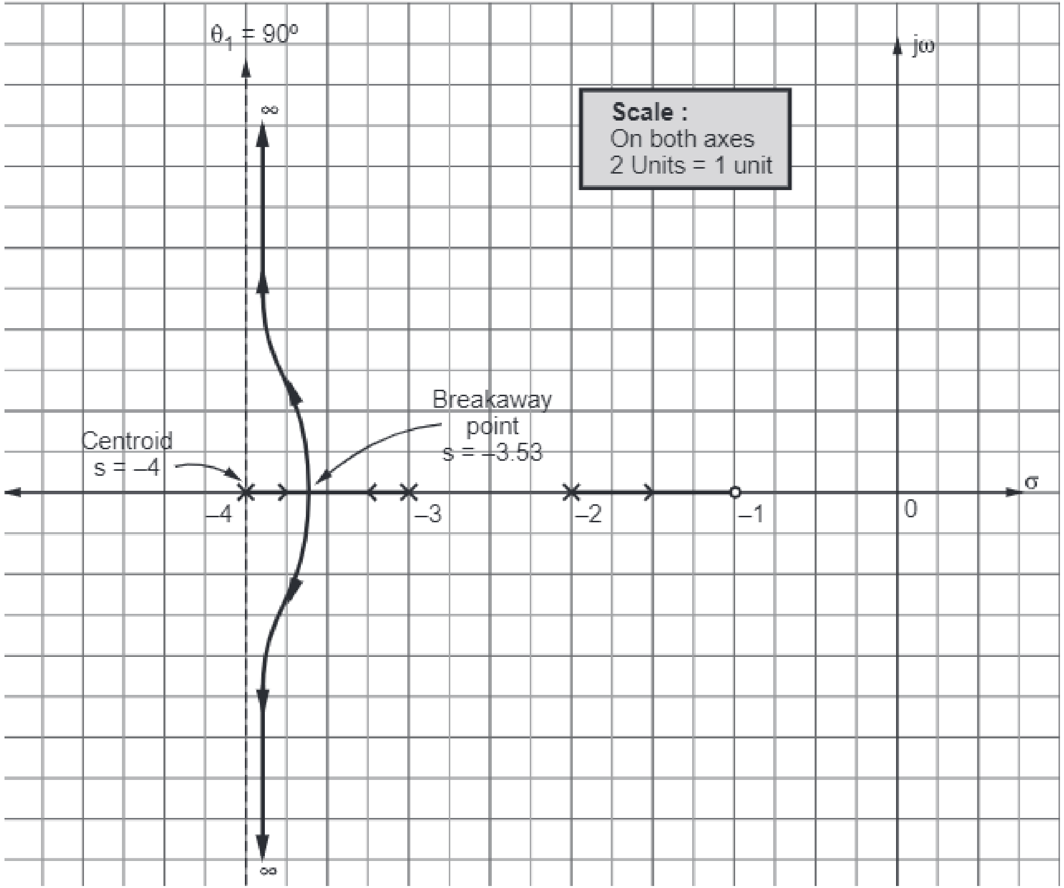
The time domain specification damping ratio ζ and ω natural frequency of oscillation of the system can be calculated

The Root locus is symmetrical about the real axis

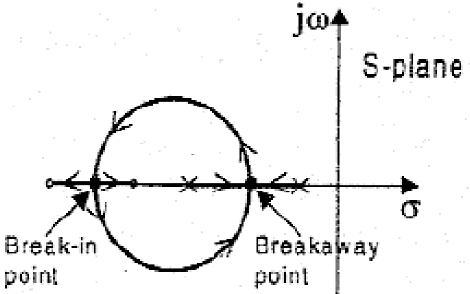
TYPICAL SKETCHES OF ROOT LOCUS PLOTS

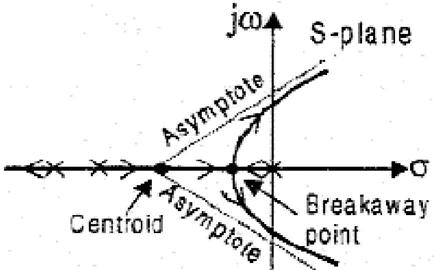


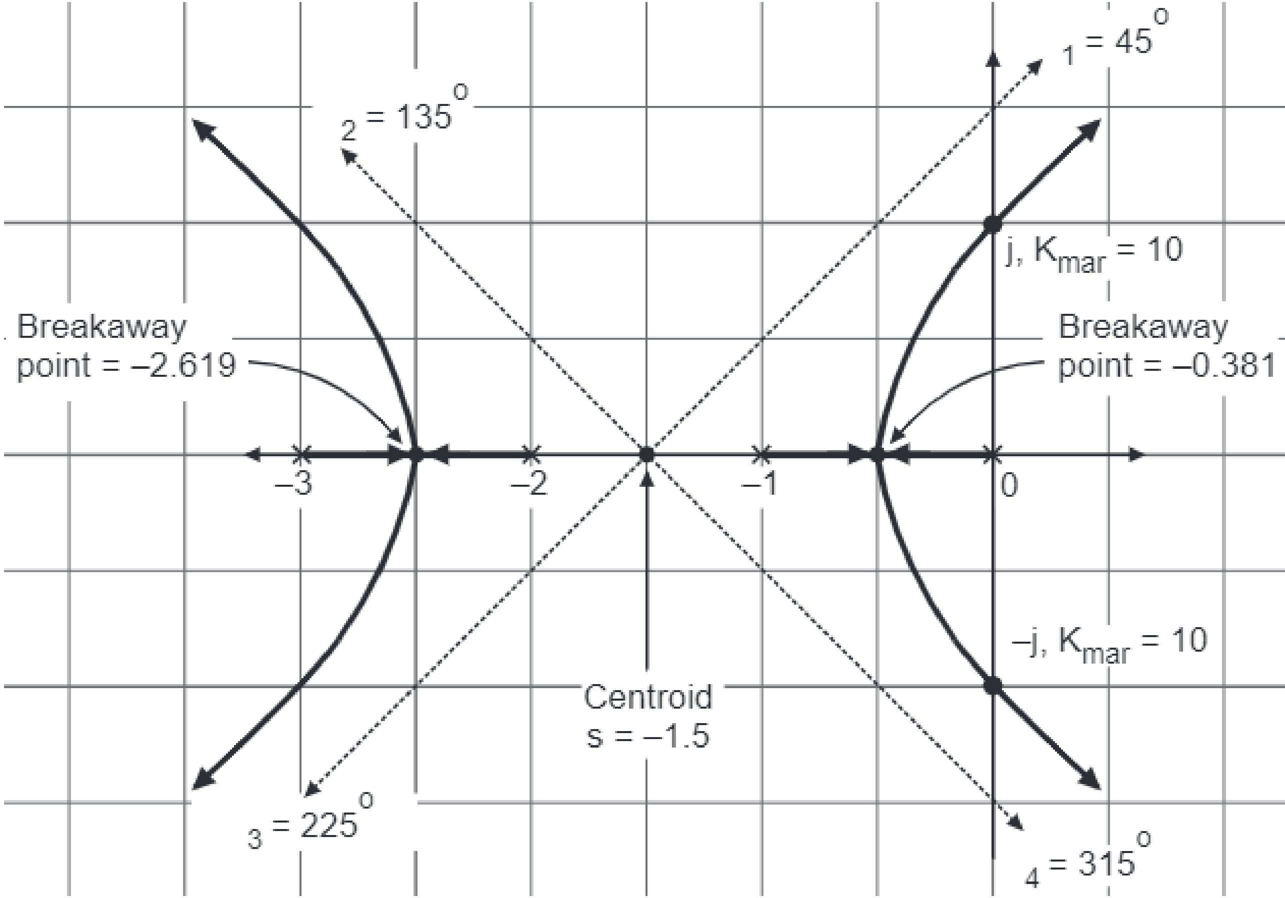




The root locus is symmetrical about the real axis







Procedure / Rule for construction of root locus

Rule 1 Locate the open loop poles and zeros in the 's' plane

Root locus branch start from open loop poles and terminate at zeros

- Number of Root locus branch will be equal to Number of open loop poles

Let n = Number of open loop poles "χ"

m = Number of finite zero "o"

then (n-m) root locus branch end at zeros "O" at Infinity

Each branch of the root locus originates from an open-loop pole corresponding to $K = 0$ and terminates at either on a finite open loop zero (or open loop zero at infinity) corresponding to $K = \infty$. The number of branches of the root locus terminating on infinity is equal to $n-m$, (i.e., the number of open loop poles minus the number of finite zeros)

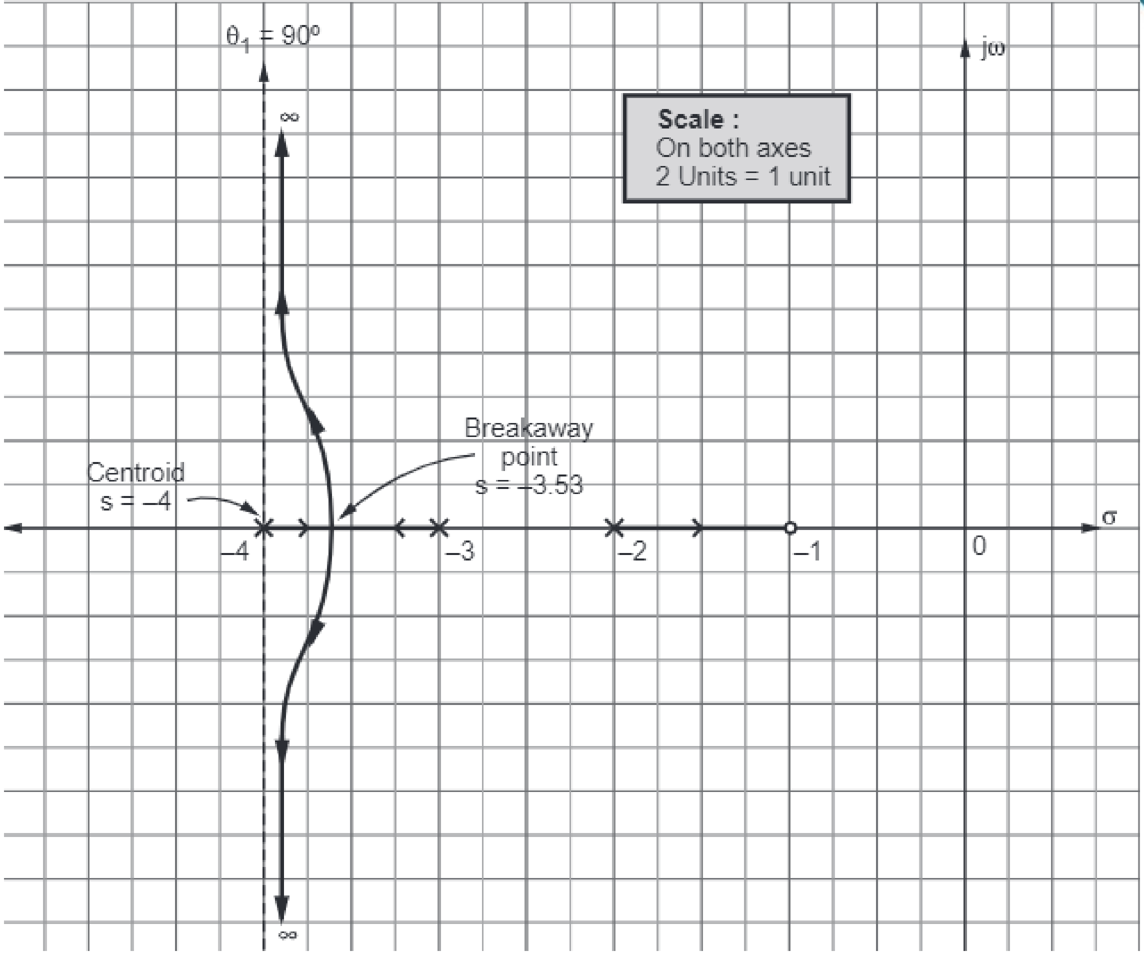
for example :- Let n = 4 and m = 2 i.e. there are 4 open Loop Poles and 2 open loop finite zero

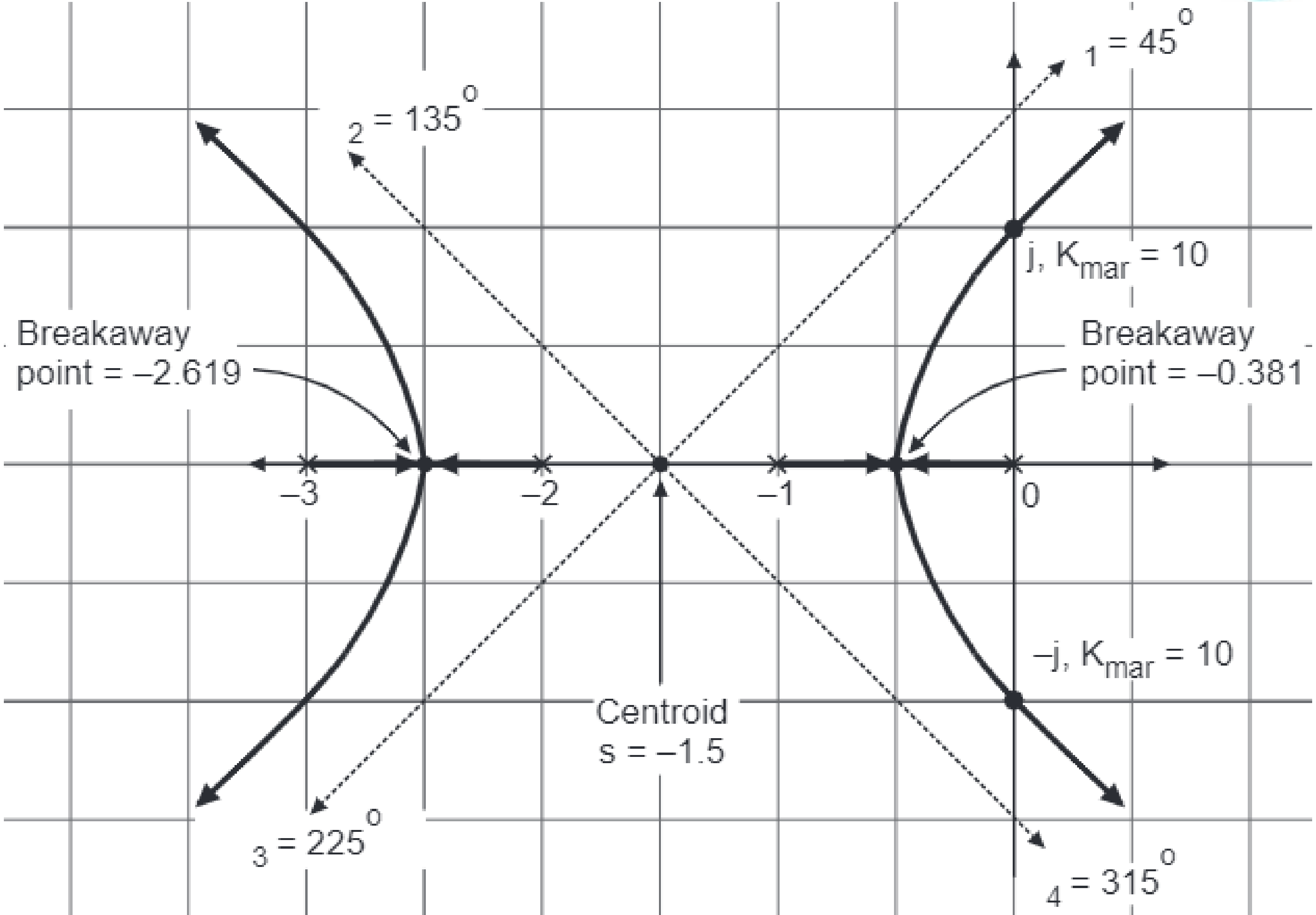
- Then there will be 4 Root locus branch that start/origin from these 4 open Loop Poles

- out of those 4 Root locus branch, 2 Root locus branch will ends at finite Zeros and remaining (n-m) root locus branches will end at zeros at infinity.

- Asymptotes give the direction of this root locus branches which ending at zeros at infinity.

- The intersection point of asymptotes Line on the real axis is known as centroid.

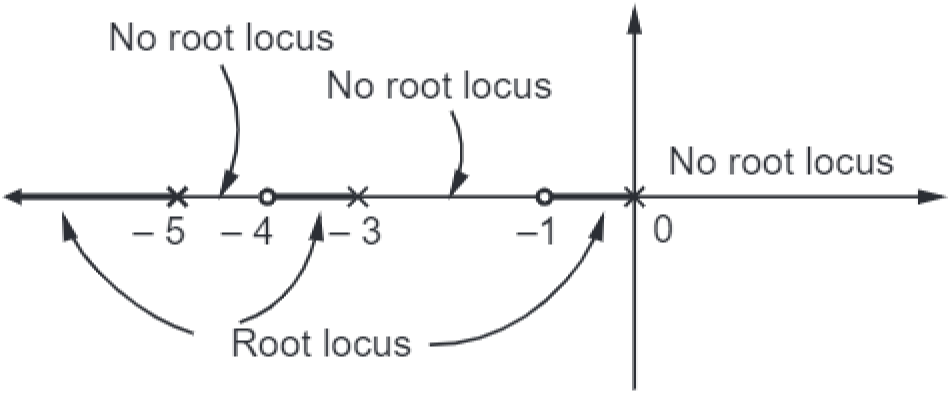

$\theta_1 = 90°$
$j\omega$
Scale :
On both axes
2 Units = 1 unit
∞
Breakaway
point
s = –3.53
Centroid
s = –4
σ
–4
–3
–2
–1
0
∞

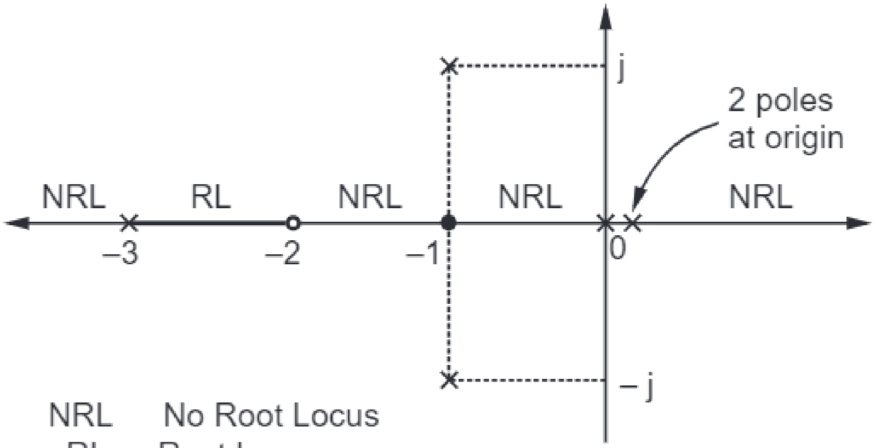

1 = 45°
2 = 135°
3 = 225°
4 = 315°
j, Kmar = 10
–j, Kmar = 10
Breakaway
point = –2.619
Breakaway
point = –0.381
–3
–2
–1
0
Centroid
s = –1.5

Rule 2 Determine the root locus on real axis

In order to determine the part of root locus on real axis, take a test point on real axis. If the total number of poles and zero on the real axis to the right of this test point is ODD number, then the test point lies on the root locus. If it is Even Number then the test point does not lie on the root locus.

Each branch of the root locus originates from an open-loop pole corresponding to $K = 0$ and terminates at either on a finite open loop zero (or open loop zero at infinity) corresponding to $K = \infty$. The number of branches of the root locus terminating on infinity is equal to $n-m$, (i.e., the number of open loop poles minus the number of finite zeros)





NRL No Root Locus
RL Root Locus

Rule 3 Determine angle of asymptotes and centroid (meeting point of asymptotes with real axis)

Angle of asymptotes and centroid

If n is the number of poles and m is the number of finite zero, (n-m) root locus branch terminate at zeros at Infinity.

These n-m root locus branches will go along an asymptotic path and meets the asymptotes at infinity. hence number of asymptotes is equal to number of root locus branches going to infinity (zeros at infinity)

Or in other words

Asymptotes are straight lines which is parallel to root locus going to infinity and meet the root locus at infinity

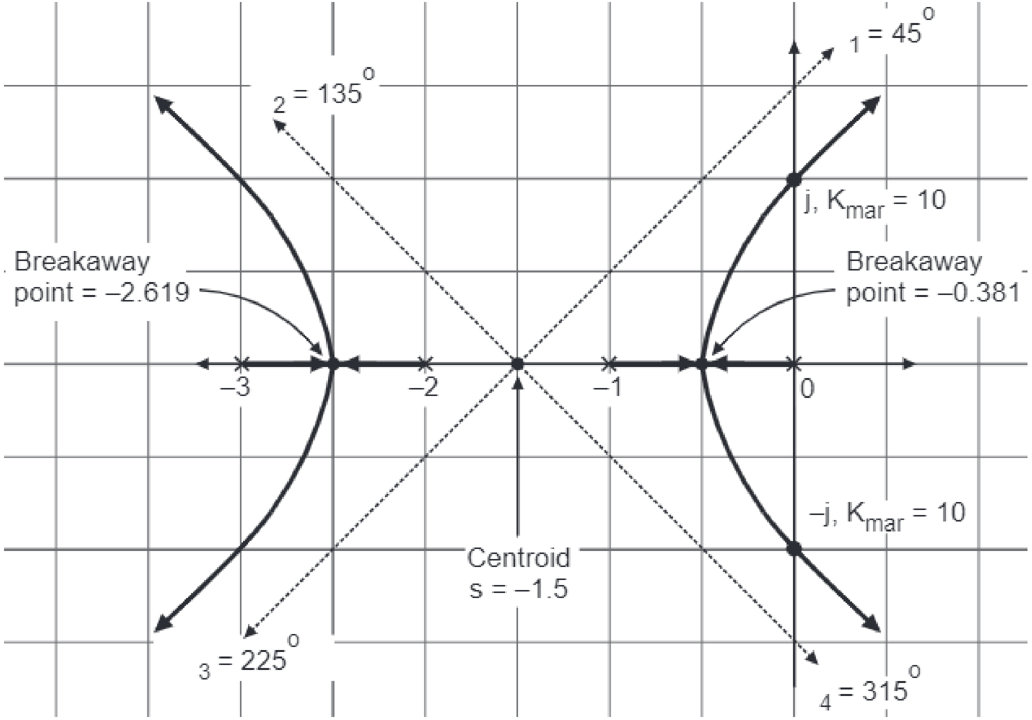
Angles of asymptotes = ±180*(2q+1) / n-m

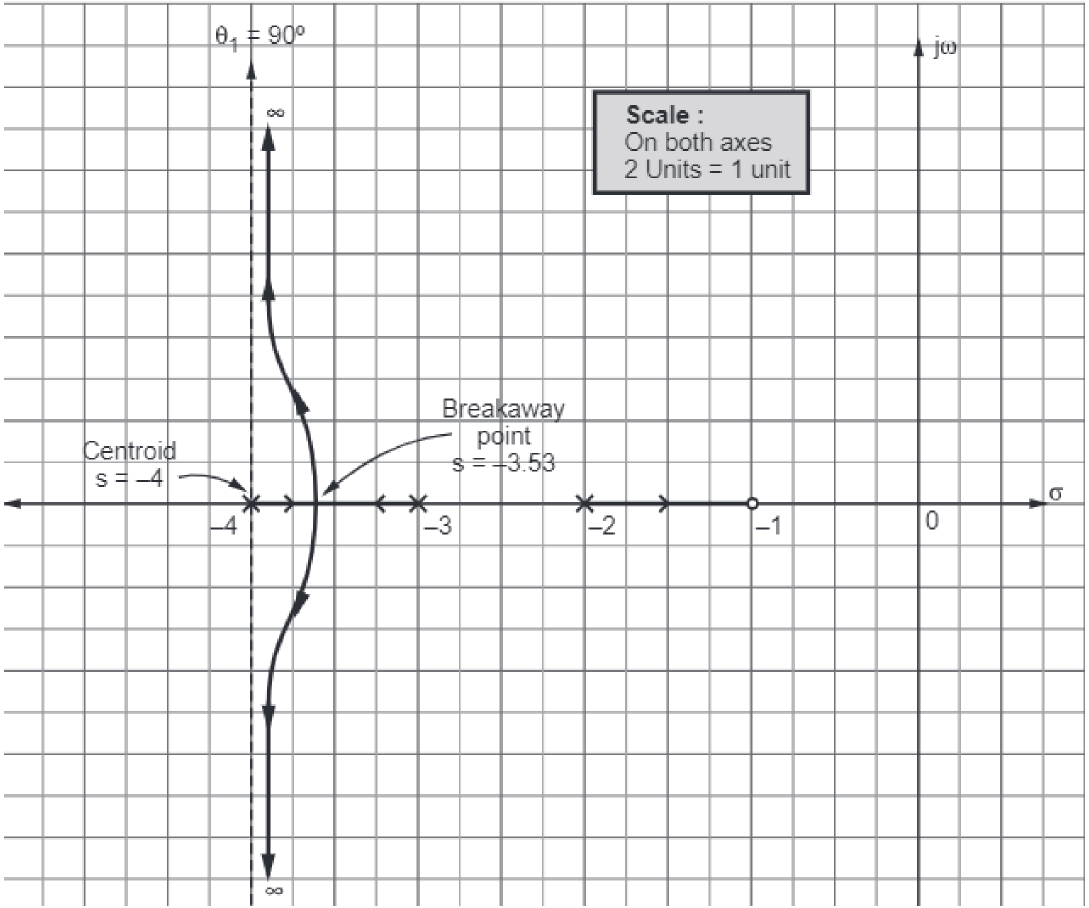
Where q=0, 1, 2, 3……(n-m)

Centroid is meeting point of asymptote with real axis, the centroid is given by,

Centroid = (sum of poles – sum of zero) / n-m

Centroid is marked on real axis and from centroid the angle of Angle of asymptotes are marked using protractor, asymptotes are drawn as dotted lines




$\theta_1 = 90^o$
∞
jω
Scale :
On both axes
2 Units = 1 unit
Breakaway
point
s = –3.53
Centroid
s = –4
σ
–4
–3
–2
–1
0
∞

Rule 4 Find Break-away and Break-in points.

Break-away and Break-in points either lie on real axis or exist as complex conjugate pairs

- if there is a root locus on real axis between 2 poles, then there exist a break-away point in between these two open loop poles
- If there is a root locus on real axis between two zeros, then there exist a break-in point in between these two open loop zeros
- If there is a root locus on real axis between pole and zero then there may be or may not be break-away point or break-in point

Follow these steps to find break-away and break-in points

Form an equation K in terms of s from the characteristic equation 1+G(S)H(S)=0

Differentiate K with respect to s and make it equal to zero i.e. dK/ds = 0 .

- Find the Roots of s.
- Substitute these values of s in equation K and determine the value of K
- If k is real and positive then s is the actual break point
- If k is not real and positive then that value of s is not the break point

The Roots of dK/ds = 0 are break-away or break-in points, provided for this value of root, the gain K value is should be positive and real

Rule No.4: Break away point

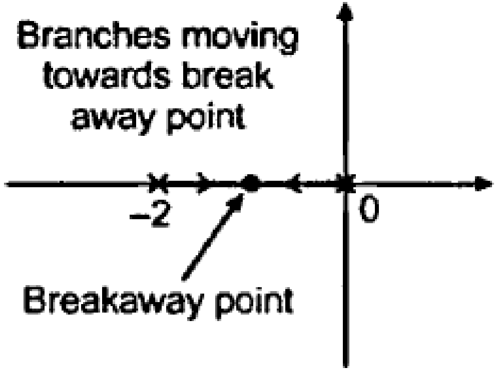
Consider $G(s)H(s) = \frac{K}{S(S+2)}$

- Open loop poles are s = 0; s = -2

- Breakaway point is a point on the root locus where multiple roots of the characteristic equation occurs for a particular value of K.

- The root locus branches always leave breakaway point at an angle of $\pm 180^0/n$,

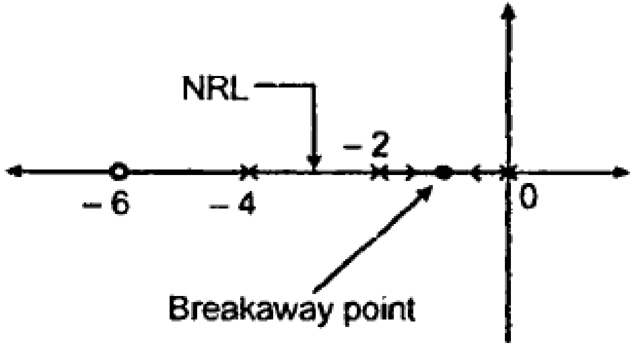
where n = number of branches approaching at break away point.



Rule No.4: Break away point

Consider $G(s)H(s) = \frac{K(s+6)}{S(S+2)(s+4)}$

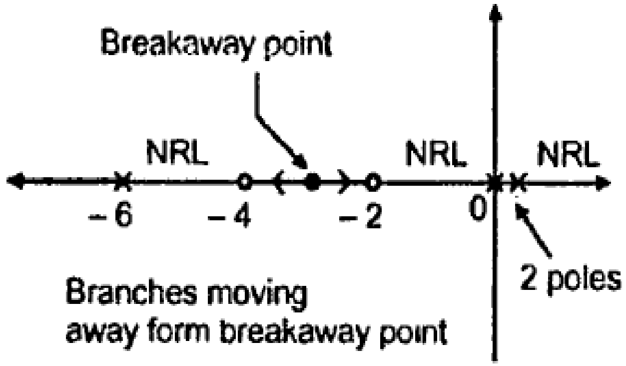
- Open loop poles are s = 0; s = -2; s = -4
- Zero at s = -6
- If there are adjacently placed poles on the real axis and the real axis between them is a part of the root locus then there exists a minimum one breakaway point in between adjacently placed poles.
- One breakaway point exists between poles 0 and -2 which are adjacent.
- The poles -2 and -4 are also adjacent but the section between them is not the part of the root locus and hence there cannot be a breakaway point between them.



Rule No.4: Break away point

Consider $G(s)H(s) = \frac{K(s+2)(s+4)}{S^2(S+6)}$

- Open loop poles; s = 0, 0, s = -6
- Zeros = s = -2, s = -4
- If there are two adjacently placed zeros on the real axis and section of real axis in between them is a part of root locus then there exists minimum one breakaway point in between adjacently placed zeros.
- In these cases the branches move away from the breakaway point towards open loop zeros. Such point is called break in point.



Rule No.4: Break away point

Steps to determine the coordinates ofbreakaway point:

Consider $G(s)H(s) = \frac{K}{S(S+1)(s+4)}$, determine the co-ordinates of valid breakaway point.

Solution:

Step 1: Construct the Characteristic equation: 1 + G(s)H(s) = 0

$$1 + \frac{K}{S(S+1)(s+4)} = 0 \quad ; \qquad S^3 + 5s^2 + 4s + K = 0;$$

Step 2: $K = - S^3 - 5s^2 - 4s$

Step 3: Differentiate above equation w.r.t. s and equate to '0'

$$\frac{dK}{ds} = -3s^2 - 10s - 4 = 0$$

Rule No.4: Break away point

Steps to determine the coordinates ofbreakaway point:

Step 4: Roots of the equation dK/ds = 0 gives us the breakaway points.

$$\frac{dK}{ds} = -3s^2 - 10s - 4 = 0$$

$$3s^2 + 10s + 4 = 0$$

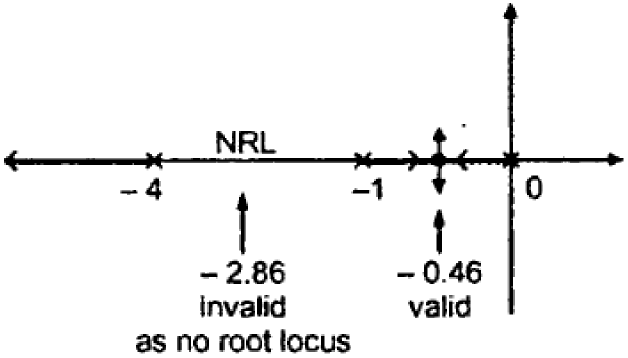
Breakaway points = $\frac{-10 \pm \sqrt{100 - 4*4*3}}{2*3} = -0.46, -2.86$

Substituting in expression for K, K = - S^3 - $5s^2$ - 4s

For s = -0.46, K = +0.8793

For s = -2.86, K = -6.064;

Hence For s = -0.46, K is positive and it is valid breakaway point for the root locus.



Rule 5 Angle of departure and angle of Arrival of root loci

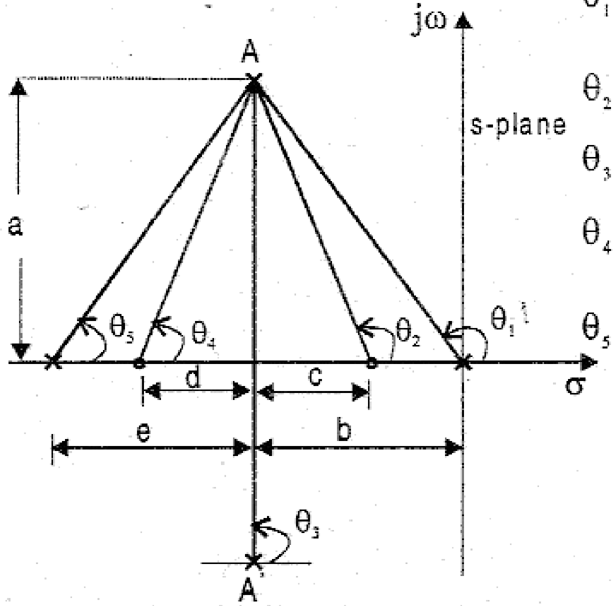
- if there is a complex pole then find angle of departure from complex pole.

- if there is a complex zero then find angle of arrival at the complex zero.

Angle of departure from a complex pole A

ϕ_d = 180°- (sum of angles of vectors drawn to the complex pole ‘A’ from all other poles)

+ (sum of angles of vector drawn to the complex pole ‘A’ from zeros)



$$\theta_1 = 180° - \tan^{-1}\frac{a}{b}$$

$$\theta_2 = 180° - \tan^{-1}\frac{a}{c}$$

$$\theta_3 = 90°$$

$$\theta_4 = \tan^{-1}\frac{a}{d}$$

$$\theta_5 = \tan^{-1}\frac{a}{e}$$

Fig : Calculation of angle of departure

$$\left.\begin{array}{c}\text{Angle of departure}\\ \text{at pole A}\end{array}\right\} = 180° - (\theta_1 + \theta_3 + \theta_5) + (\theta_2 + \theta_4)$$

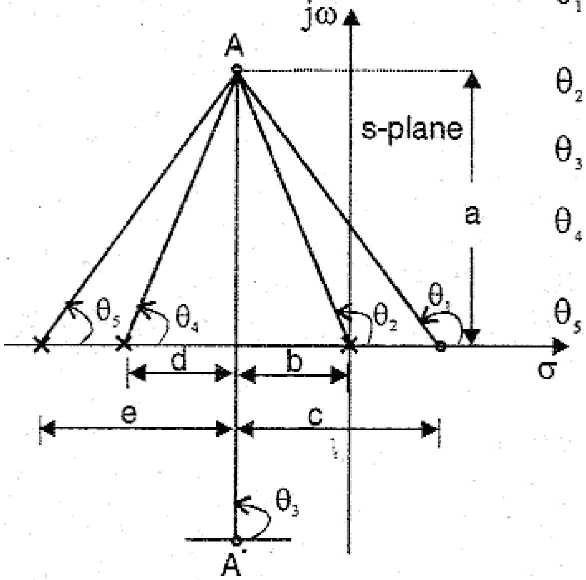
$$\left.\begin{array}{c}\text{Angle of departure}\\ \text{at pole A}^*\end{array}\right\} = -[\text{Angle of departure at pole A}]$$

The angle can be measured using protractor

Angle of Arrival at a complex Zero A

$\varnothing_A = 180^\circ$- (sum of angles of vectors drawn to the complex Zero ‘A’ from all other zeros)

+ (sum of angles of vectors drawn to the complex Zero ‘A’ from poles)



$$\theta_1 = 180^\circ - \tan^{-1}\frac{a}{b}$$

$$\theta_2 = 180^\circ - \tan^{-1}\frac{a}{c}$$

$$\theta_3 = 90^\circ$$

$$\theta_4 = \tan^{-1}\frac{a}{d}$$

$$\theta_5 = \tan^{-1}\frac{a}{e}$$

Fig : Calculation of angle of arrival

$$\left.\begin{array}{c}\text{Angle of arrival}\\ \text{at zero B}\end{array}\right\} = 180^\circ - (\theta_1 + \theta_3) + (\theta_2 + \theta_4 + \theta_5)$$

$$\left.\begin{array}{c}\text{Angle of arrival}\\ \text{at zero B}^*\end{array}\right\} = -[\text{Angle of arrival at zero B}]$$

Rule 6

Find The Point Where The Root Locus May Cross (intersects) The Imaginary jω Axis

- The Point Where The Root Locus Intersects The Imaginary Axis Can Be Found By Following Method's

Method-1

Put S=jω in the characteristic equation 1+G(S)*H(S)=0 and Separate the real part and imaginary part
Two equations are obtained by one by Equating real part to 0 and other by equating imaginary part to 0
Solve the two equation to get ω and K

- **The value of ω gives the point where the root locus crosses imaginary axis**

- **The value of K gives the value of open loop gain at there crossing points . Also this value of K is the limiting/Marginal value of K for stability of the system**

Rule 6 : Find The Point Where The Root Locus May Cross (intersects) The Imaginary Axis

Method-2 : intersection of root locus branch with jω axis can be determined through Routh Hurwitz Criterion

Consider the characteristic equation 1+G(s)H(s) = 0

$$0.1s^3 + 0.65s^2 + s + K = 0$$

The marginal value of K is value which makes any row other than s^0 as row of zeros.

S^3	0.1	1
S^2	0.65	K
S^1	$\frac{0.65*1 - 0.1K}{K}$	0
S^0	K	---

$$0.65 - 0.1\,K_{mar} = 0$$

$$\boxed{K_{mar} = 6.5}$$

Range of values of K, $0 < K < 6.5$.

To find frequency, find out the auxiliary equation at K_{mar}

$$A(s) = 0.65s^2 + K = 0\ ;$$

$$0.65s^2 + 6.5 = 0 \qquad \because\ K_{mar} = 6.5$$

$$s^2 = -10$$

$$s = \pm j\,3.162$$

$$s = \pm j\omega$$

ω = Frequency of oscillations

= 3.162 rad/sec.

Rule No.6: Intersection of root locus with Imaginary axis

Consider $G(s)H(s) = \frac{K}{S(S+1)(S+4)}$

Solution:

Step 1: Consider the characteristic equation 1+G(s)H(s) = 0

$$1 + \frac{K}{S(S+1)(S+4)} = 0$$

$$S^3+5S^2+4S+K = 0$$

S^3	1	4
S^2	5	K
S^1	$\frac{20-K}{5}$	0
S^0	K	

The marginal value of K is value which makes any row other than s^0 as row of zeros.

$$\frac{20-K}{5} = 0 \ ; \qquad Kmar = 20$$

The auxiliary equation ; A(s) = $5S^2$ + K = 0 and K=K_{mar} = 20

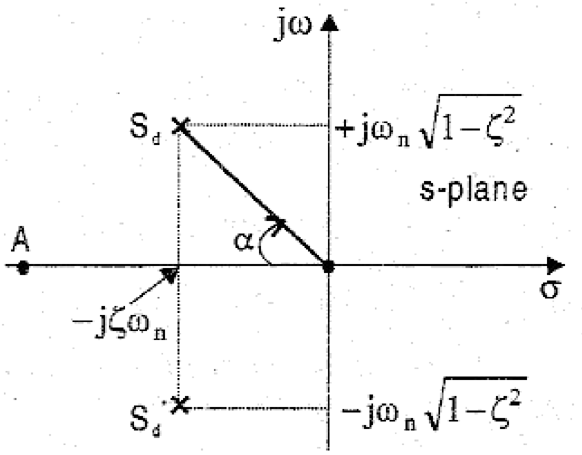
$5S^2+20 = 0$; $S^2 = -4$; $S = \pm j2$

To find frequency, find out the auxiliary equation at K_{mar}

$s = \pm j\omega$

ω = Frequency of oscillations

Rule 7 To determine value of Open Loop Gain K at any point of root locus



$$\alpha = \cos^{-1}\zeta$$

To fix a dominant pole on root locus, draw a line at an angle of $\cos^{-1}\zeta$ with respect to negative real axis. The meeting point of this line with root locus will give the location of dominant pole. The value of K corresponding to dominant pole can be obtained from magnitude condition.

Let, K_{sd} be the value of gain at dominant pole s_d,

$$\text{Now,}\quad K_{sd} = \frac{\text{Product of length of vectors from open loop poles to dominant pole}}{\text{Product of length of vectors from open loop zeros to dominant pole}}$$

Root Locus Technique: Example 1

Consider the open loop transfer function

$G(s)H(s) = \frac{K}{s(s+1)(s+2)(s+3)}$, Draw the root locus curve.

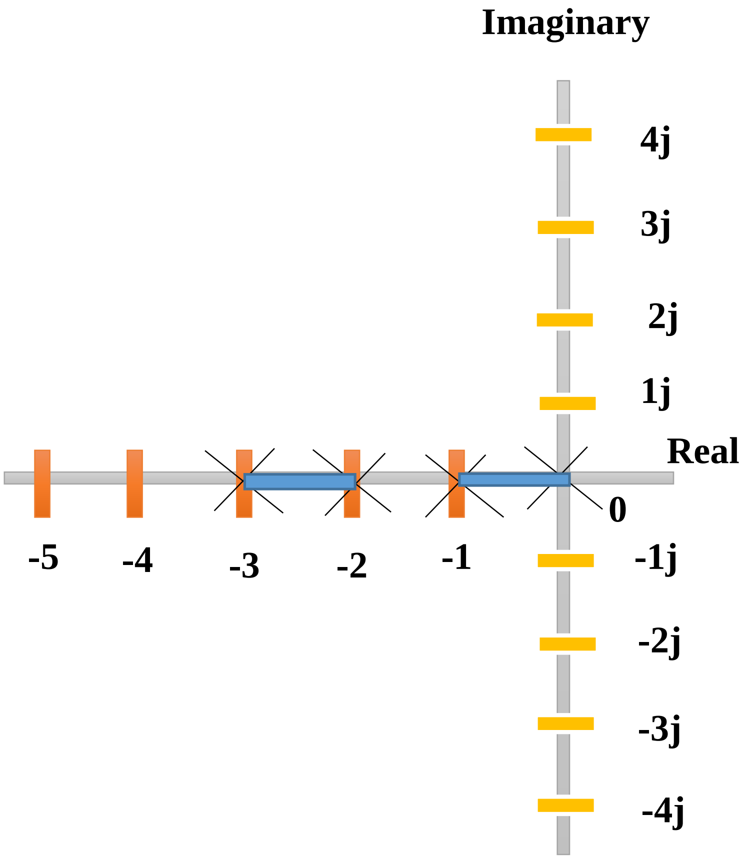
Solution:

Step 1: Poles = P = 4;   s = 0, s = -1, s = -2, s = -3   and Z = 0

Step 2: Number of branches approaching to infinity

N = P – Z = 4.

Step 3: Root locus branch on real axis.



Root Locus Technique

Step 4: Angle of asymptotes.

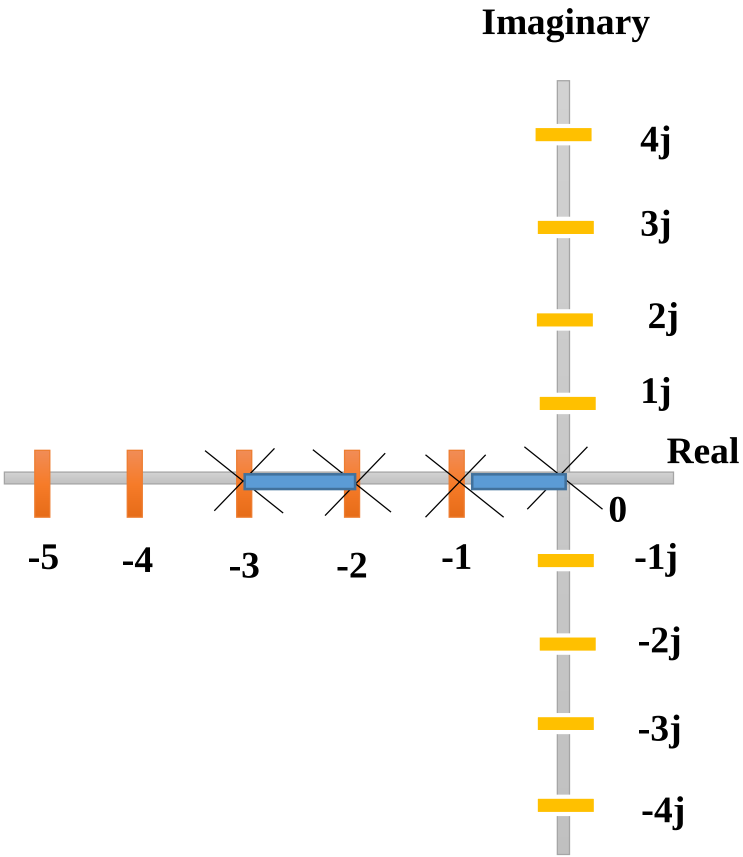
$$\theta = \frac{(2q+1)180°}{P-Z}; \quad q = 0, 1, 2, \ldots$$

For q = 0, $\theta_1 = \frac{180°}{4} = 45°$

For q = 1, $\theta_2 = \frac{3*180°}{4} = 135°$

For q = 2, $\theta_3 = \frac{5*180°}{4} = 225°$

For q = 3, $\theta_4 = \frac{7*180°}{4} = 315°$



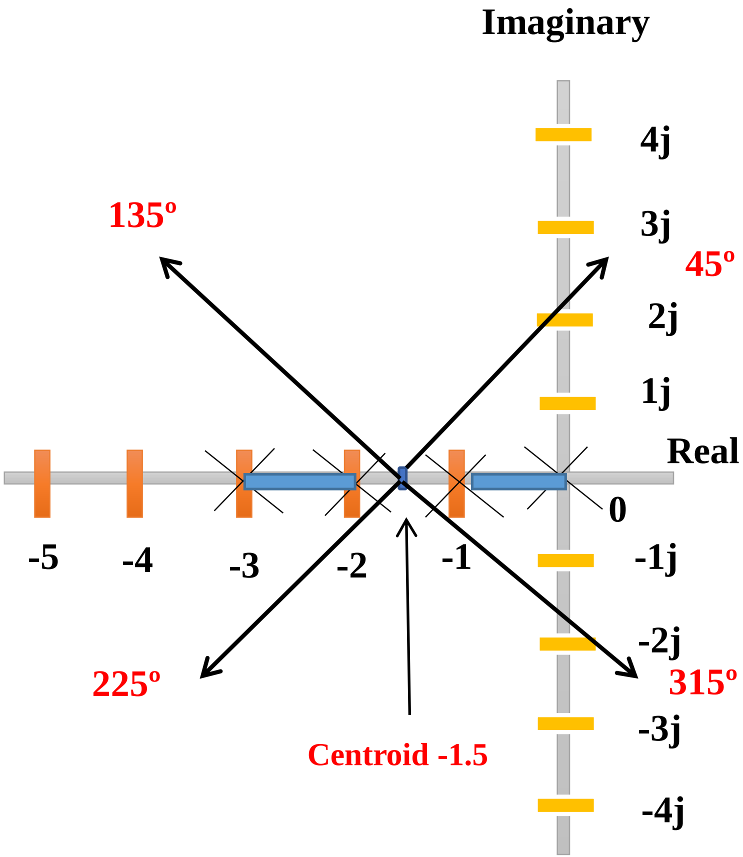
Root Locus Technique

Poles = P = 4;   s = 0, s = -1, s = -2, s = -3

Step 5: Centroid

$$\sigma = \frac{\sum Real\ parts\ of\ poles\ - \sum Real\ parts\ of\ zeros}{P - Z}$$

$$\sigma = \frac{0 - 1 - 2 - 3}{4} = -\frac{6}{4} = -1.5$$



Root Locus Technique

Step 6: Breakaway point

Characteristic equation = 1+G(s)H(s) = 0

$$1+\frac{K}{s(s+1)(s+2)(s+3)}=0$$

$$S^4+6S^3+11S^2+6S+K = 0$$

$$K = -S^4-6S^3-11S^2-6S$$

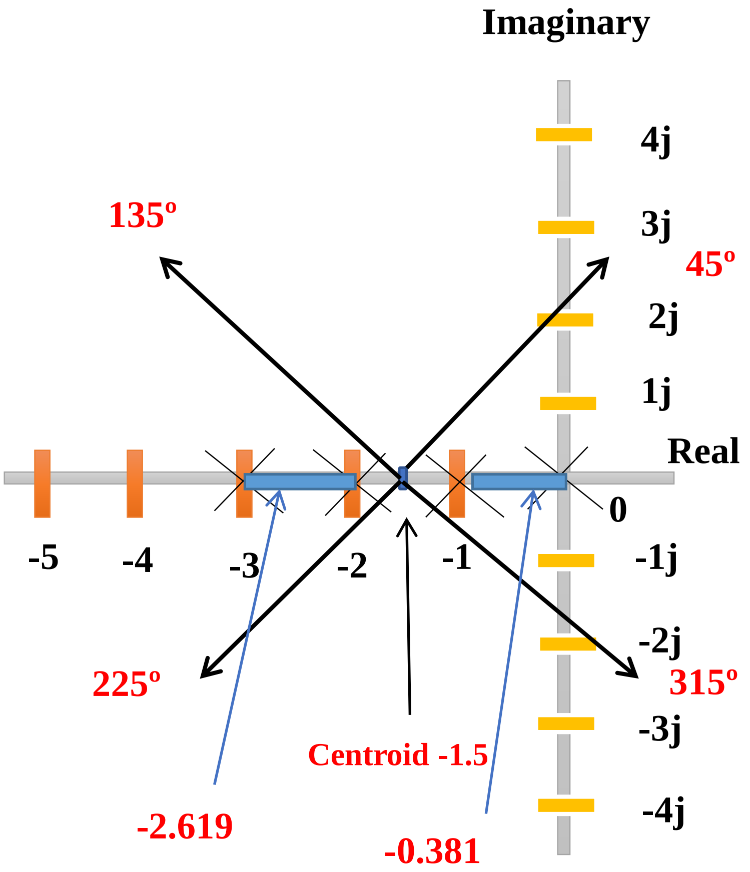
$$dK/ds = -4S^3 - 18S^2 - 22S - 6 = 0$$

$$4S^3+18S^2+22S+6 = 0$$

Breakaway points = -1.5, -0.381, -2.619

There is no root locus between -1 and -2 and hence -1.5 is not valid breakaway point.

For s = -0.381, K = 1 and For s = -2.619, K = 1



Root Locus Technique

Step 7: Intersection with Imaginary axis

Characteristic equation = 1+G(s)H(s) = 0

$S^4+6S^3+11S^2+6S+K = 0$

S^4	1	11	K
S^3	6	6	0
S^2	10	K	0
S^1	(60-6K)/10	0	0
S^0	K		

$$\frac{60-6K}{10} = 0; \quad Kmar = 10$$

The auxiliary equation = A(s) = $10S^2 + K = 0$

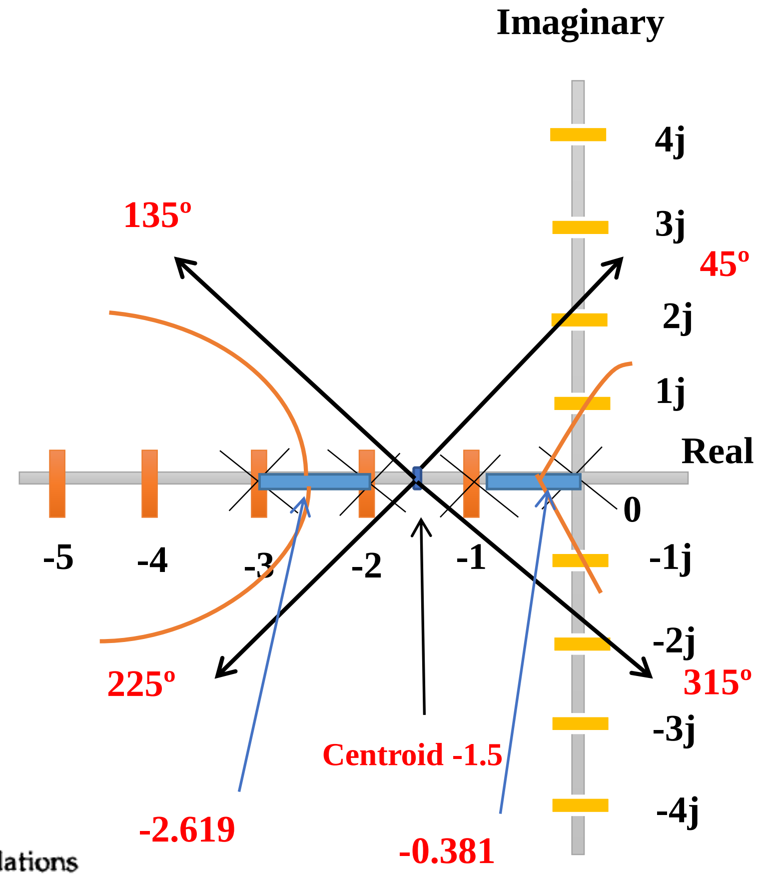
$10S^2 + 10 = 0$; $s = \pm 1j$

To find frequency, find out the auxiliary equation at K_{mar}

The marginal value of K is value which makes any row other than s^0 as row of zeros.

$s = \pm j\omega$

ω = Frequency of oscillations



Root Locus Technique

Program:

$$G(s)H(s) = \frac{K}{s(s+1)(s+2)(s+3)}$$

$$G(s)H(s) = \frac{K}{s^4 + 6s^3 + 11s^2 + 6s}$$

```
num=[1];
den=[1 6  11 6 0];
sys=tf(num,den);
printsys(num,den);
[r,k]=rlocus(sys)
rlocus(sys)
```

	Theoretical Values	Simulated Values
Poles	0, -1, -2, -3	
Breakaway Point	-0.38, -2.69	
Gain	10	
Imaginary Axis Crossover	± 1j	

Program:

$$G(s)H(s) = \frac{K}{s(s+1)(s+2)(s+3)}$$

$$G(s)H(s) = \frac{K}{s^4 + 6s^3 + 11s^2 + 6s}$$

```
clc
num=[1];
den=[1 6 11 6 0];
sys=tf(num, den);
pzmap(sys)
figure
rlocus(sys)
title('root locus')
```

clc #clears all the text from the Command Window, resulting in a clear screen
num=[1]; #Coefficients of the numerator
den=[1 6 11 6 0]; #Coefficients of the denominator
sys=tf(num,den); #creates a continuous-time transfer function with numerator(s) and denominator(s) specified by num and den
pzmap(sys) #pole zero map title('pole zero map of c(t)')
figure
rlocus(sys)#calculates and plots the root locus of the SISO model sys
title('root locus')

clc #clears all the text from the Command Window, resulting in a clear screen

num=[1]; #Coefficients of the numerator

den=[1 6 11 6 0]; #Coefficients of the denominator

sys=tf(num,den); #creates a continuous-time transfer function with numerator(s) and denominator(s) specified by num and den

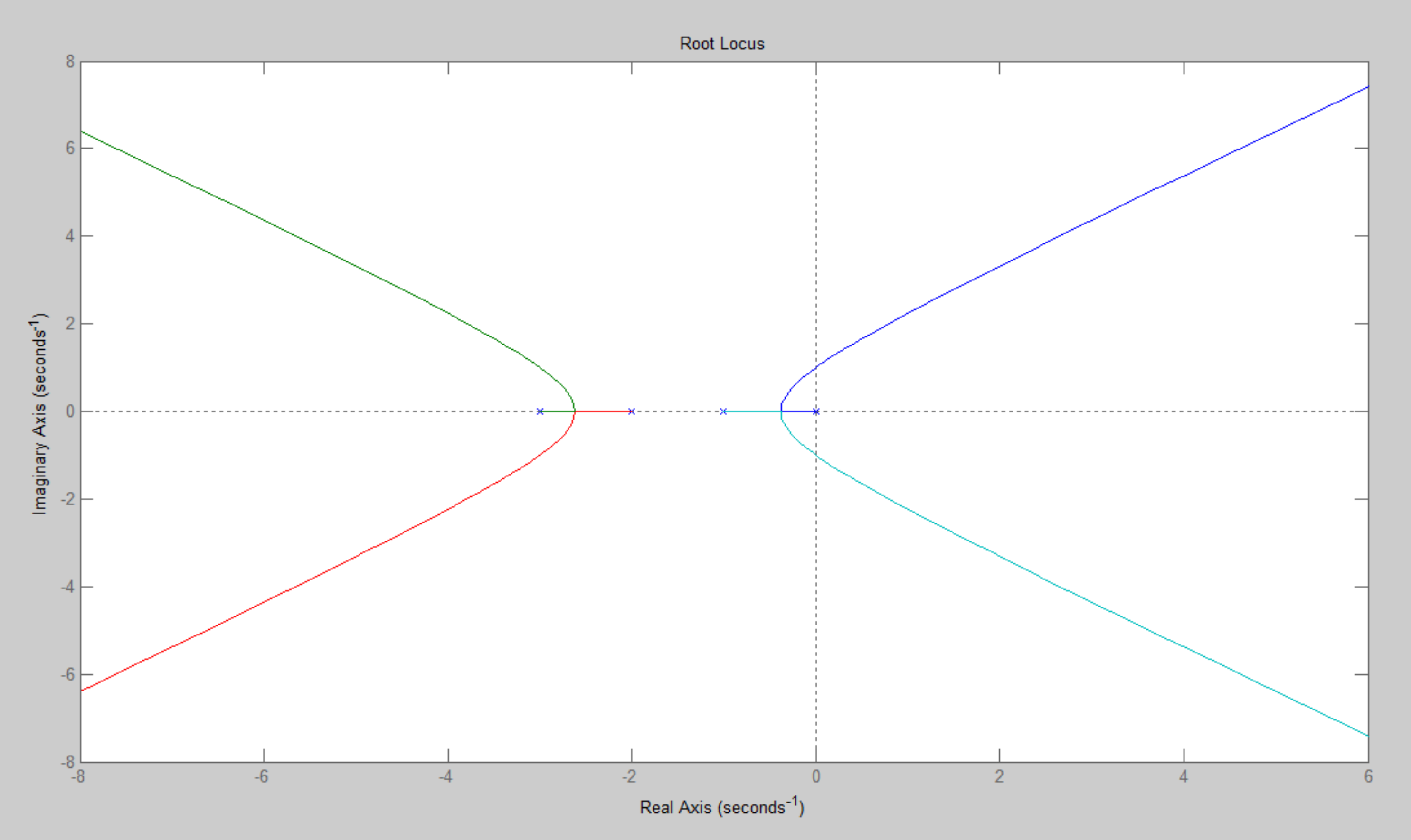
pzmap(sys)#pole zero map title('pole zero map of c(t)')

figure

rlocus(sys)#calculates and plots the root locus of the SISO model sys

title('root locus')

Root Locus Technique



```
clc
num=[1];
den=[1 6 11 6 0];
sys=tf(num, den);
pzmap(sys)
rlocus(sys)
```

clc #clears all the text from the Command Window, resulting in a clear screen
num=[1]; #Coefficients of the numerator
den=[1 6 11 6 0]; #Coefficients of the denominator
sys=tf(num,den); #creates a continuous-time transfer function with numerator(s) and denominator(s) specified by num and den
zpk(sys); #zpk is used to create zero-pole-gain models (ZPK objects) or to convert TF or SS models to zero-pole-gain form
rlocus(sys)#calculates and plots the root locus of the SISO model sys

Solved Example-2

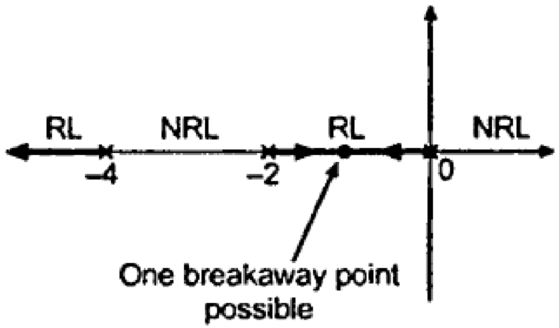
Consider $G(s)H(s) = \frac{K}{S(S+2)(S+4)}$, Draw the root locus.

Solution:

Step 1: Number of branches approaching to infinity

P = 3, s = 0; s = -2; s = -4;　　　　　Z = 0

N = P – Z = 3; Three root locus branches will terminate at zero @ infinity



Step 2: Locate the root locus path on real axis.

Step 3: Angle of asymptotes

$$\theta = \frac{(2q+1)180^0}{P-Z}, \qquad q = 0, 1, 2.$$

For q=0, $\theta_1 = \frac{180^0}{3} = 60$; For q=1, $\theta_2 = \frac{(2+1)180^0}{3} = 180^0$; For q=2, $\theta_3 = \frac{(4+1)180^0}{3} = 300$

Solved Example

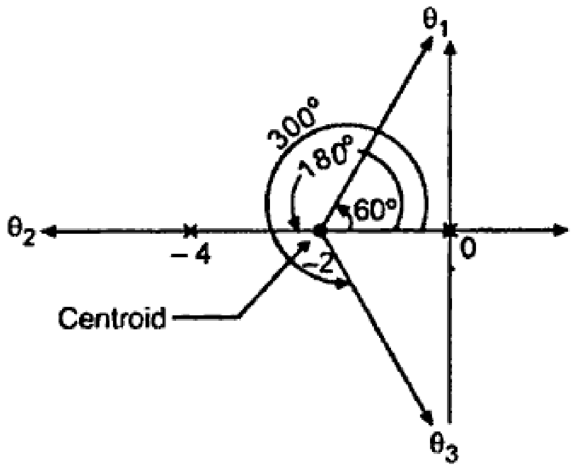
Angle of asymptotes: 60^0, 180^0, 300^0

P = 3, s = 0; s = -2; s = -4;   Z = 0

Step 4: Centroid

$$\sigma = \frac{\sum Real\ parts\ of\ poles\ - \sum Real\ parts\ of\ zeros}{P\ - Z}$$

$$\sigma = \frac{0-2-4}{3} = -\frac{6}{3} = -2$$



Solved Example

Step 5: To find Breakaway point

Characteristic equation 1+G(s)H(s) = 0

$$1+\frac{K}{S(S+2)(S+4)}=0$$

$S^3+6S^2+8S+K = 0$

$K = -S^3-6S^2-8S$

$$\frac{dK}{ds}=-3S^2-12S-8=0$$

$3S^2 + 12S+8 = 0$

Breakaway points = -0.845, -3.15

For s = -3.15; K = -3.079

For s = -0.845; K = +3.079

For s = -0.845, K is positive hence -0.845 is valid breakaway point.

Solved Example

Step 6: Intersection point with imaginary axis

Characteristic equation 1+G(s)H(s) = 0

$S^3+6S^2+8S+K = 0$

$$\frac{48-K}{6} = 0; \quad Kmar = 48$$

The auxiliary equation = A(s) = $6S^2 + K = 0$

$6S^2 + 48 = 0$

$S^2 = -8$

$S = \pm j2.828$

S^3	1	8
S^2	6	K
S^1	$\frac{48-K}{6}$	0
S^0	K	

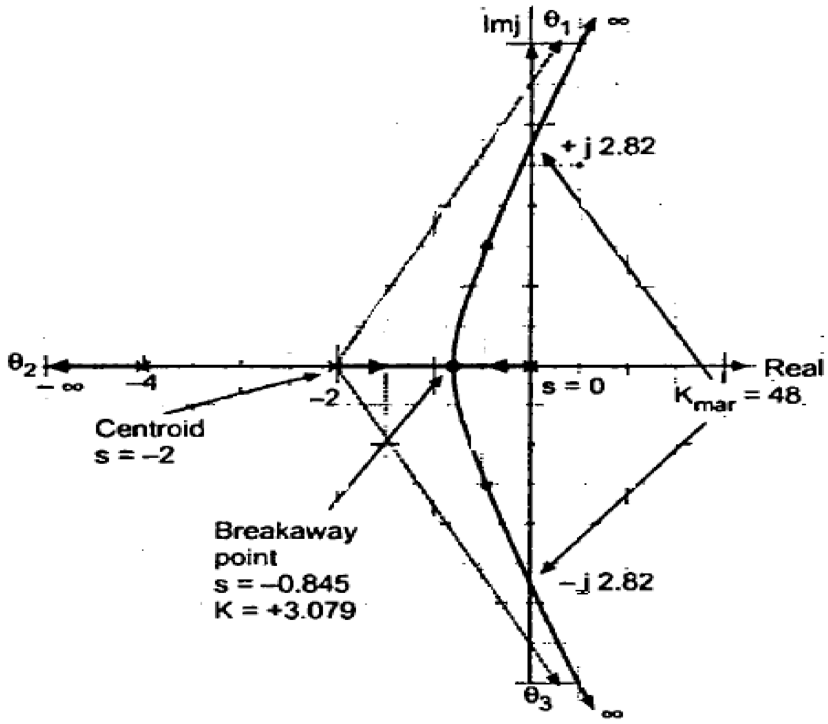
The marginal value of K is value which makes any row other than s^0 as row of zeros.

To find frequency, find out the auxiliary equation at K_{mar}

$s = \pm j\omega$

ω = Frequency of oscillations

Solved Example



Example 3

A feedback control system has an open loop transfer function $G(s)H(s) = \frac{K}{S(S+3)(S^2+2S+2)}$. Draw the root locus.

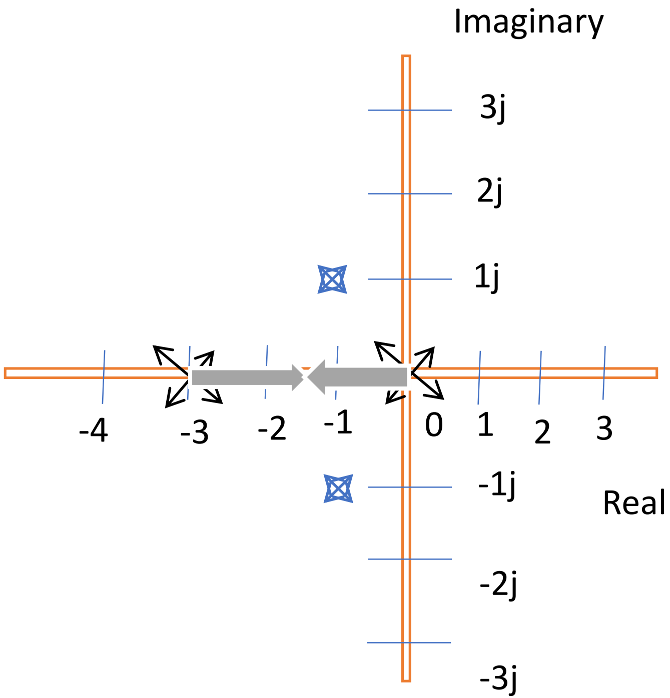
Solution:

Step 1: Number of poles = P = 4 ; s = 0, s = -3, s = -1 + j, s = -1 – j

Number of zeros = Z = 0

Number of branches approaching towards infinity = N = P-Z = 4

Step 2: Locate the root locus on real axis



Example 3

Step 3: Angle of asymptotes

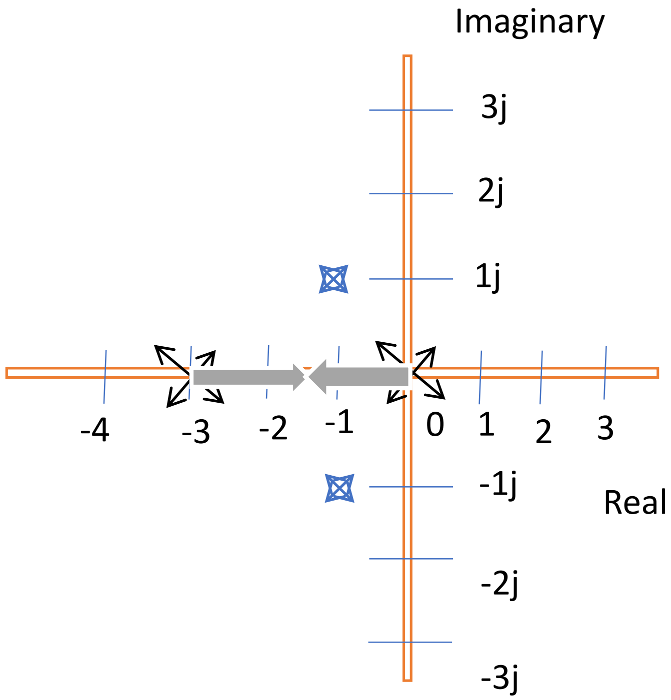
$$\theta = \frac{(2q+1)180°}{P-Z}; \quad q = 0, 1, 2, \ldots$$

For q = 0, $\theta_1 = \frac{180°}{4} = 45°$

For q = 1, $\theta_2 = \frac{3*180°}{4} = 135°$

For q = 2, $\theta_3 = \frac{5*180°}{4} = 225°$

For q = 3, $\theta_4 = \frac{7*180°}{4} = 315°$



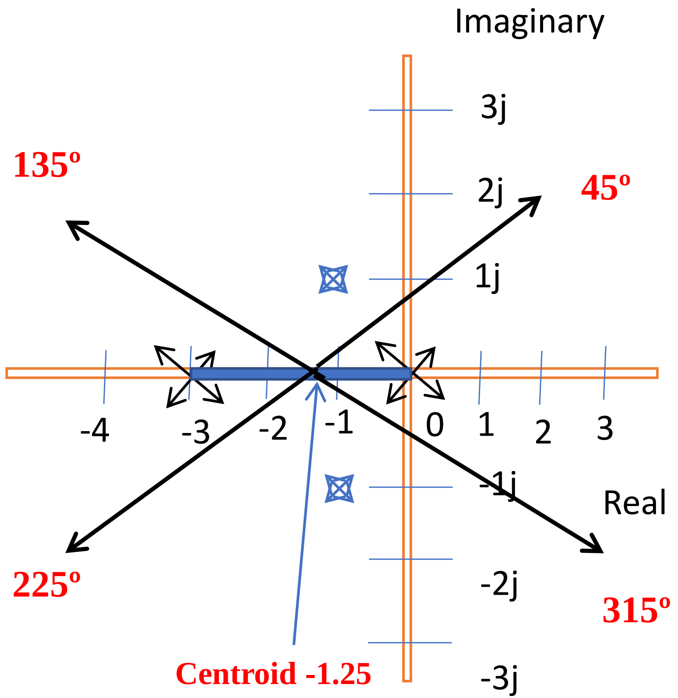
Example 3

Step 4: Centroid

$$\sigma = \frac{\sum Real\ parts\ of\ poles\ - \sum Real\ parts\ of\ zeros}{P - Z}$$

Number of poles = P = 4 ;   s = 0, s = -3, s = -1 + j, s = -1 – j

$$\sigma = \frac{0-3-1-1}{4} = -\frac{5}{4} = -1.25$$



Example 3

Step 5: Breakaway point

Characteristic equation = 1+G(s)H(s) = 0

$$1 + \frac{K}{s(s+3)(s^2+2s+2)} = 0$$

$$S^4+5S^3+8S^2+6S+K = 0$$

$$K = -S^4-5S^3-8S^2-6S$$

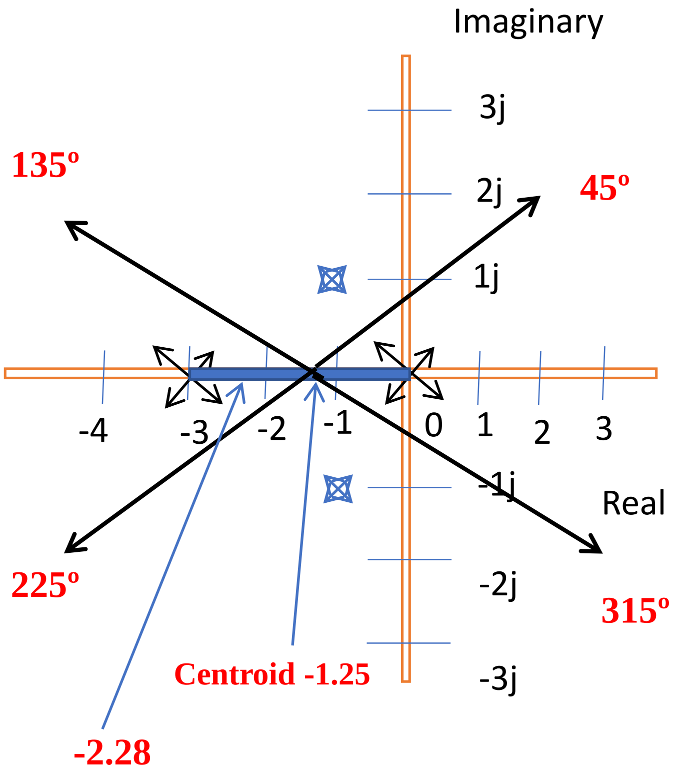
$$dK/ds = -4S^3 - 15S^2 - 16S - 6 = 0$$

$$4S^3+15S^2+16S+6 = 0$$

Breakaway points = -2.28

For s = -2.28, K = 4.3315

Hence s = -2.28 is valid breakaway point



Example 3

Step 6: Intersection with imaginary axis

Characteristic equation = 1+G(s)H(s) = 0

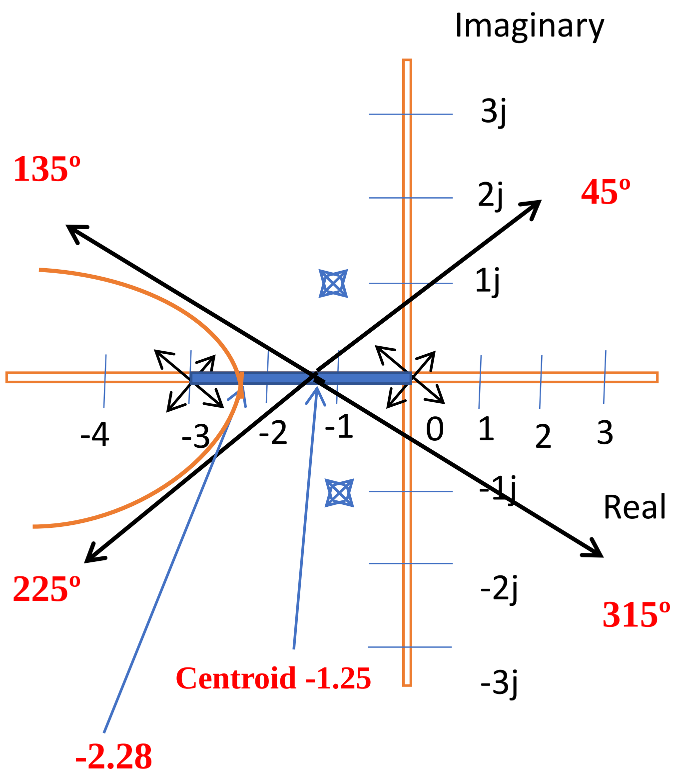
$S^4+5S^3+8S^2+6S+K = 0$

S^4	1	8	K
S^3	5	6	0
S^2	6.8	K	0
S^1	(40.8-5K)/6.8	0	0
S^0	K		

$$\frac{40.8-5K}{6.8} = 0; \quad Kmar = \frac{40.8}{5} = 8.16$$

The auxiliary equation = A(s) = $6.8S^2 + K = 0$

$6.8S^2 + 8.16 = 0$; $s = \pm 1.095\,j$



The marginal value of K is value which makes any row other than s^0 as row of zeros.

To find frequency, find out the auxiliary equation at K_{mar}

$s = \pm j\omega$

ω = Frequency of oscillations

Example 3

Step 6: Angle of departure

$$\varphi_{P1} = 180^\circ - \tan^{-1}\left(\frac{1}{1}\right) = 135^\circ$$

$$\varphi_{P2} = 90^\circ$$

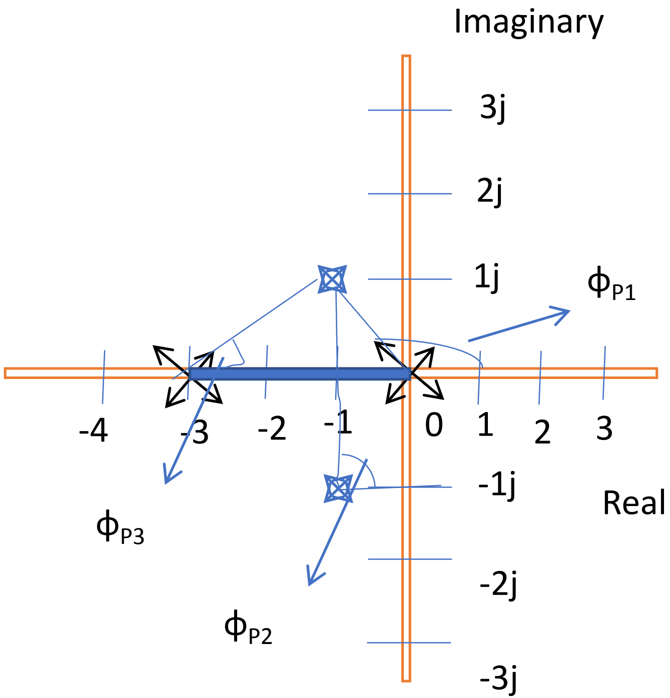
$$\varphi_{P3} = \tan^{-1}\left(\frac{1}{2}\right) = 26.56^\circ$$

$$\sum \varphi_P = 135^\circ + 90^\circ + 26.56^\circ = 251.56^\circ \qquad \sum \varphi_z = 0^\circ$$

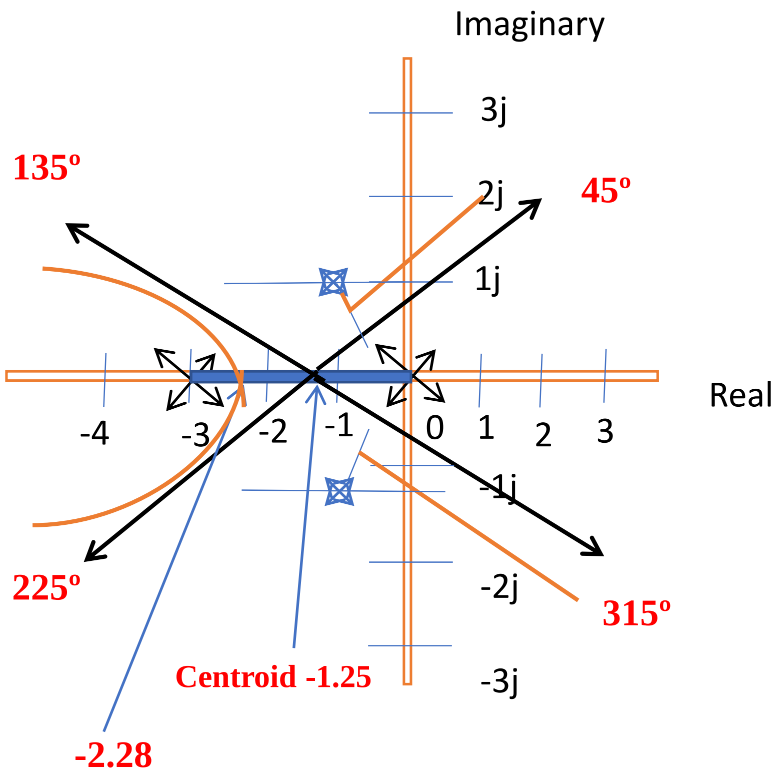
$$\varphi_d = 180^\circ - \left(\sum \varphi_P - \sum \varphi_Z\right)$$

$$\varphi_d = 180^\circ - 251.56^\circ = -71.56^\circ$$

- **At $-1 + 1j = -71.56^\circ$**
- **At -1-1j = +71.56º**



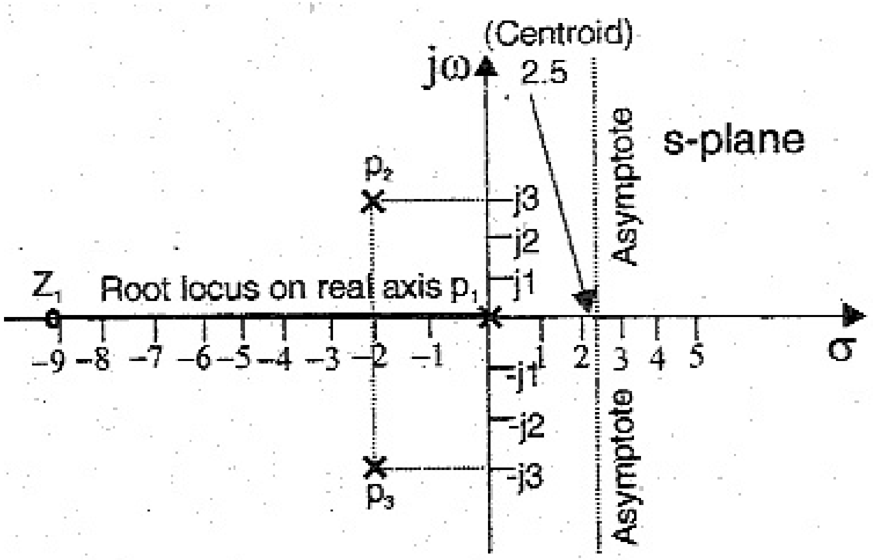
Example 3



Imaginary
3j
2j
1j
-1j
-2j
-3j
Real
-4
-3
-2
-1
0
1
2
3
135º
45º
225º
315º
Centroid -1.25
-2.28

Plot root locus for the T.F $G(s) = \dfrac{K(s+9)}{s(s^2+4s+11)}$.

$$s = \frac{-4 \pm \sqrt{4^2 - 4\times 11}}{2} = -2 \pm j2.64$$



Step 3: Angle of asymptotes

$$q = 0, \quad \text{Angles} = \pm\frac{180^\circ}{2} = \pm 90^\circ$$

$$q = 1, \quad \text{Angles} = \pm\frac{180^\circ \times 3}{2} = \pm 270^\circ = \mp 90^\circ$$

$$q = 2, \quad \text{Angles} = \pm\frac{180^\circ \times 5}{2} = \pm 450^\circ = \pm 90^\circ$$

$$\text{Centroid} = \frac{\text{Sum of poles} - \text{Sum of zeros}}{n - m} = \frac{0 - 2 + j2.64 - 2 - j2.64 - (-9)}{2} = 2.5$$

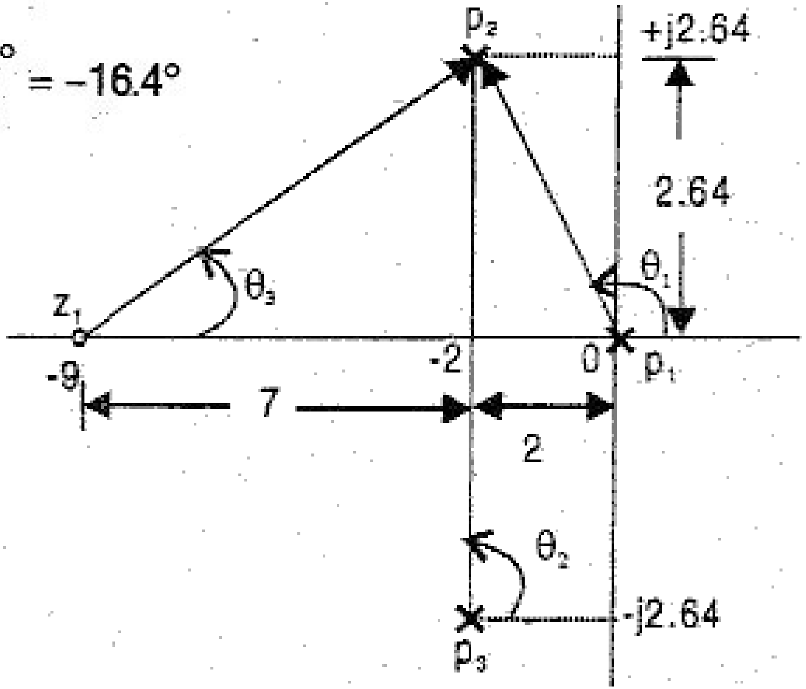
$$\left.\begin{array}{l}\text{Angle of departure from}\\ \text{the complex pole } p_2\end{array}\right\} = 180^\circ - (\theta_1 + \theta_2) + \theta_3$$

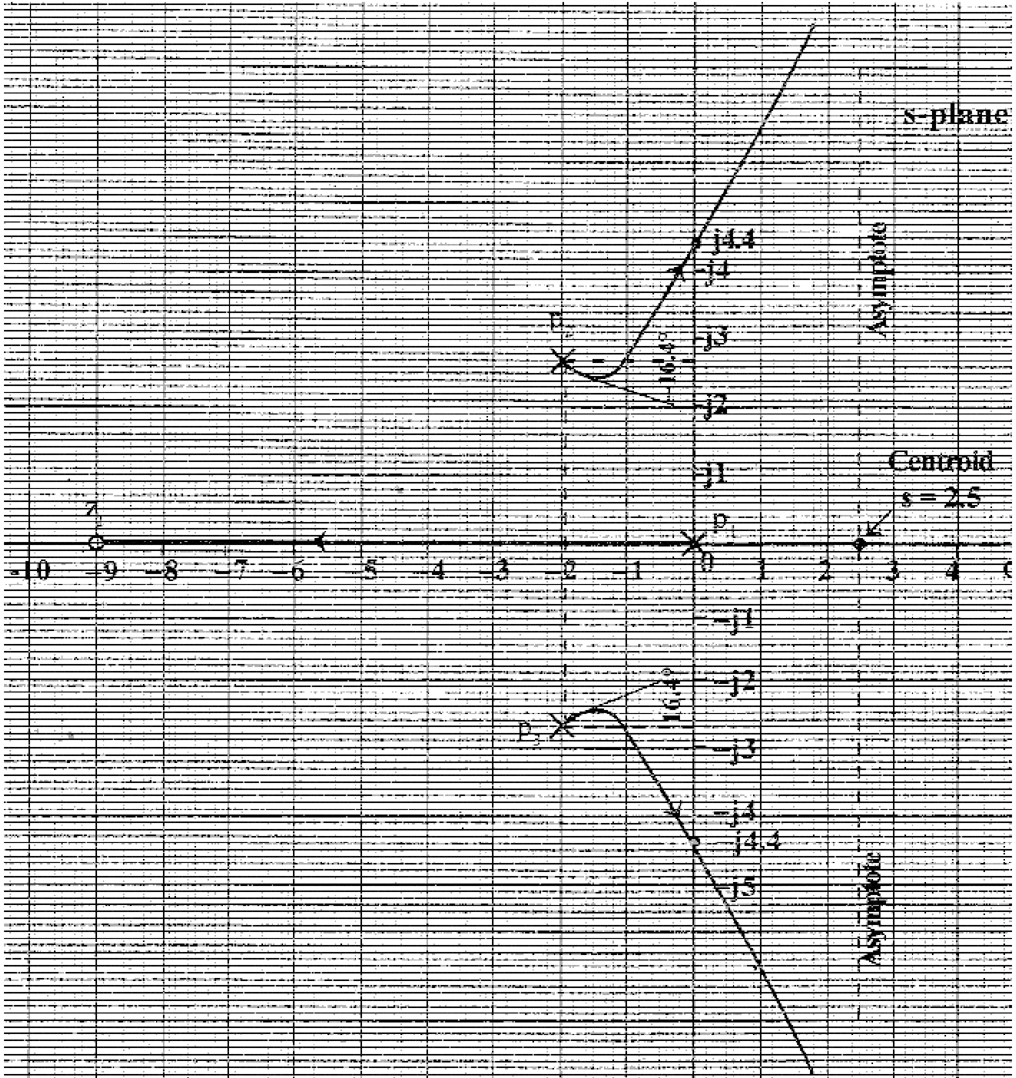
$$= 180^\circ - (127.1^\circ + 90^\circ) + 20.7^\circ = -16.4^\circ$$

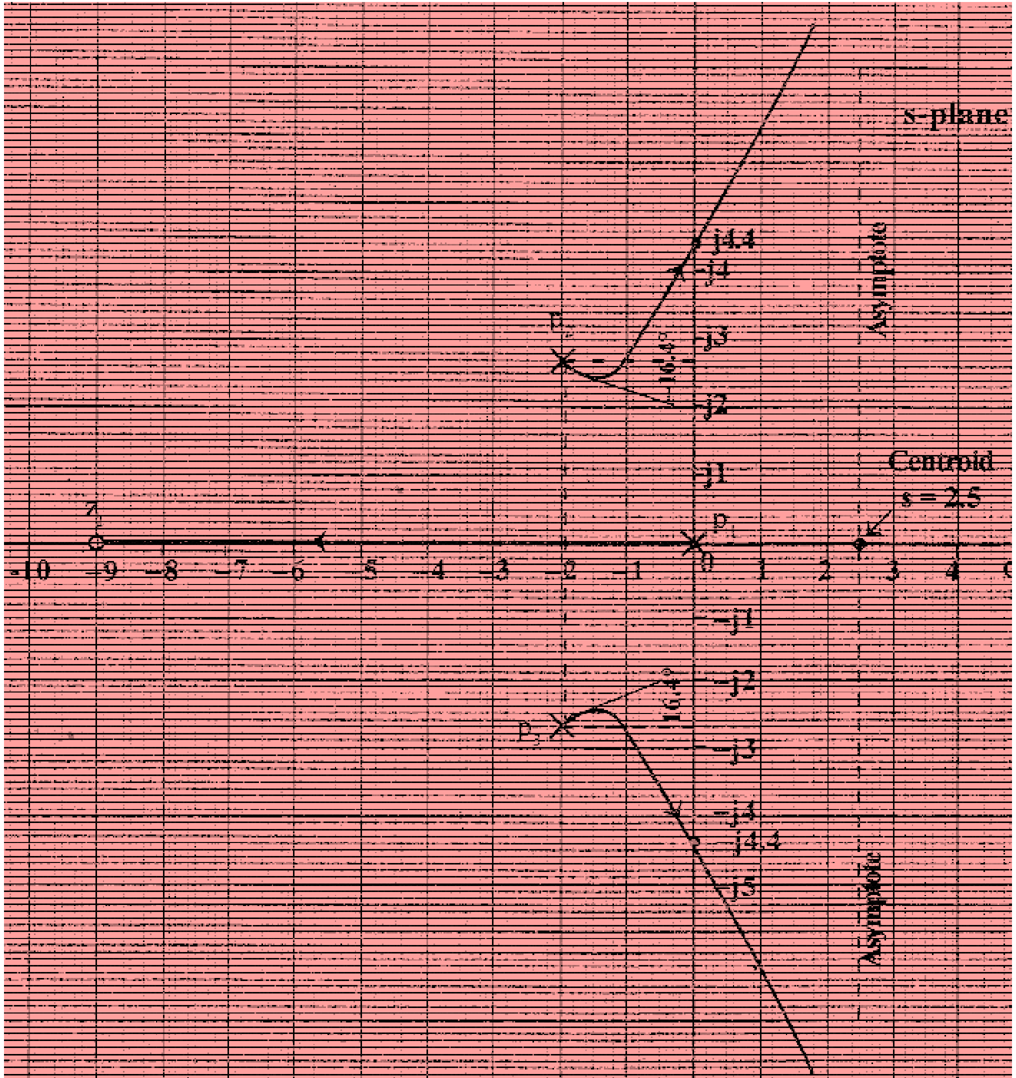
$$\theta_1 = 180^\circ - \tan^{-1}\frac{2.64}{2} = 127.1^\circ$$

$$\theta_2 = 90^\circ$$

$$\theta_3 = \tan^{-1}\frac{2.64}{7} = 20.7^\circ$$




s-plane
j4.4
j4
j3
j2
j1
p_2
p_1
p_3
z_1
16.4°
16.4°
Asymptote
Asymptote
Centroid
s = 2.5
-10
-9
-8
-7
-6
-5
-4
-3
-2
-1
0
1
2
3
4
-j1
-j2
-j3
-j4
-j4.4
-j5


s-plane
j4.4
j4
j3
j2
j1
16.4°
p_2
Asymptote
Centroid
s = 2.5
z_1
p_1
-10
-9
-8
-7
-6
-5
-4
-3
-2
-1
0
1
2
3
4
-j1
-j2
-j3
-j4
-j4.4
-j5
p_3
16.4°
Asymptote

FREQUENCY RESPONSE ANALYSIS

The response of a system for the sinusoidal input is called sinusoidal response

The ratio of sinusoidal response and sinusoidal input is called sinusoidal transfer function of the system and general, it is denoted by $T(j\omega)$

The sinusoidal transfer function is the frequency domain representation of the system, and so it is also called frequency domain transfer function

The sinusoidal transfer, $T(j\omega)$ can be obtained as shown below

1. Construct a physical model of a system using basic elements/parameters.

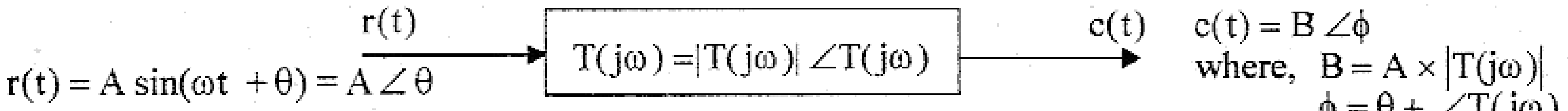
2. Determine the differential equations governing the system from the physical model of the system.

3. Take Laplace transform of differential equations in order to convert them to s-domain equation

4. Determine s-domain transfer function, $T(s)$, which is ratio of s-domain output and input.

5. Determine the frequency domain transfer function, $T(j\omega)$ by replacing s by $j\omega$ in the s-domain transfer function, $T(s)$.

Consider a linear time invariant system with frequency domain transfer function, T(jω) shown in fig



System with sinusoidal transfer function T(jω).

The advantages of frequency response analysis

1. The absolute and relative stability of the closed loop system can be estimated from the knowledge of their open loop frequency response.

2. The practical testing of systems can be easily carried with available sinusoidal signal generators and precise measurement equipment's .

3. The transfer function of complicated systems can be determined experimentally by frequency response tests

4. The design and parameter adjustment of the open loop transfer function of a system for specified closed loop performance is carried out more easily in frequency domain.

5. When the system is designed by use of the frequency response analysis, the effects of noise, disturbance and parameters variations are relatively easy to visualize and incorporate corrective measures

6. The frequency response analysis and designs can be extended to certain nonlinear control systems.

Frequency Domain specifications

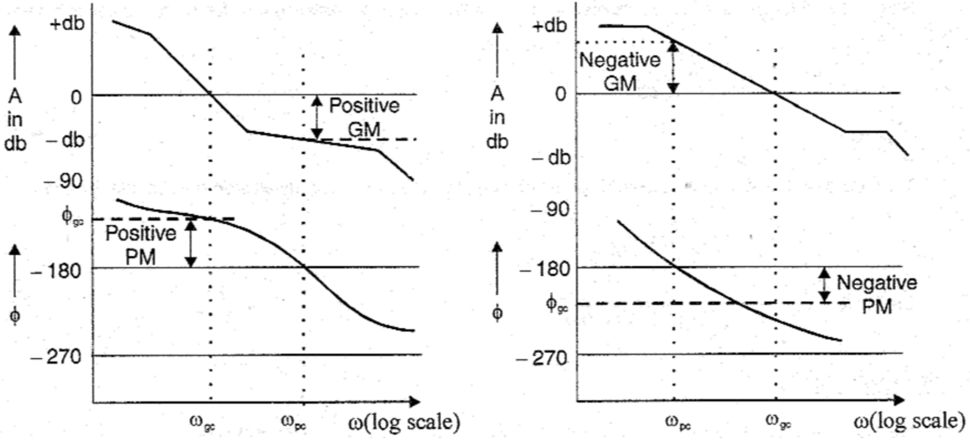
In designing a control system, we require that the system meets the performance specifications. Performance specifications are the constraints imposed on the mathematical models describing system characteristics

In frequency response analysis, following are the frequency-domain specifications.

i) **Gain Margin**: It is defined as the reciprocal of the magnitude of open-loop transfer function $|G(j\omega)H(j\omega)|$ at the frequency where the phase angle is -180°. The frequency at which the phase angle is -180° is called phase cross over frequency (ω_{pc} or ω_{π}).

Gain margin is a measure of relative stability of the system. The positive value of gain margin corresponds to stable system and the negative value of gain margin leads to unstable system. For satisfactory performance gain margin should be greater than 6 dB.

Gain margin and Phase margin for stable and unstable system on Bode plot

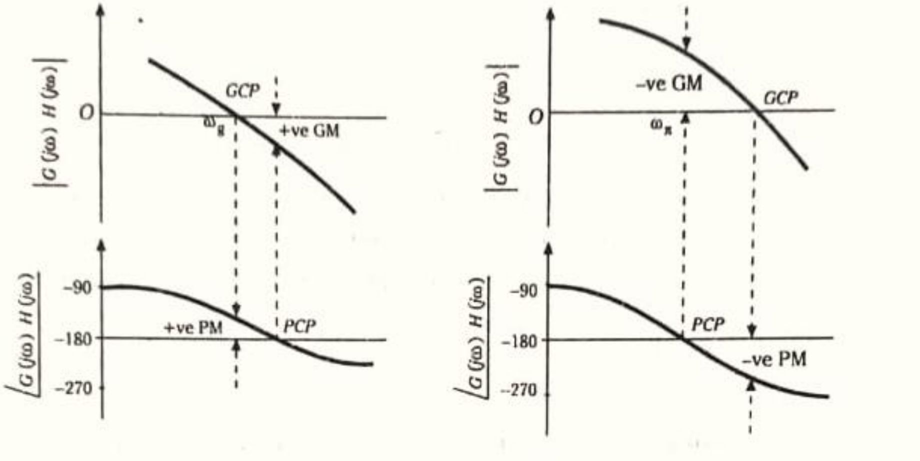


$$\text{Gain Margin, } K_g = \frac{1}{|G(j\omega_{pc})|}$$

The gain margin in db can be expressed as,

$$K_g \text{ in db} = 20 \log K_g = 20 \log \frac{1}{|G(j\omega_{pc})|}$$

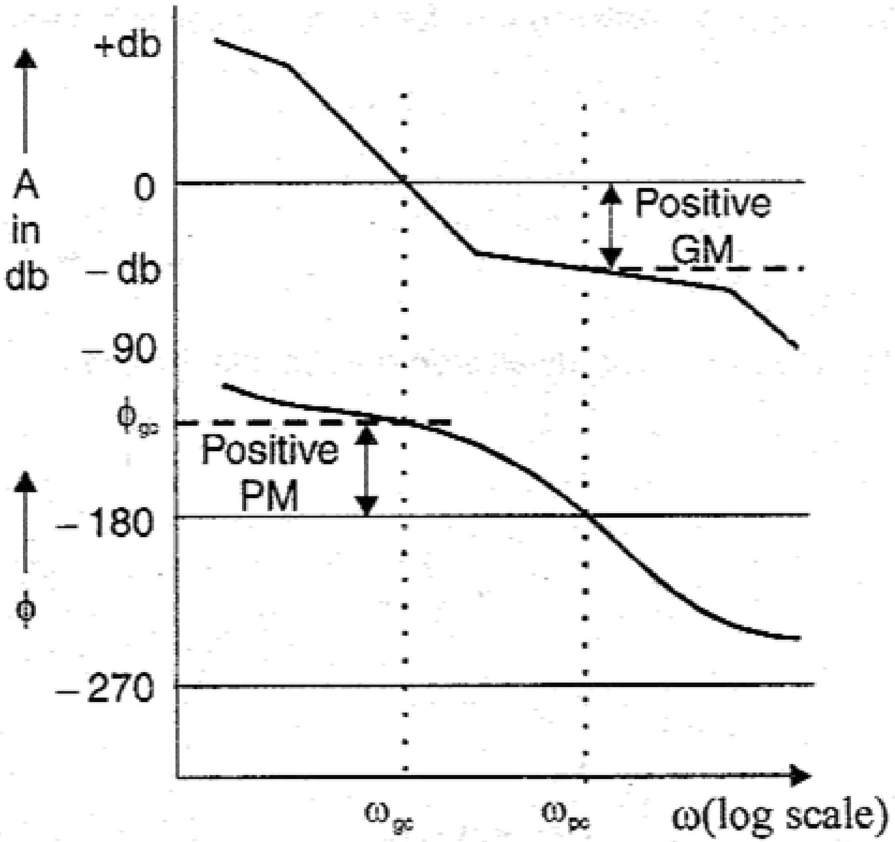
Note : $|G(j\omega_{pc})|$ *is the magnitude of* $G(j\omega)$ at $\omega = \omega_{pc}$

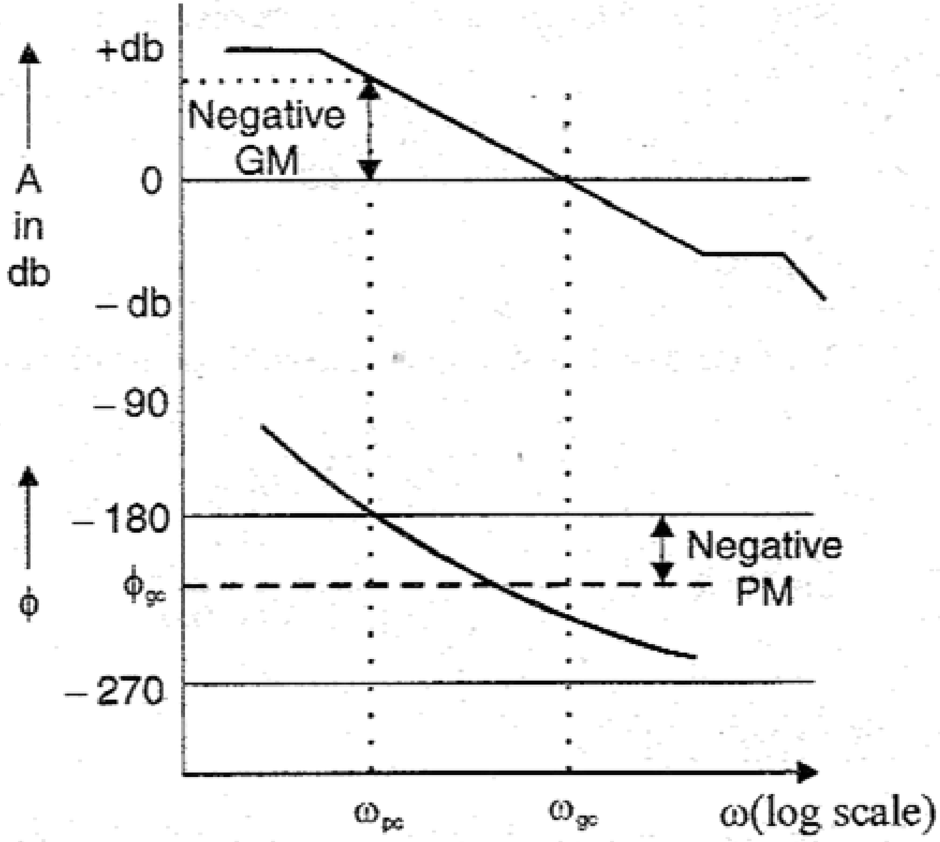


(a) Stable system (b) Unstable system

Bode diagram showing GM and PM for stable and unstable system

Gain margin and Phase margin for stable and unstable system on Bode plot





ii) Phase margin: It is defined as 180° plus the phase angle Φ of the open loop transfer function at gain cross over frequency

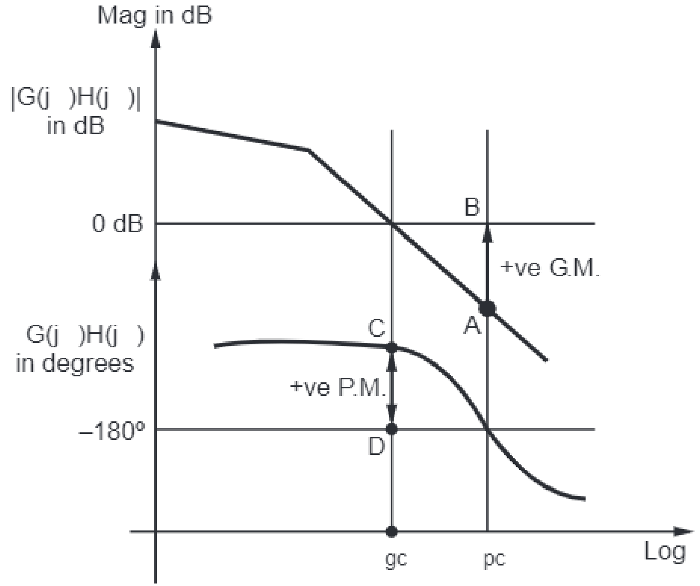
$$\text{Phase margin, } \gamma = 180^\circ + \phi_{gc},$$

$$\text{where, } \phi_{gc} = \angle G(j\omega_{gc})$$

> *Note :* $\angle G(j\omega_{gc})$ *is the phase angle of* $G(j\omega)$ at $\omega = \omega_{gc}$

where ω_{gc} or ω_g is called gain cross-over frequency. For a minimum phase system to be stable, the phase margin must be positive. Phase margin is also a measure of relative stability for satisfactory performance, the phase margin should lie between 30° and 60°.

Stability Conditions :



Gain Margin and Phase Margin

Gain Margin:

$$GM = -20 \text{ Log } |G(j\omega)H(j\omega)| \text{ at } \omega = \omega_{pc}$$

Phase Margin:

$$PM = \text{Angle } (G(j\omega)H(j\omega)) \text{ at } \omega = \omega_{gc}$$

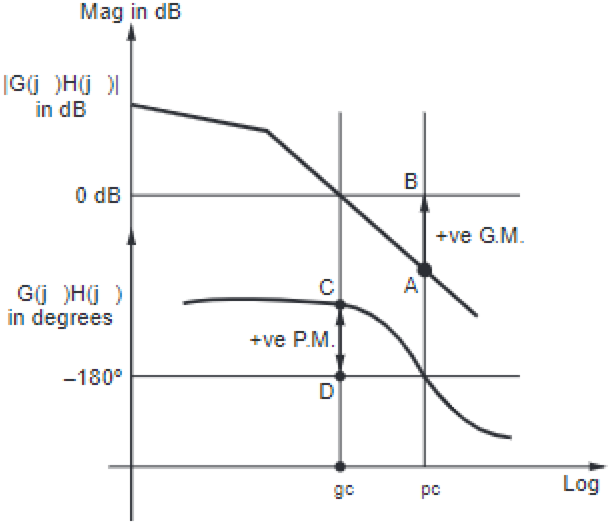


Fig. 11.7.1 $\omega_{gc} < \omega_{pc}$ G.M. and P.M. positive, stable system

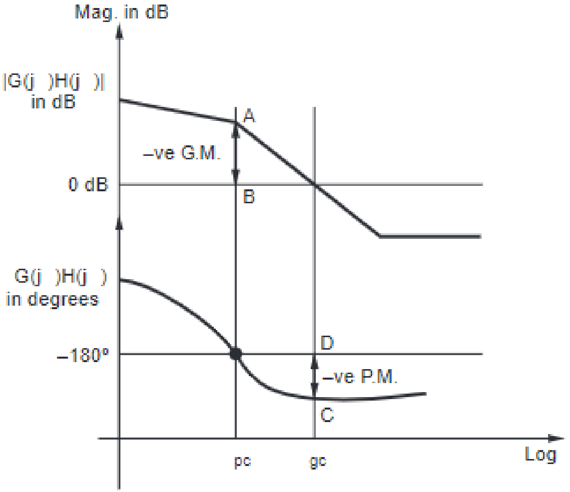


Fig. 11.7.2 $\omega_{gc} > \omega_{pc}$ G.M. and P.M. negative, unstable system

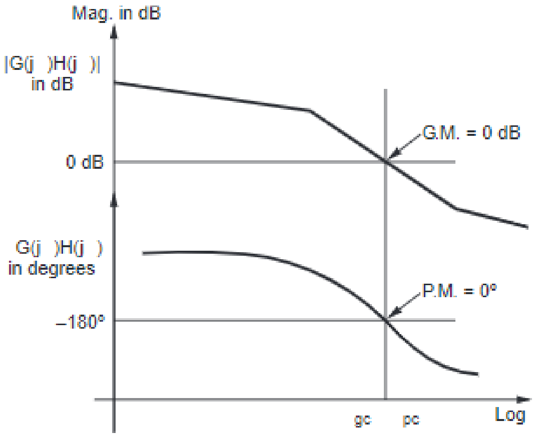
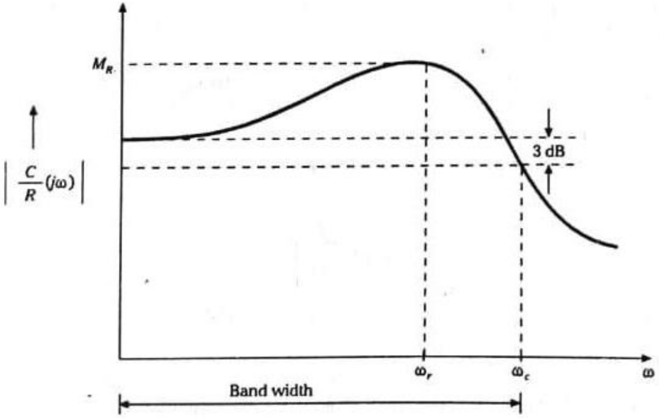


Fig. 11.7.3 $\omega_{gc} = \omega_{pc}$ G.M. and P.M. zero, marginally stable system

iii) Resonant peak 'M_R': It is defined as the maximum value of the magnitude of the closed-loop frequency response

$$M_R = \max \left| \frac{C}{R}(j\omega) \right|$$

The resonant peak is indicative of the relative stability of the system.



Performance criteria on frequency domain

For satisfactory transient performance, the value of **M_R** should be in the range 1.0 < **M_R** < 1.4 (0 dB < **M_R** < 3 dB). For the values of **M_R** > 1.5, the transient response may exhibit several overshoots. At the resonant frequency, the resonant peak **M_R** is given by the relation $M_R = \frac{1}{2\zeta\sqrt{1-\zeta^2}}$

iv) Resonant frequency ω_r

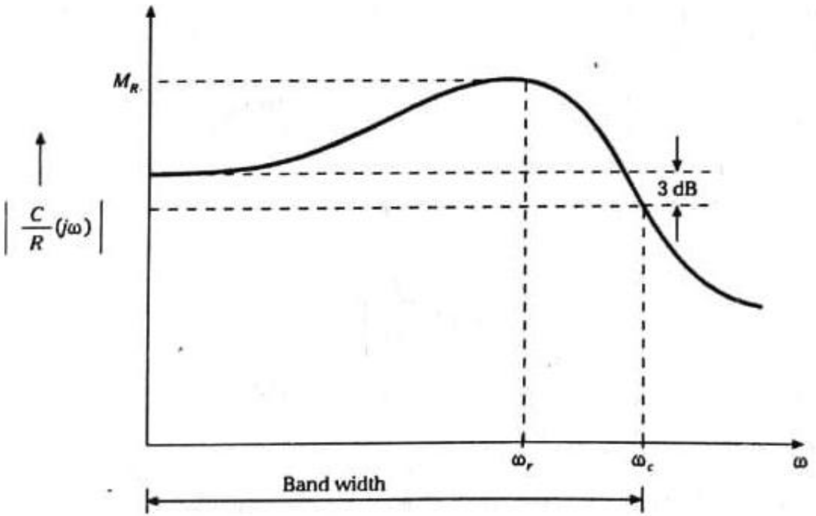
It is the frequency at which resonant peak M_R occurs.

It is an indicative of speed of transient response. The smaller the values of $\boldsymbol{\omega_r}$ more sluggish the time response is. The resonant frequency is given by the relation. $\omega_r = \omega_n \sqrt{1-2\zeta^2}$

v) Cut-off rate 'ω_c

It is frequency rate at which the magnitude ratio decreases beyond the cut-off frequency $\boldsymbol{\omega_c}$.

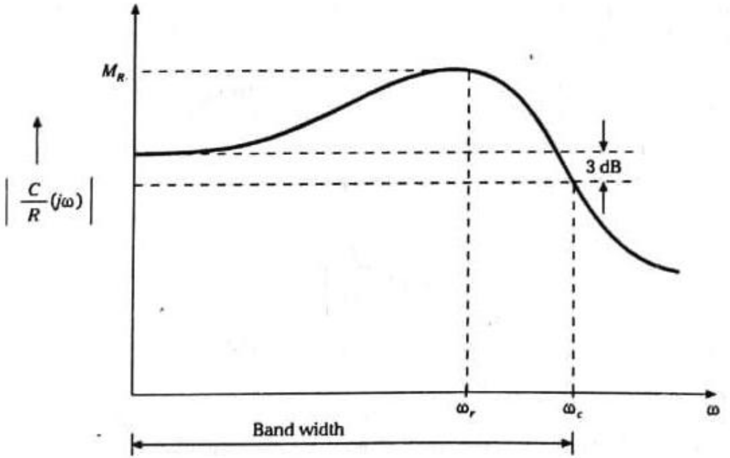
It indicates the ability of a system to distinguish the signal from unwanted signal



Performance criteria on frequency domain

vi) Bandwidth 'ω_b': It is defined as the range of frequencies over which the system will respond satisfactorily. Often the bandwidth corresponds to range of frequencies over which the 'magnitude ratio does not differ by more than -3 dB as shown in Fig. from its value at a specified frequency. The bandwidth is given

$$\omega_b = \omega_n \left[1 - 2\zeta^2 + \sqrt{2 - 4\zeta^2 + 4\zeta^4} \right]^{1/2}$$



Performance criteria on frequency domain

The bandwidth indicates how well the system will follow the input signal. For the system to follow the input accurately, larger bandwidth is desired. However larger bandwidth demands for costlier high performance components. The Bandwidth is also indicative of rise time in transient response for a given damping factor. A large bandwidth corresponds to small rise time or fast response

Bode Plot

The Bode plot is a frequency response plot of the sinusoidal transfer function of a system.

Frequency response of a system can be represented graphically by its magnitude $|G(j\omega)|$ and phase response $\Phi(\omega)$. Such a plot is called Bode plot or logarithmic plot or corner plot.

This method of plotting frequency response employs logarithms of function so that multiplications and division are reduced to addition and subtraction

Bode plot is a straight forward approach in the analysis of complex transfer functions with many poles and zeros.

A Bode plot consists of two graphs. One is a plot of logarithm of the magnitude of a sinusoidal transfer function versus log ω. The other is a plot of the phase angle of a sinusoidal transfer function versus log ω

The log magnitude is expressed in dbs (The decibel is a quantity which control engineering took from acoustics and is used to compare sound intensities) and the frequency in radians per second

Since log magnitude is also a function of frequency, a convenient way to express frequency bands are necessary

The standard representation of the logarithmic magnitude of open loop transfer function of G(jω) is 20 log |G(j ω)| where the base of the logarithm is 10

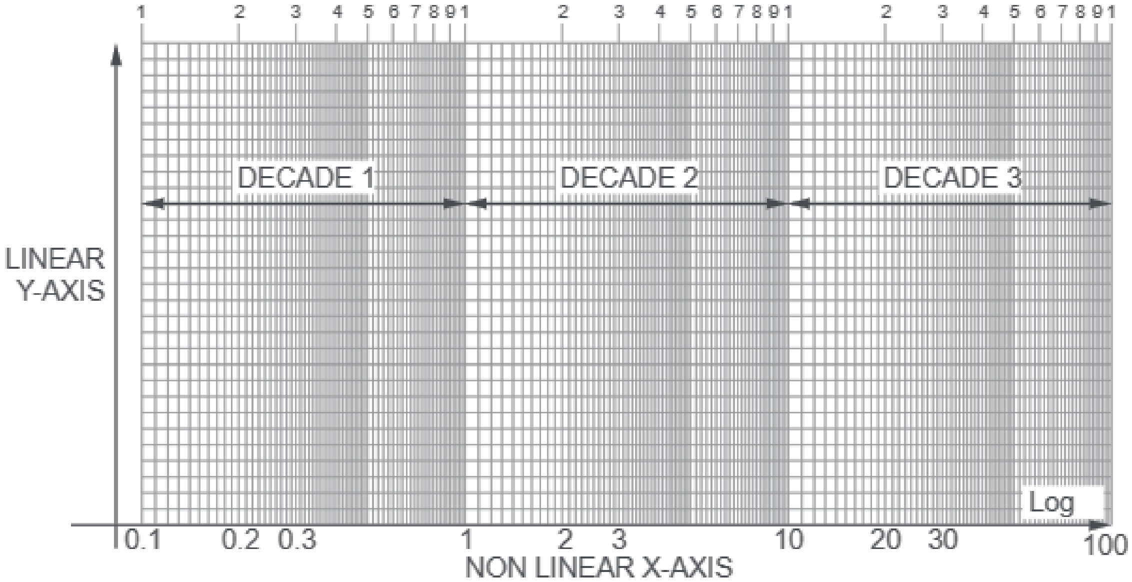
The unit used in this (representation of the magnitude is the decibel, usually abbreviated db

The curves are drawn on semilog paper , using the log scale (abcissa) for frequency and the linear scale (ordinate) for either magnitude (in decibels) or phase angle (in degrees)

The main advantage of the bode plot is that multiplication of magnitudes can be converted into addition

When the frequency varies from ω_1 to ω_2 where $\omega_2 = 10\ \omega_1$, then the frequency band is referred to as a decade

The band from 1 Hz to 10 Hz or from 2 Hz to 20 Hz is one decade We observe that if G(jω) increases by tenfold or one decade, then the log magnitude increases by 20 db.



Method for sketching an approximate log-magnitude curve

Consider the open loop transfer function which is in Time constant Form, $G(s) = \dfrac{K(1+sT_1)}{s(1+sT_2)(1+sT}$

$$G(j\omega) = \frac{K(1+j\omega T_1)}{j\omega(1+j\omega T_2)(1+j\omega T_3)}$$

$$= \frac{K\angle 0^\circ \sqrt{1+\omega^2 T_1^2}\ \angle \tan^{-1}\omega T_1}{\omega\angle 90^\circ \sqrt{1+\omega^2 T_2^2}\ \angle \tan^{-1}\omega T_2 \sqrt{1+\omega^2 T_3^2}\ \angle \tan^{-1}\omega T_3}$$

The magnitude of $G(j\omega) = |G(j\omega)| = \dfrac{K\sqrt{1+\omega^2 T_1^2}}{\omega\sqrt{1+\omega^2 T_2^2}\ \sqrt{1+\omega^2 T_3^2}}$

The phase angle of the $G(j\omega) = \angle G(j\omega) = \tan^{-1}\omega T_1 - 90^\circ - \tan^{-1}\omega T_2 - \tan^{-1}\omega T_3$

The magnitude of G(jω) can be expressed in decibels as shown in below

$$|G(j\omega)| \text{ in db} = 20 \log |G(j\omega)|$$

$$= 20 \log \left[\frac{K \sqrt{1+\omega^2 T_1^2}}{\omega \sqrt{1+\omega^2 T_2^2} \sqrt{1+\omega^2 T_3^2}} \right]$$

$$= 20 \log \left[\frac{K}{\omega} \times \sqrt{1+\omega^2 T_1^2} \times \frac{1}{\sqrt{1+\omega^2 T_2^2}} \times \frac{1}{\sqrt{1+\omega^2 T_3^2}} \right]$$

$$= 20 \log \frac{K}{\omega} + 20 \log \sqrt{1+\omega^2 T_1^2} + 20 \log \frac{1}{\sqrt{1+\omega^2 T_2^2}} + 20 \log \frac{1}{\sqrt{1+\omega^2 T_3^2}}$$

$$= 20 \log \frac{K}{\omega} + 20 \log \sqrt{1+\omega^2 T_1^2} - 20 \log \sqrt{1+\omega^2 T_2^2} - 20 \log \sqrt{1+\omega^2 T_3^2}$$

it is clear that, when the magnitude is expressed in db, the multiplication is converted to addition. Hence in magnitude plot, the db magnitudes of individual factors of G(jω) can be added

Therefore to sketch the magnitude plot, a knowledge of the magnitude variations of individual a factor is essential. The magnitude plot and phase plot of various factors, of G(jω) are explained in the following section

The basic factors that very frequently occur in a typical transfer function G(jω) are,

$$G(s) = \frac{K\,(1+sT_1)}{s\,(1+sT_2)\,(1+sT}$$

$$G(j\omega) = \frac{K\,(1+j\omega T_1)}{j\omega\,(1+j\omega T_2)\,(1+j\omega T_3)}$$

1. Constant gain, K
2. Integral factor, $\frac{K}{j\omega}$ or $\frac{K}{(j\omega)^n}$
3. Derivative factor, $K \times j\omega$ or $K \times (j\omega)^n$
4. First order factor in denominator, $\frac{1}{1+j\omega T}$ or $\frac{1}{(1+j\omega T)^m}$
5. First order factor in numerator, $(1+j\omega T)$ or $(1+j\omega T)^m$
6. Quadratic factor in denominator, $\left[\frac{1}{1+2\zeta(j\omega/\omega_n)+(j\omega/\omega_n)^2}\right]$
7. Quadratic factor in numerator, $\left[1+2\zeta\left(\frac{j\omega}{\omega_n}\right)+\left(\frac{j\omega}{\omega_n}\right)^2\right]$

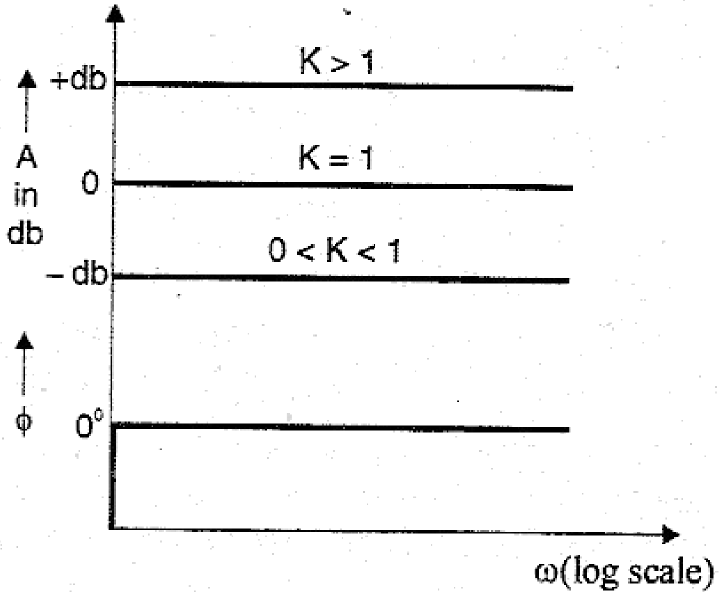
1. Constant gain, K

Let, $G(s) = K$

$$\therefore G(j\omega) = K = K\angle 0°$$

$$A = |G(j\omega)| \text{ in db} = 20 \log K$$

$$\phi = \angle G(j\omega) = 0°$$



- The magnitude plot for a constant gain K is a horizontal straight line at the magnitude of 20 log K db. and independent of $\text{Log}_{10}\omega$
- The phase plot is straight line at 0°.

When $K > 1$,	20 log K is positive.
When $0 < K < 1$,	20 log K is negative.
When $K = 1$,	20 log K is zero.

Integral Factor

Let, $G(s) = \dfrac{K}{s}$

$$\therefore G(j\omega) = \frac{K}{j\omega} = \frac{K}{\omega} \angle -90^\circ$$

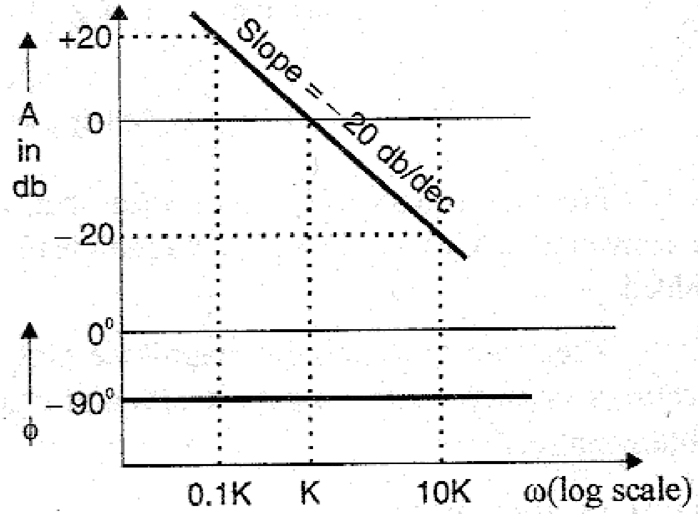
$$A = |G(j\omega)| \text{ in db} = 20 \log (K/\omega)$$

$$\phi = \angle G(j\omega) = -90^\circ$$

When $\omega = 0.1$ K, $\quad A = 20 \log (1/0.1) = 20$ db

When $\omega =$ K, $\quad A = 20 \log 1 = 0$ db

When $\omega = 10$ K, $\quad A = 20 \log (1/10) = -20$ db



From the above analysis it is evident that For **n =1,** $G(s) = K/s^n$

- ➢ The magnitude plot of the integral factor is a straight line with a slope of –20 db/decade and passing through zero db, when ω = K.
- ➢ Since the ∟G(jω) is a constant and independent of ω the phase plot is a straight line at angle -90°.

When an integral factor has multiplicity of n, then,

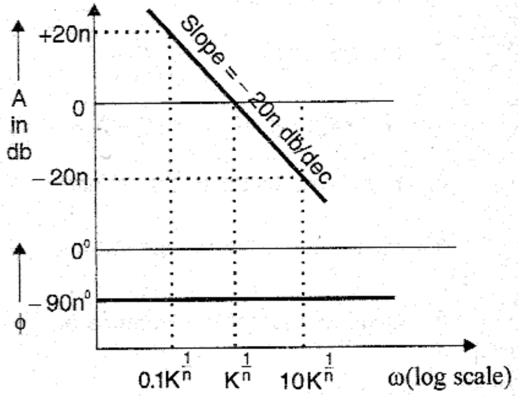
$$G(s) = K/s^n$$

$$G(j\omega) = K/(j\omega)^n = K/\omega^n \angle -90n^\circ$$

$$A = |G(j\omega)| \text{ in db} = 20 \log \frac{K}{\omega^n}$$

$$= 20\log\left(\frac{K^{\frac{1}{n}}}{\omega}\right)^n = 20\, n \log\left(\frac{K^{\frac{1}{n}}}{\omega}\right)$$

$$\phi = \angle G(j\omega) = -90\, n^\circ$$



Bode plot of integral factor, $K/(j\omega)^n$.

Now the Magnitude plot of the integral factor is a straight line with a slope of -20n db/dec and passing through zero db when $\omega = K^{1/n}$. The phase plot is a straight line at -90n°.

- For **n =2,** The magnitude plot of the integral factor is a straight line with a slope of –40 db/decade and phase plot is a straight line at angle -180°. and so on
- For n = 3, –60 db/decade and -270°

Derivative Factor

Let, $G(s) = Ks$

$$\therefore G(j\omega) = K\, j\omega = K\omega \angle 90^\circ$$

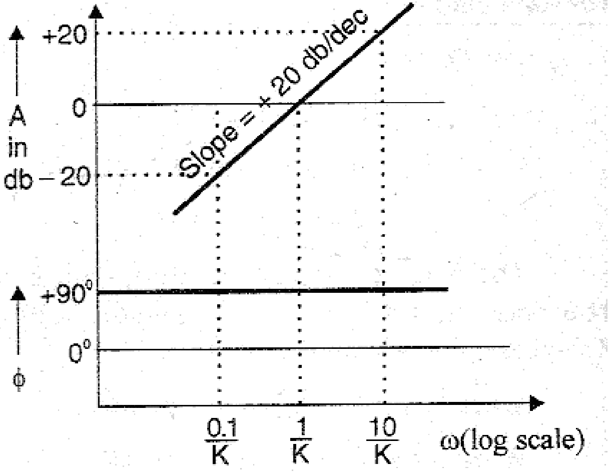
$$A = |G(j\omega)| \text{ in db} = 20 \log (K\omega)$$

$$\phi = \angle G(j\omega) = +90^\circ$$

When $\omega = 0.1/K$, $A = 20 \log (0.1) = -20$ db

When $\omega = 1/K$, $A = 20 \log 1 = 0$ db

When $\omega = 10/K$, $A = 20 \log 10 = +20$ db



Bode plot of derivative factor, $K \times j\omega$.

From the above analysis it is evident that the For **n =1,** $G(s) = K s^n$

Magnitude plot of the derivative factor is a straight line with a slope of +20 db/decade and passing through zero db when $\omega = 1/K$. Since the $\angle G(j\omega)$ is a constant and independent of ω, the phase plot is a straight line at +90°.

Derivative Factor

When an Derivative Factor has multiplicity of n, then,

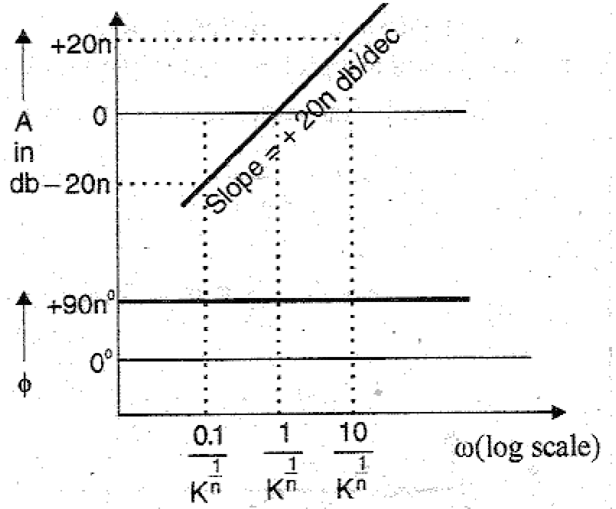
$$G(s) = K\,s^n$$

$$\therefore\; G(j\omega) = K(j\omega)^n = K\omega^n \angle 90n^\circ$$

$$A = |G(j\omega)| \text{ in db} = 20\log(K\omega^n)$$

$$= 20\log(K^{1/n}\,\omega)^n = 20\,n\log(K^{1/n}\,\omega)$$

$$\phi = \angle G(j\omega) = 90n^\circ$$



Bode plot of derivative factor, $K(j\omega)^n$.

Now the magnitude plot of the derivative factor is a straight line with a slope of +20n db/decade and passing through zero db when $\omega = 1/K^{1/n}$. The phase plot is a straight line at +90n°.

- ➢ For **n =2,** The magnitude plot of the integral factor is a straight line with a slope of +40 db/decade and phase plot is a straight line at angle +180°. and so on
- ➢ **For n = 3**, +60 db/decade and +270°

First order factor in denominator

$$G(s) = \frac{1}{1+sT}$$

$$\therefore G(j\omega) = \frac{1}{1+j\omega T} = \frac{1}{\sqrt{1+\omega^2T^2}} \angle -\tan^{-1}\omega T$$

Let, $A = |G(j\omega)|$ in db.

$$\therefore\ A = |G(j\omega)|_{\text{in db}} = 20\log\frac{1}{\sqrt{1+\omega^2T^2}} = -20\log\sqrt{1+\omega^2T^2}$$

At very low frequencies, $\omega T << 1$;

At very high frequencies, $\omega T >> 1$;

$$\therefore A = -20\log\sqrt{1+\omega^2T^2} \approx -20\log 1 = 0$$

$$\therefore A = -20\log\sqrt{1+\omega^2T^2} \approx -20\log\sqrt{\omega^2T^2} = -20\log\omega T$$

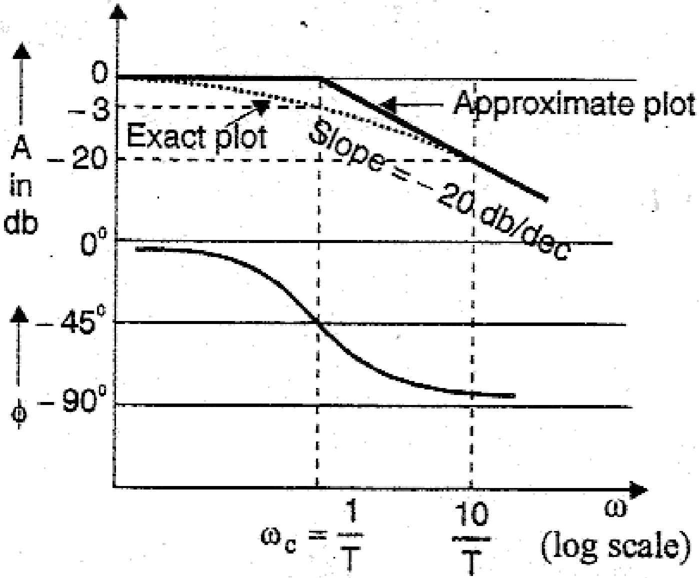
$$\text{At } \omega = \frac{1}{T},\quad A = -20\log 1 = 0$$

$$\text{At } \omega = \frac{10}{T},\quad A = -20\log 10 = -20\text{ db}$$

The above analysis shows that the magnitude plot of the factor

1/(1+jωT) can be approximated by two straight lines

one is a straight line at 0 db for the frequency range, $0 < \omega < 1/T$, and the other is a straight line with slope -20 db/dec for the frequency range, $1/T < \omega < \infty$. The two straight lines are asymptotes of the exact curve.



First order factor in denominator

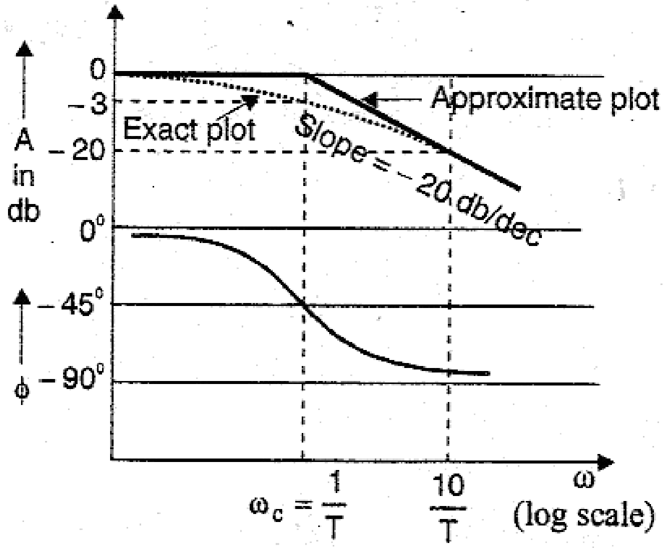
The frequency at which the two asymptotes meet is called corner frequency or break frequency. For the factor 1/(1+jωT) the frequency, ω=1/T is the corner frequency, ωc . It divides the frequency response curve into two regions, a curve for low frequency region and a curve for high frequency region

Phase angle, $\phi = \angle G(j\omega) = -\tan^{-1}\omega T$

At the corner frequency, $\omega = \omega_c = \frac{1}{T}$, $\phi = -\tan^{-1}\omega T = -\tan^{-1} 1 = -45^\circ$

As $\omega \to 0$, $\phi \to 0^\circ$

As $\omega \to \infty$, $\phi \to -90^\circ$



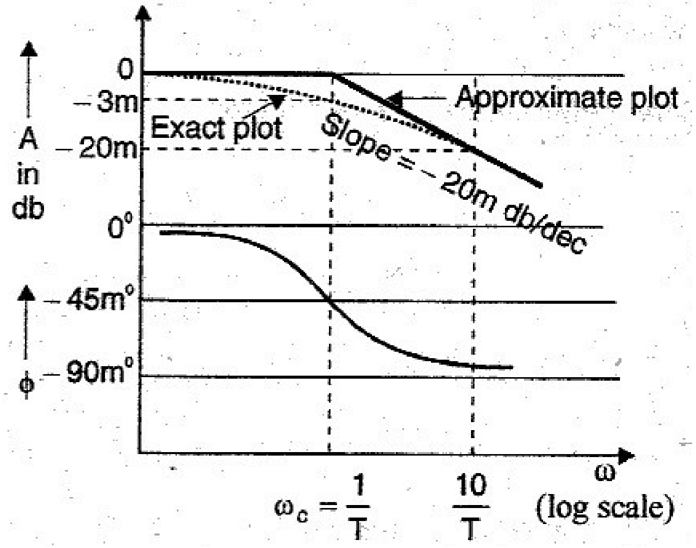
The phase angle of the factor, 1/(1+ jωT), varies from 0° to -90° as ω is varied from zero to infinity. The phase plot is a curve passing through -45° at ωc .

When the first order factor in the denominator has a multiplicity of m, then,

$$G(s) = \frac{1}{(1+sT)^m} \; ; \qquad \therefore G(j\omega) = \frac{1}{(1+j\omega T)^m} = \frac{1}{\left(\sqrt{1+\omega^2T^2}\right)^m \angle m \tan^{-1}\omega T}$$

$$A = |G(j\omega)| \text{ in db} = 20 \log \frac{1}{\left(\sqrt{1+\omega^2T^2}\right)^m} = -20\, m \log\sqrt{1+\omega^2T^2}$$

$$\phi = \angle G(j\omega) = -m \tan^{-1}\omega T$$



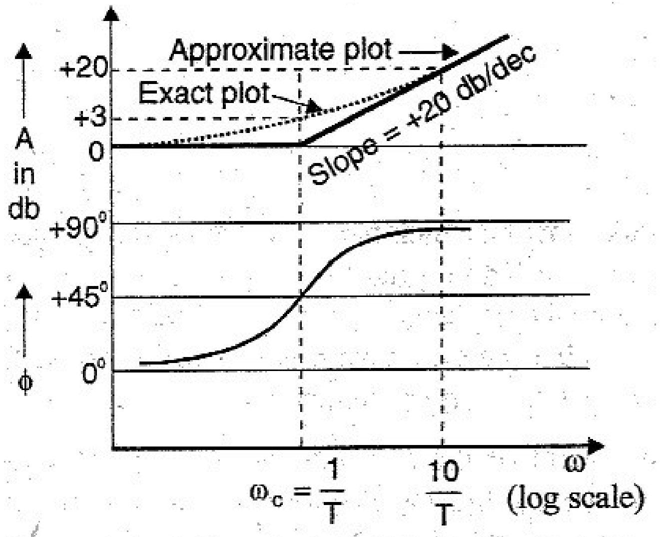
First order factor in Numerator

$$G(s) = 1 + sT$$

$$G(j\omega) = 1 + j\omega T \;= \sqrt{1+\omega^2T^2}\;\angle \tan^{-1}\omega T$$

$$A = |G(j\omega)| \text{ in db} = 20\log\sqrt{1+\omega^2T^2}$$

$$\phi = \angle G(j\omega) = \tan^{-1}\omega T$$



Bode plot of the factor (1+jωT).

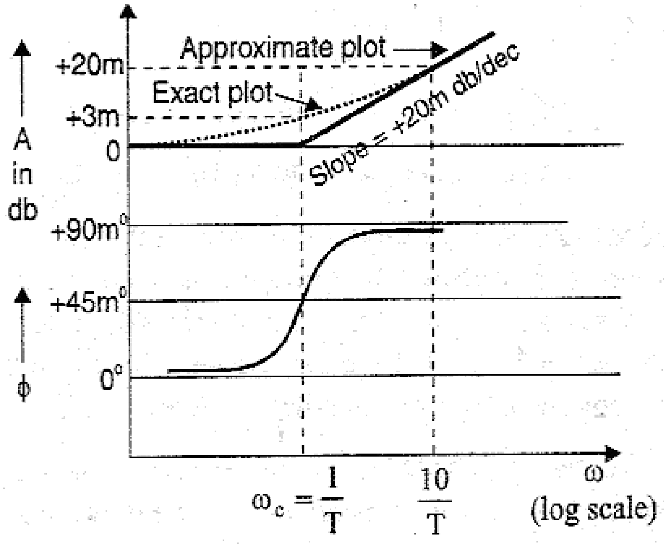
When the first order factor in the Numerator has a multiplicity of m

$$G(s) = (1 + sT)^m$$

$$G(j\omega) = (1 + j\omega T)^m \;= \left(\sqrt{1+\omega^2T^2}\right)^m \angle m\tan^{-1}\omega T$$

$$A = |G(j\omega)| \text{ in db} = 20\log\left(\sqrt{1+\omega^2T^2}\right)^m = 20m\log\sqrt{1+\omega^2T^2}$$

$$\phi = \angle G(j\omega) = m\tan^{-1}\omega T$$



Bode plot of the factor $(1+j\omega T)^m$.

Quadratic (Second order) factor in denominator

$$G(s)=\frac{\omega_n^2}{s^2+2\zeta\omega_n s+\omega_n^2}=\frac{1}{1+2\zeta\frac{s}{\omega_n}+\left(\frac{s}{\omega_n}\right)^2}\qquad \therefore G(j\omega)=\frac{1}{1+j\frac{2\zeta\omega}{\omega_n}+\left(\frac{j\omega}{\omega_n}\right)^2}=\frac{1}{1-\left(\frac{\omega}{\omega_n}\right)^2+j2\zeta\frac{\omega}{\omega_n}}=\frac{1}{\sqrt{\left(1-\frac{\omega^2}{\omega_n^2}\right)^2+4\zeta^2\frac{\omega^2}{\omega_n^2}}}\angle-\tan^{-1}\frac{2\zeta\frac{\omega}{\omega_n}}{1-\frac{\omega^2}{\omega_n^2}}$$

Let, $A=|G(j\omega)|$ in db.

$$A=20\log\frac{1}{\sqrt{\left(1-\frac{\omega^2}{\omega_n^2}\right)^2+4\zeta^2\frac{\omega^2}{\omega_n^2}}}=-20\log\sqrt{\left(1-\frac{\omega^2}{\omega_n^2}\right)^2+4\zeta^2\frac{\omega^2}{\omega_n^2}}$$

$$=-20\log\sqrt{1+\frac{\omega^4}{\omega_n^4}-2\frac{\omega^2}{\omega_n^2}+4\zeta^2\frac{\omega^2}{\omega_n^2}}=-20\log\sqrt{1-\frac{\omega^2}{\omega_n^2}(2-4\zeta^2)+\frac{\omega^4}{\omega_n^4}}$$

At very low frequencies when $\omega \ll \omega_n$, the magnitude is,

$$A=-20\log\sqrt{1-\frac{\omega^2}{\omega_n^2}(2-4\zeta^2)+\frac{\omega^4}{\omega_n^4}}\approx-20\log 1=0$$

At very high frequencies when $\omega \gg \omega_n$, the magnitude is,

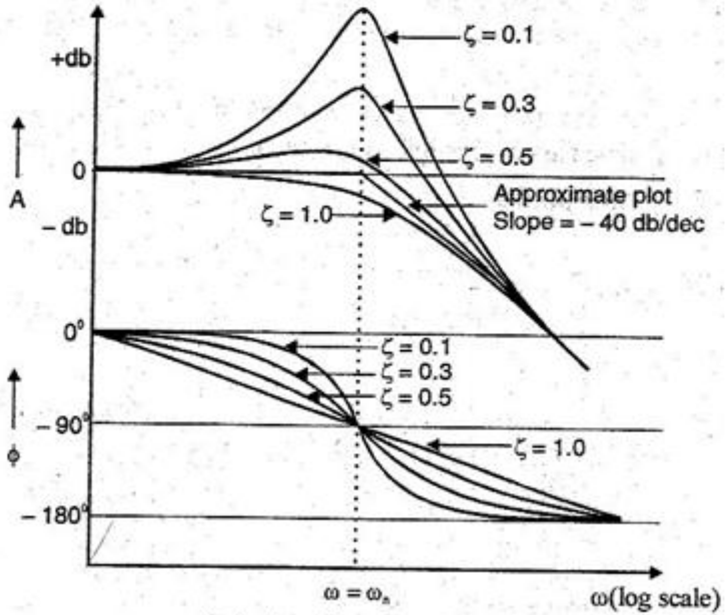
$$A=-20\log\sqrt{1-\frac{\omega^2}{\omega_n^2}(2-4\zeta^2)+\frac{\omega^4}{\omega_n^4}}\approx-20\log\sqrt{\frac{\omega^4}{\omega_n^4}}=-20\log\frac{\omega^2}{\omega_n^2}=-20\log\left(\frac{\omega}{\omega_n}\right)^2$$

$$\therefore A=-40\log\frac{\omega}{\omega_n}$$

At $\omega=\omega_n$, $A=-40\log 1=0$ db

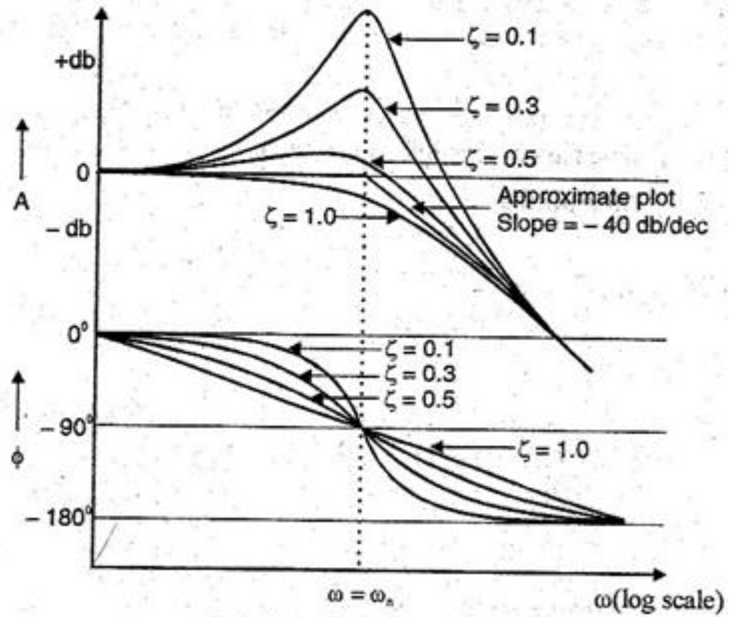
At $\omega=10\omega_n$, $A=-40\log 10=-40$ db

The magnitude plot of the quadratic factor in the denominator an be approximated by two straight lines, one is a straight line at 0 db for the frequency range $0 < \omega < \omega n$. and the other is a straight line with slope -40 db/dec for the frequency range $\omega n < \omega < \infty$. The two straight are asymptotes of the exact curve. The frequency at which the two asymptotes meet is called the corner frequency. For the quadratic factor, the frequency ωn is the corner frequency ωc ,



: Bode plot of quadratic factor in denominator.

The two asymptotes of the exact curve are independent of the damping ratio, ζ. In the exact magnitude plot, resonant peak occurs near the corner frequency and the magnitude of resonant peak depends on ζ. Lower the value of ζ , larger will be the resonant peak. Hence by this approximation the error at the corner frequency depends on damping ratio ζ. The phase plot is obtained by calculating the phase angle of $G(j\omega)$ for various values of ω.



: *Bode plot of quadratic factor in denominator.*

$$\phi = \angle G(j\omega) = -\tan^{-1}\left(\frac{2\zeta\frac{\omega}{\omega_n}}{1-\frac{\omega^2}{\omega_n^2}}\right)$$

$$\text{As } \omega = \omega_n, \quad \phi = -\tan^{-1}\frac{2\zeta}{0} = -\tan^{-1}\infty = -90^\circ$$

$$\text{As } \omega \to 0, \quad \phi \to 0$$

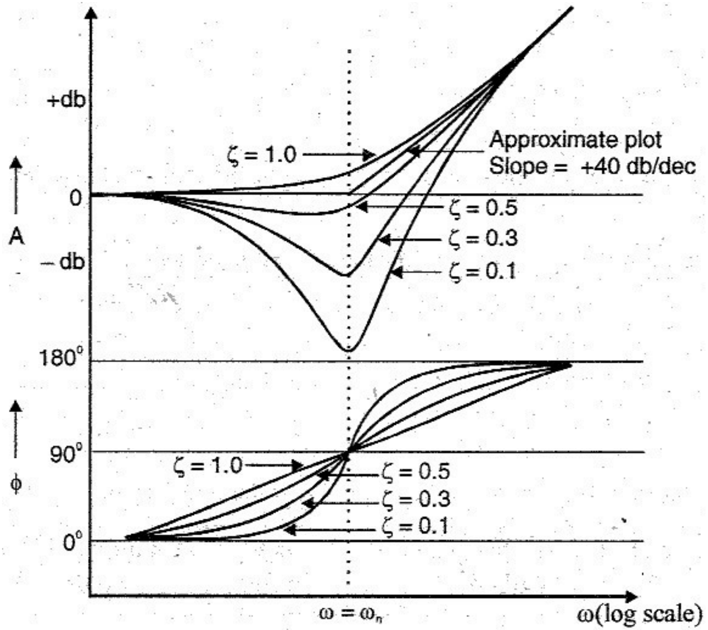
$$\text{As } \omega \to \infty, \quad \phi \to -180^\circ$$

ATME College of Engineering

Quadratic (Second order) factor in Numerator

$$G(s)=\frac{s^2+2\zeta\omega_n s+\omega_n^2}{\omega_n^2}=1+2\zeta\left(\frac{s}{\omega_n}\right)+\left(\frac{s}{\omega_n}\right)^2$$

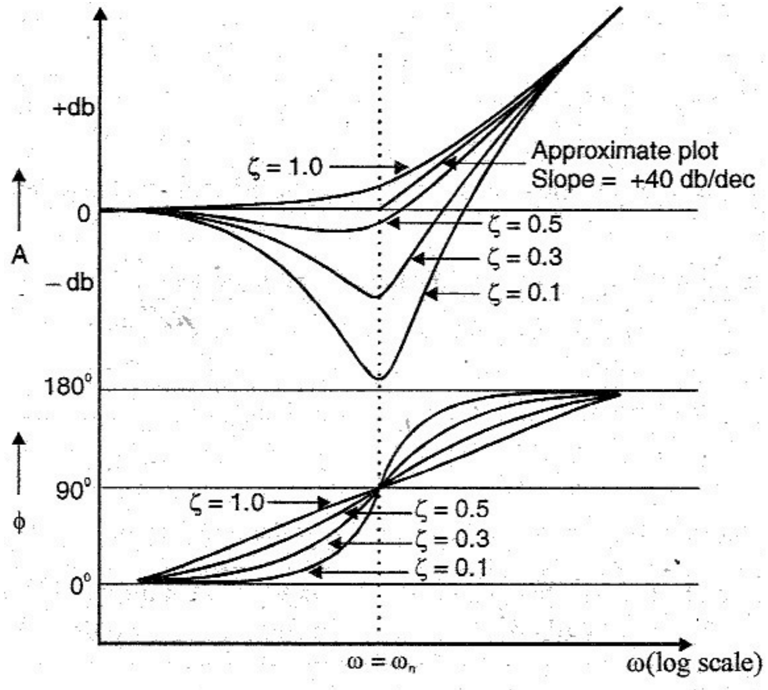
$$G(j\omega)=1+j2\zeta\frac{\omega}{\omega_n}+\left(\frac{j\omega}{\omega_n}\right)^2=\sqrt{\left(1-\frac{\omega^2}{\omega_n^2}\right)^2+4\zeta^2\frac{\omega^2}{\omega_n^2}}\ \angle\tan^{-1}\frac{2\zeta\frac{\omega}{\omega_n}}{1-\frac{\omega^2}{\omega_n^2}}$$



Bode plot of quadratic factor in numerator.

The magnitude plot of the quadratic factor in the Numerator an be approximated by two straight lines, one is a straight line at 0 db for the frequency range $0 < \omega < \omega n$. and the other is a straight line with slope +40db/dec for the frequency range $\omega n < \omega < \infty$. The two straight are asymptotes of the exact curve. The frequency at which the two asymptotes meet is called the corner frequency. For the quadratic factor, the frequency ωn is the corner frequency ωc, Due to this approximation the error at the corner frequency depends on ζ.

The phase angle varies from 0 to +180°, as ω is varied from 0 to ∞. At the corner frequency the phase angle is +90° and independent of ζ, but at all other frequency it depends on ζ.



Bode plot of quadratic factor in numerator.

PROCEDURE FOR MAGNITUDE PLOT OF BODE PLOT

From the analysis, the following conclusions can be obtained.

1. The constant gain K, integral and derivative factors Contribute gain (Magnitude) at all frequencies.

2. In approximate plot the first, quadratic and higher order factors contribute gain (magnitude) only when the frequency is greater than the corner frequency.

Hence the low frequency response upto the lowest corner frequency is decided by K or K / $(j\omega)^n$, $K(j\omega)^n$ term . Then at every corner frequency the slope of the magnitude plot is altered by the first, Quadratic and higher order terms. Therefore the magnitude plot can be started with K or $K/(j\omega)^n$ or $K(j\omega)^n$ term and then the db magnitude of every first and higher order terms are added one by one in the increasing order of the corner frequency.

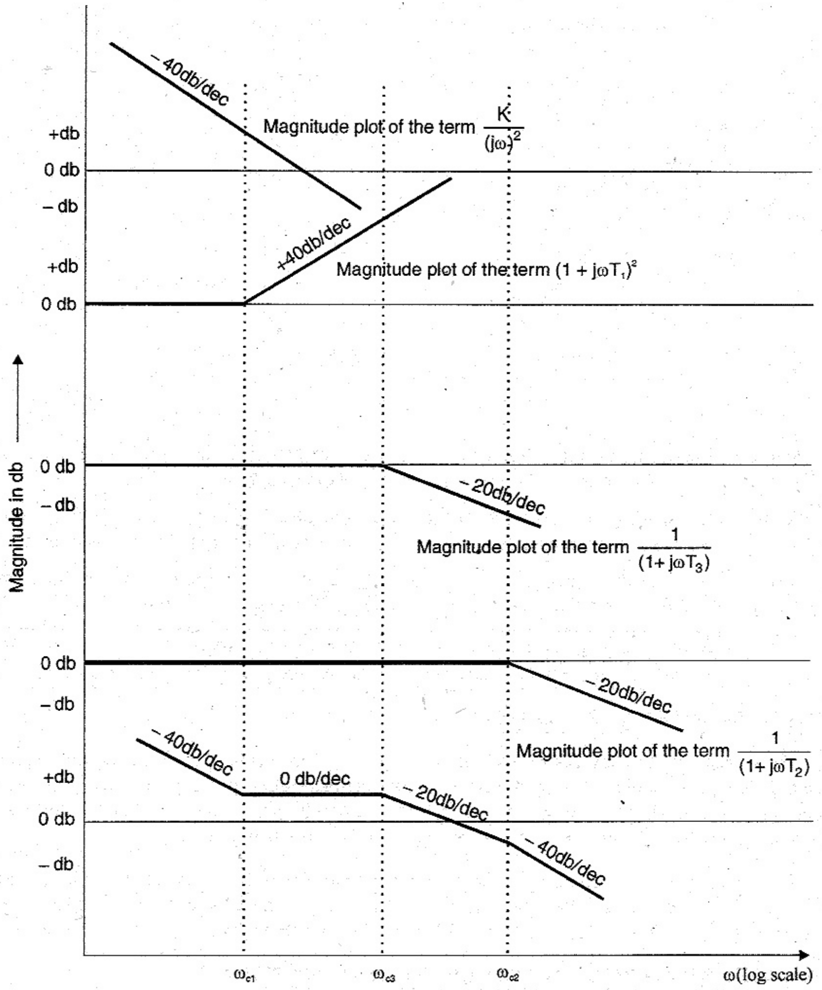
Let, $G(s) = \dfrac{K(1+sT_1)^2}{s^2(1+sT_2)(1+sT_3)}$

$$\therefore G(j\omega) = \frac{K(1+j\omega T_1)^2}{(j\omega)^2(1+j\omega T_2)(1+j\omega T_3)}$$

Let, $T_2 < T_3 < T_1$. The corner frequencies are, $\omega_{c1} = \dfrac{1}{T_1}$, $\omega_{c2} = \dfrac{1}{T_2}$, $\omega_{c3} = \dfrac{1}{T_3}$.

Let, $\omega_{c1} < \omega_{c3} < \omega_{c2}$.

The magnitude plot of the individual terms of G(jω), and their combined magnitude plot are shown an fig



: *Magnitude plot of bode plot of,* $G(j\omega) = \frac{K(1+j\omega T_1)^2}{(j\omega)^2(1+j\omega T_2)(1+j\omega T_3)}$.

The step by step procedure for plotting the magnitude plot is given below

Step 1 Convert the transfer function into Bode form or time constant form. The Bode form of the transfer function is

$$G(s) = \frac{K(1+sT_1)}{s(1+sT_2)\left(1+\frac{s^2}{\omega_n^2}+2\zeta\frac{s}{\omega_n}\right)} \xrightarrow{s=j\omega} G(j\omega) = \frac{K(1+j\omega T_1)}{j\omega(1+j\omega T_2)\left(1-\frac{\omega^2}{\omega_n^2}+j2\zeta\frac{\omega}{\omega_n}\right)}$$

Step 2 : List the corner frequencies in the increasing order and prepare a table as shown below

Term	Corner frequency rad/sec	Slope db/dec	Change in slope db/dec

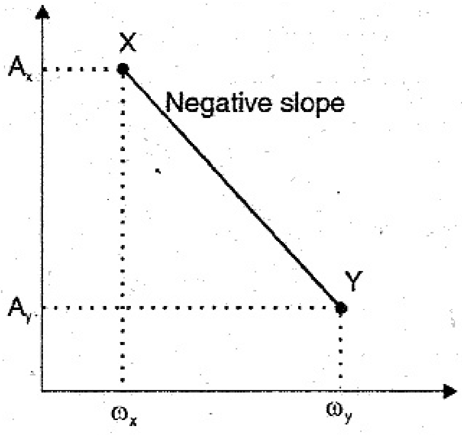
In the above table enter K or K / $(j\omega)^n$, $K(j\omega)^n$ as the first term and the other terms in the increasing order of corner frequencies. Then enter the corner frequency, slope contributed by each term and change in slope at every corner frequency.

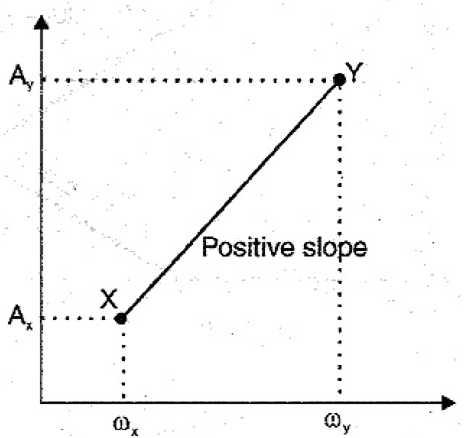
Step 3: Choose an arbitrary frequency ω_L, which is lesser than the lowest corner frequency. Calculate the db magnitude of K or K / $(j\omega)^n$, $K(j\omega)^n$ at ω_L, and at the lowest corner frequency

Step 4: Then calculate the gain (db magnitude) at every corner frequency one by one by using the formula

$$\text{Gain at } \omega_y = \text{change in gain from } \omega_x \text{ to } \omega_y + \text{Gain at } \omega_x$$

$$= \left[\text{Slope from } \omega_x \text{ to } \omega_y \times \log\frac{\omega_y}{\omega_x}\right] + \text{Gain at } \omega_x$$





Step 5: Choose an arbitrary frequency ω_H, which is greater than the highest corner frequency. Calculate the gain at ω_H by using the formula in step 4

Step 6: In a semilog graph sheet, mark the required range of frequency on x-axis (log scale) and the range of db magnitude on y-axis (ordinary scale) after choosing proper units.

Step7: Mark all the points obtained in steps 3, 4, and 5 on the graph and join the points by straight lines. Mark the slope at every part of the graph.

Note: The magnitude plot obtained above is an approximate plot. If an exact plot is needed then appropriate corrections should be made at every corner frequencies.

PROCEDURE FOR PHASE PLOT OF BODE PLOT

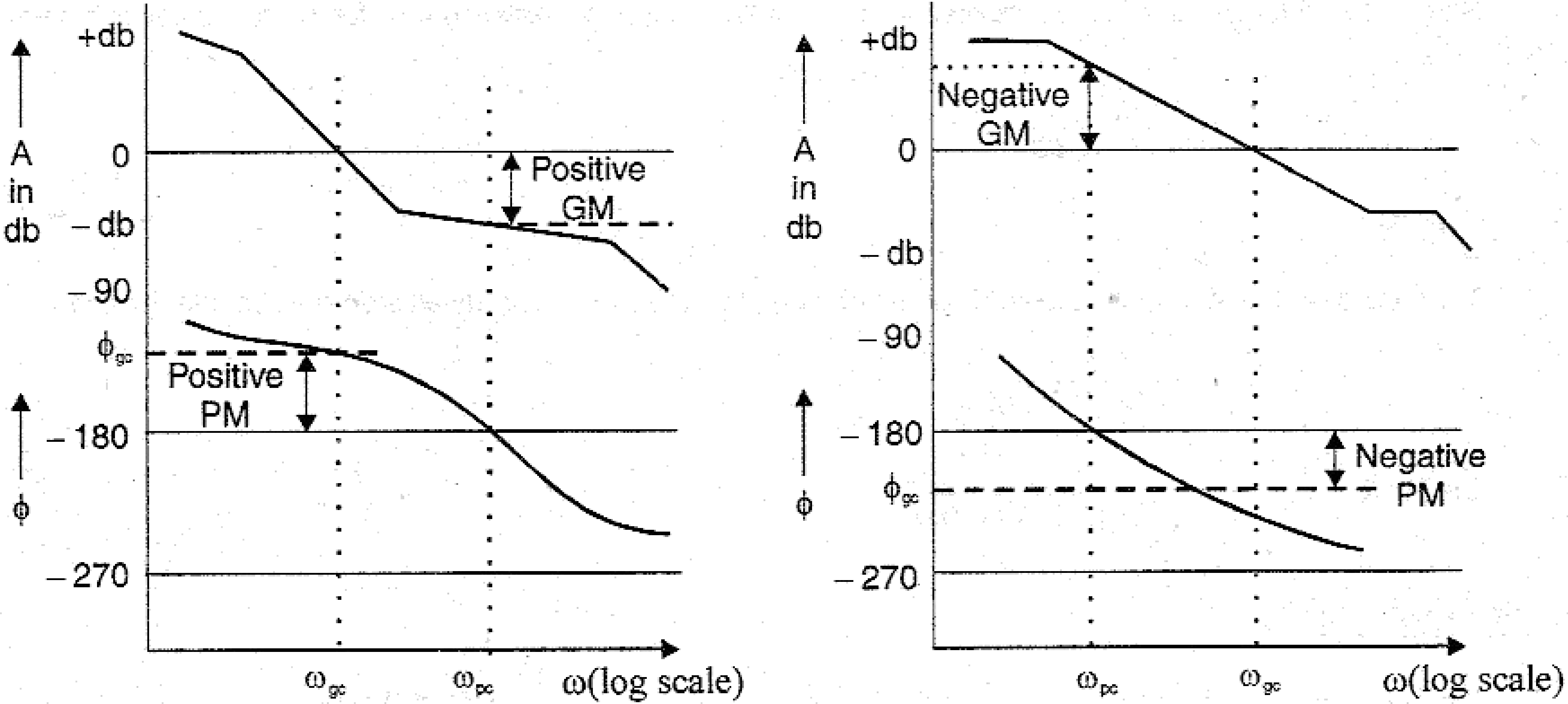
The phase plot is an exact plot and no approximations are made while drawing the phase plot. Hence the exact phase angles of G(jω) are computed for various values of ω and tabulated. The choice of frequencies are preferably the frequencies chosen for magnitude plot. Usually the magnitude plot and phase plot are drawn in a single semilog- sheet on a common frequency scale.

Take another y-axis in the graph where the magnitude plot is drawn and in this y-axis mark the desired range of phase angles after choosing proper units. From the tabulated values of w and phase angles, mark all the points on the graph. Join the points by a smooth curve

DETERMINATION OF GAIN MARGIN AND PHASE MARGIN FROM BODE PLOT

The gain margin in db is given by the negative of db magnitude of $G(j\omega)$ at the phase cross-over frequency, ω_{pc} The ω_{pc} is the frequency at which phase of $G(j\omega)$ is -180°. If the db magnitude of $G(j\omega)$ at ω_{pc} is negative then gain margin is positive and vice versa.

Let Φ_{gc} be the phase angle of $G(j\omega)$ at gain cross over frequency ω_{gc}. The ω_{gc} is the frequency at which the db magnitude of $G(j\omega)$ is zero. Now the phase margin, γ is given by, $\gamma = 180° + \Phi_{gc}$. If Φ_{gc} is less negative than -180° then phase margin is positive and vice versa



Bode plot showing phase margin (PM) and gain margin (GM).

GAIN ADJUSTMENT IN BODE PLOT

In the open loop transfer function $G(j\omega)$ the constant K contributes only magnitude. Hence by changing the value of K the system gain can be adjusted to meet the desired specifications. The desired specifications are gain margin, phase margin, ω_{pc} and ω_{gc}.

In a system transfer function if the value of K required to be estimated to satisfy a desired specification then draw the bode plot of the system with K = 1. The constant K can add 20logK to every point of the magnitude plot and due to this addition the magnitude plot will shift vertically up or down. Hence shift the magnitude plot vertically up or down to meet the desired specification. Equate the vertical distance by which the magnitude plot is shifted to 20logK and solve for K.

Plot the Bode diagram for the following transfer and obtain the gain and phase cross over frequencies.

$$G(s) = \frac{10}{s\,(1+0.4s)\,(1+0.1s)}$$

The sinusoidal transfer function of G(jω) is obtained by replacing s by jω in the given transfer function.

$$\therefore G(j\omega) = \frac{10}{j\omega\,(1+j0.4\omega)\,(1+j0.1\omega)}$$

MAGNITUDE PLOT

The corner frequencies are, $\omega_{c1} = \frac{1}{0.4} = 2.5 \text{ rad/sec}$ & $\omega_{c2} = \frac{1}{0.1} = 10 \text{ rad/sec}$

The various terms of G(jω) are listed in table-1 in the increasing order of their corner frequencies. Also the table shows the slope contributed by each term and the change in slope at the corner frequency.

$$\therefore G(j\omega) = \frac{10}{j\omega\,(1+j0.4\omega)\,(1+j0.1\omega)}$$

MAGNITUDE PLOT

The corner frequencies are, $\omega_{c1} = \frac{1}{0.4} = 2.5 \text{ rad/sec}$ $\qquad \omega_{c2} = \frac{1}{0.1} = 10 \text{ rad/sec}$

The various terms of G(jω) are listed in table-1 in the increasing order of their corner frequencies. Also the table shows the slope contributed by each term and the change in slope at the corner frequency.

TABLE-1

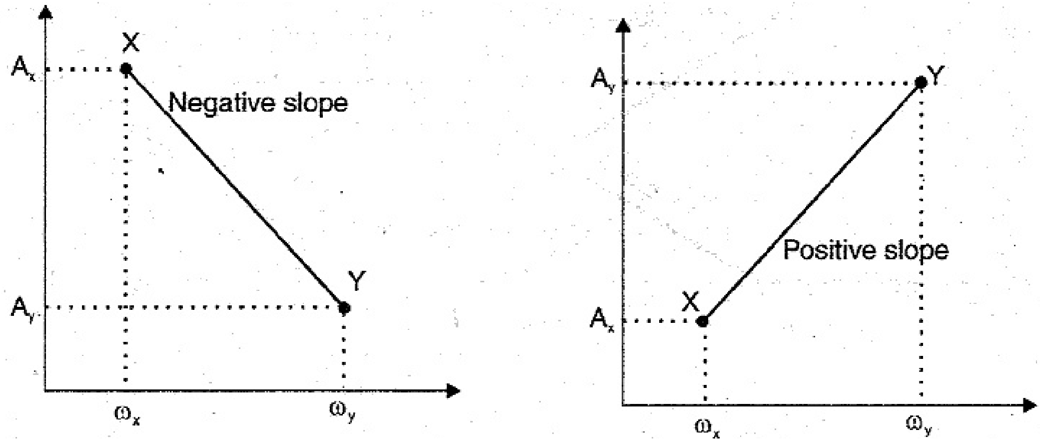
Term	Corner frequency rad/sec	Slope db/dec	Change in slope db/dec
$\frac{10}{j\omega}$	–	-20	
$\frac{1}{1+j0.4\omega}$	$\omega_{c1} = \frac{1}{0.4} = 2.5$	-20	$-20-20=-40$
$\frac{1}{1+j0.1\omega}$	$\omega_{c2} = \frac{1}{0.1} = 10$	-20	$-40-20=-60$

Step 3: Choose an arbitrary frequency ωL, which is lesser than the lowest corner frequency. Calculate the db magnitude of Integral Factor '10/jω' at ωL and at the lowest corner frequency ωc1

Step 4: Then calculate the gain (db magnitude) at every corner frequency one by one by using the below formula

$$\text{Gain at } \omega_y = \text{change in gain from } \omega_x \text{ to } \omega_y + \text{Gain at } \omega_x$$

$$= \left[\text{Slope from } \omega_x \text{ to } \omega_y \times \log\frac{\omega_y}{\omega_x}\right] + \text{Gain at } \omega_x$$



Step 5: Choose an arbitrary frequency ω_H, which is greater than the highest corner frequency. Calculate the gain at ω_H by using the formula in step 4

Choose a low frequency ω_L, such that $\omega_L < \omega_{c1}$ and choose a high frequency ω_H , such that $\omega_H > \omega_{c2}$

$\omega_L = 0.1$ rad/sec, and $\omega_H = 50$ rad/sec

$A = |G(j\omega)|$ in db

Let us calculate A at ω_L , ω_{c1}, ω_{c2} and ω_H

Choose a low frequency ω_L, such that $\omega_L < \omega_{C1}$ and choose a high frequency ω_H , such that $\omega_H > \omega_{c2}$

$\omega_L = 0.1$ rad/sec and $\omega_H = 50$ rad/sec

$A = |G(j\omega)|$ in db   Let us calculate A at ω_L , ω_{c1}, ω_{c2} and ω_H

$$\text{At } \omega = \omega_l,\quad A = 20\log\left|\frac{10}{j\omega}\right| = 20\log\frac{10}{0.1} = 40\text{ db}$$

$$\text{At } \omega = \omega_{c1},\quad A = 20\log\left|\frac{10}{j\omega}\right| = 20\log\frac{10}{2.5} = 12\text{ db}$$

$$\text{At } \omega = \omega_{c2},\quad A = \left[\text{Slope from } \omega_{c1} \text{ to } \omega_{c2} \times \log\frac{\omega_{c2}}{\omega_{c1}}\right] + A_{(\text{at } \omega = \omega_{c1})} = -40 \times \log\frac{10}{2.5} + 12 = -12\text{ db}$$

$$\text{At } \omega = \omega_h,\quad A = \left[\text{Slope from } \omega_{c2} \text{ to } \omega_h \times \log\frac{\omega_h}{\omega_{c2}}\right] + A_{(\text{at } \omega = \omega_{c2})} = -60 \times \log\frac{50}{10} + (-12) = -54\text{ db}$$

Let the points a, b, c and d be the points corresponding to frequencies ω_L , $\omega c1$, $\omega c2$ and ω_H, respectively on the magnitude plot

In a semilog graph sheet choose a scale of 1unit = 10db on y-axis

The frequencies are marked in decades from 0.1 to 100 rad/sec on logarithmic scales in x-axis.

Fix the points a, b, c and d on the graph. Join the points by a straight line and mark the slope in the respective region.

PHASE PLOT

$$\therefore G(j\omega) = \frac{10}{j\omega\,(1+j0.4\omega)\,(1+j0.1\omega)}$$

The phase angle of $G(j\omega)$ as a function of ω is given by $\phi = -90° - \tan^{-1} 0.4\omega - \tan^{-1} 0.1\omega$

The phase angle of $G(j\omega)$ are calculated for various values of ω and listed in table-2

Table-2

ω rad/sec	$\tan^{-1} 0.4\omega$ deg	$\tan^{-1} 0.1\omega$ deg	$\phi = \angle G(j\omega)$ deg	Points in phase plot
0.1	2.29	0.57	$-92.86 \approx -92$	e
1	21.80	5.71	$-117.5 \approx -118$	f
2.5	45.0	14.0	$-149 \approx -150$	g
4	57.99	21.8	$-169.79 \approx -170$	h
10	75.96	45.0	$-210.96 \approx -210$	i
20	82.87	63.43	$-236.3 \approx 236$	j

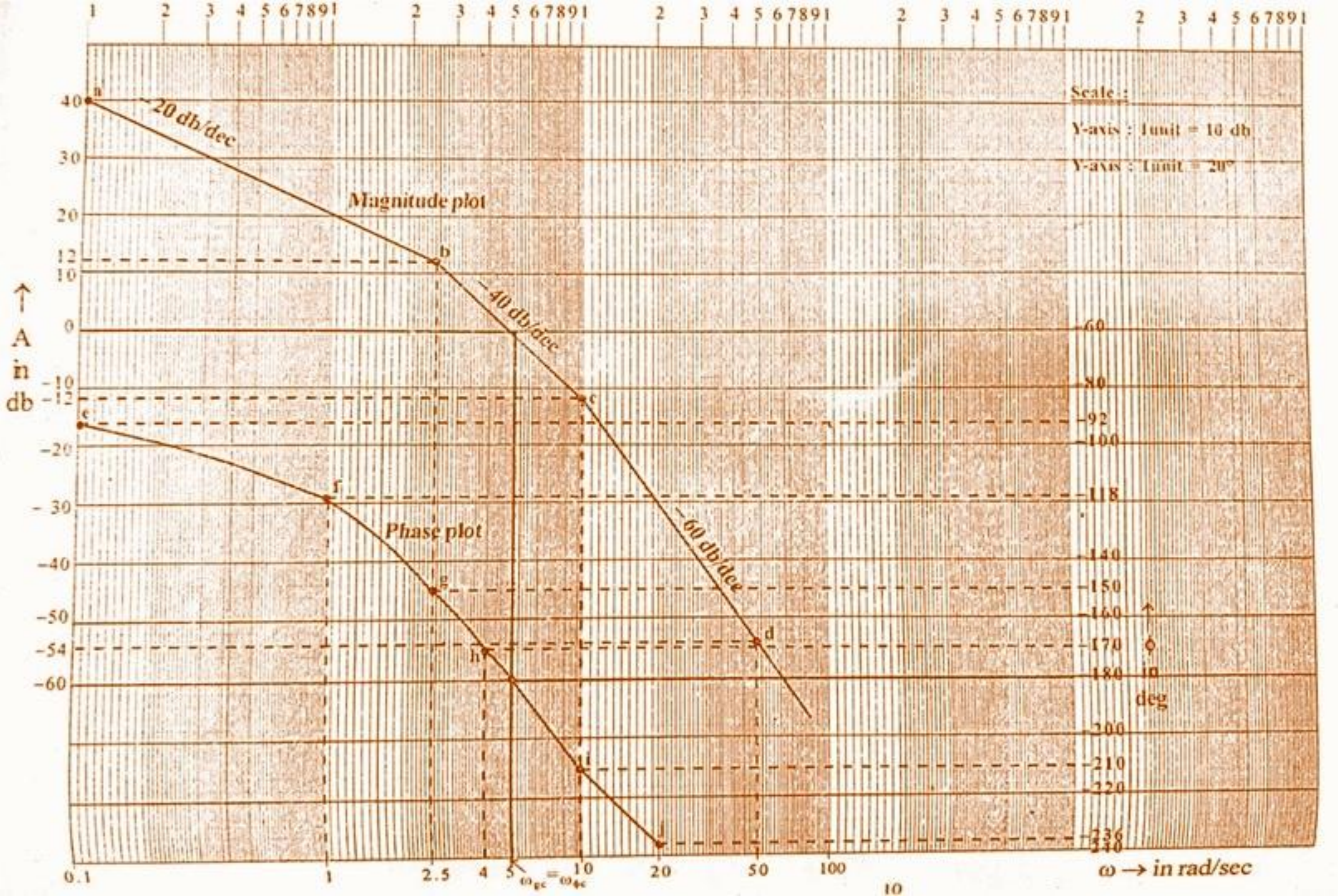
on the same semilog graph sheet choose a scale of 1unit = 20° on the y-axis on the right side of semilog graph sheet.

Mark the calculated phase angle on the graph sheet. Join the points by a smooth curve

From the graph, the gain and phase cross over frequencies are found to be 5 rad/sec

Gain cross-over frequency = 5 rad/sec.

Phase cross-over frequency = 5 rad/sec.


Scale :
Y-axis : 1unit = 10 db
Y-axis : 1unit = 20°
-20 db/dec
Magnitude plot
-40 db/dec
-60 db/dec
Phase plot
A in db
ϕ in deg
ω → in rad/sec
$\omega_{gc} = \omega_{\phi c}$
0.1
1
2.5
4
5
10
20
50
100
40
30
20
12
10
0
-10
-12
-20
-30
-40
-50
-54
-60
-60
-80
-92
-100
-118
-140
-150
-160
-170
-180
-200
-210
-220
-236
-240
a
b
c
d
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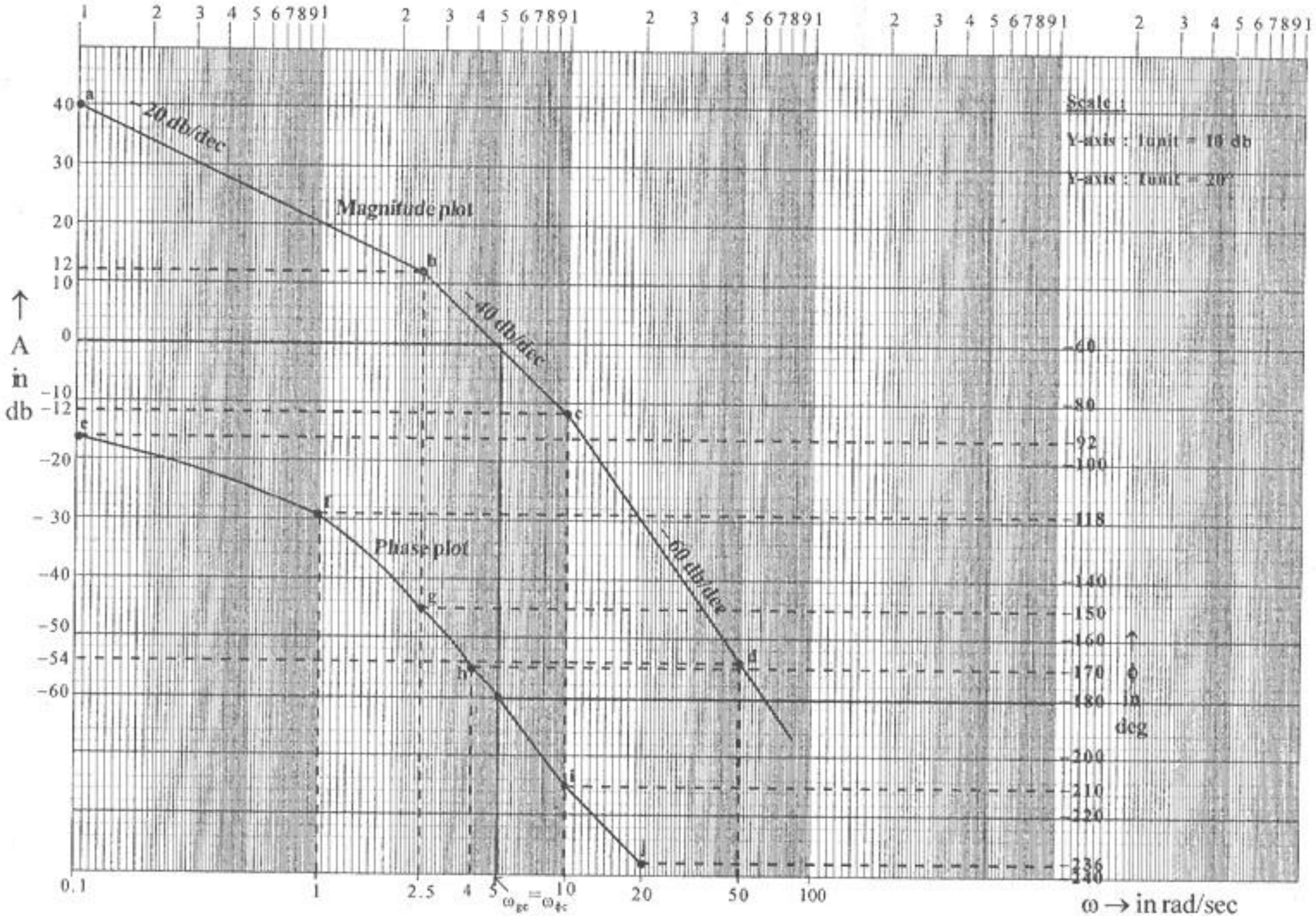
f
g
h
i
j


Scale :
Magnitude plot
Phase plot
-20 db/dec
-40 db/dec
-60 db/dec
A in db
φ in deg
ω → in rad/sec
$\omega_{gc} = \omega_{pc}$
a
b
c
d
e
f
g
h
i
j
40
30
20
12
10
0
-10
-12
-20
-30
-40
-50
-54
-60
-60
-80
-92
-100
-118
-140
-150
-160
-170
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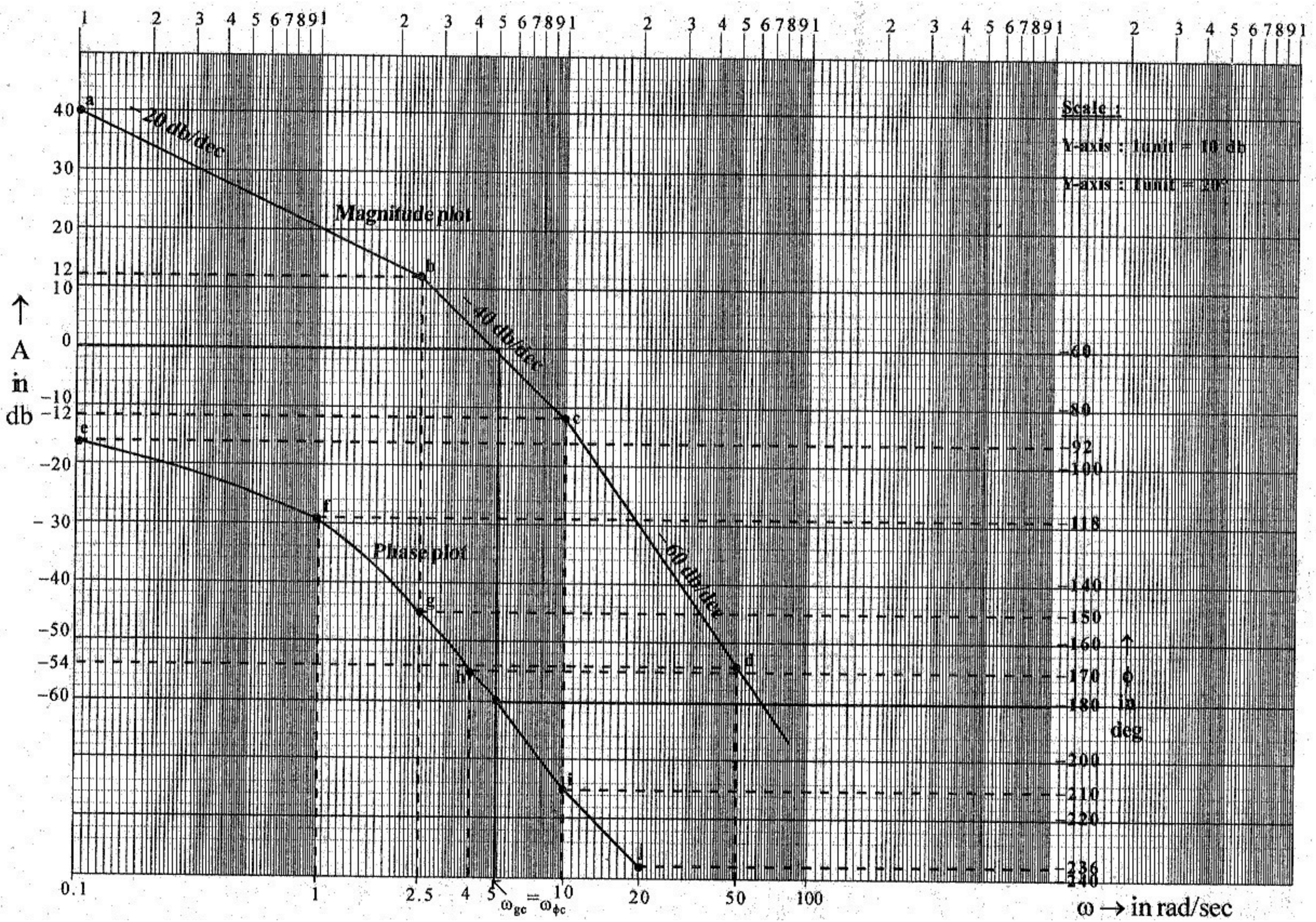
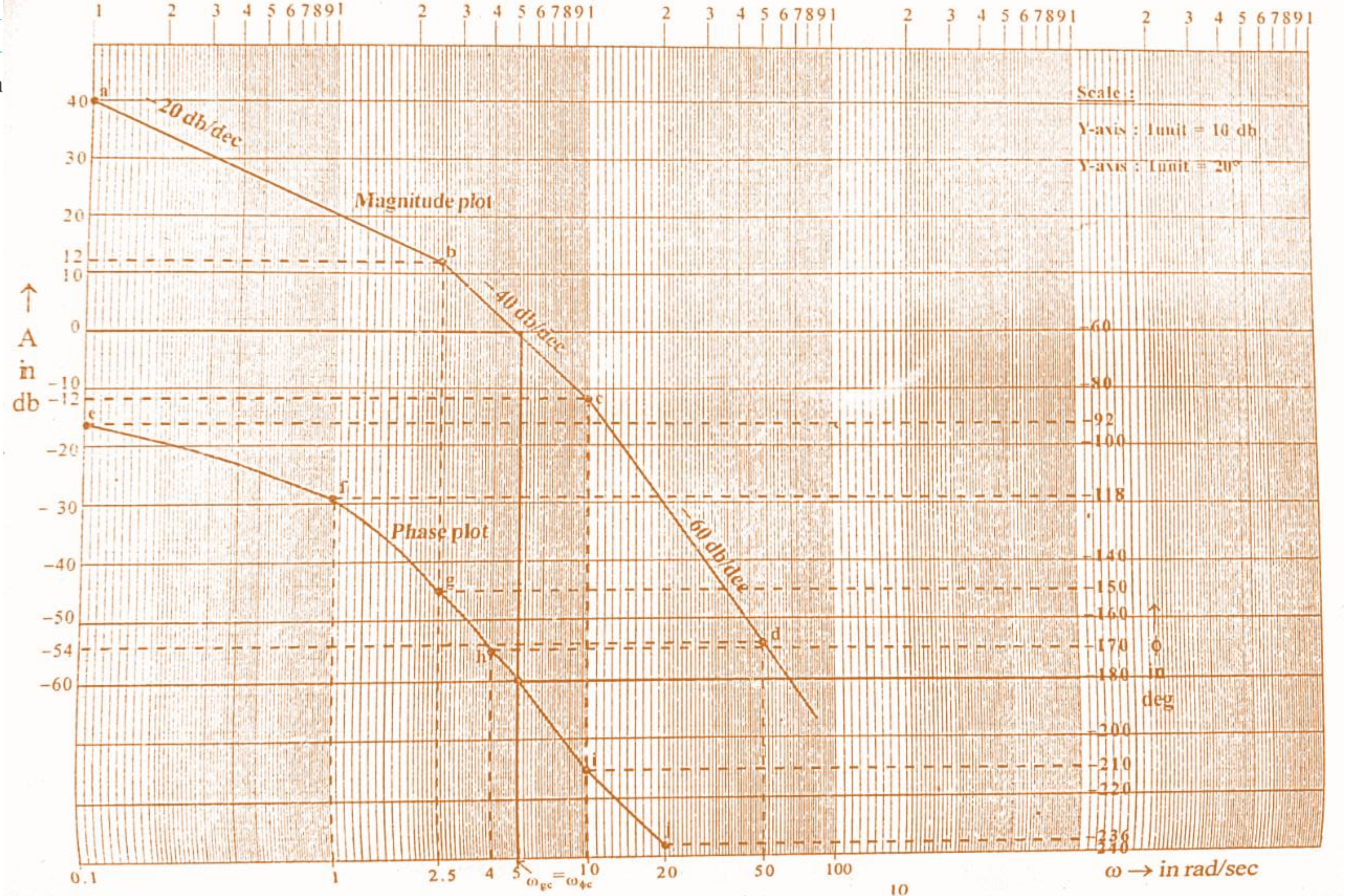
5
10
20
50
100


-20 db/dec
Magnitude plot
-40 db/dec
-60 db/dec
Phase plot
A in db
φ in deg
ω → in rad/sec
$\omega_{gc} = \omega_{\phi c}$
Scale :
a
b
c
d
e
f
g
h
i
j
0.1
1
2.5
4
5
10
20
50
100
40
30
20
12
10
0
-10
-12
-20
- 30
-40
-50
-54
-60
-60
-80
-92
-100
-118
-140
-150
-160
-170
-180
-200
-210
-220
-236
-240



Scale :
Y-axis : 1unit = 10 db
Y-axis : 1unit = 20°
-20 db/dec
Magnitude plot
-40 db/dec
-60 db/dec
Phase plot
A in db
φ in deg
ω → in rad/sec
ωgc = ωφc

Gain Margin and Phase Margin

Gain Margin:

GM = - 20 Log |G(jω)H(jω)| at $\omega = \omega_{pc}$

Phase Margin:

PM = Angle (G(jω)H(jω)) at $\omega = \omega_{gc}$

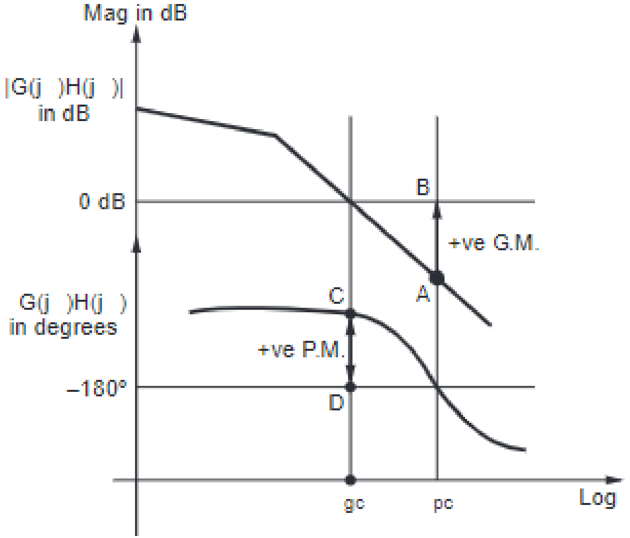


Fig. 11.7.1 $\omega_{gc} < \omega_{pc}$ G.M. and P.M. positive, stable system

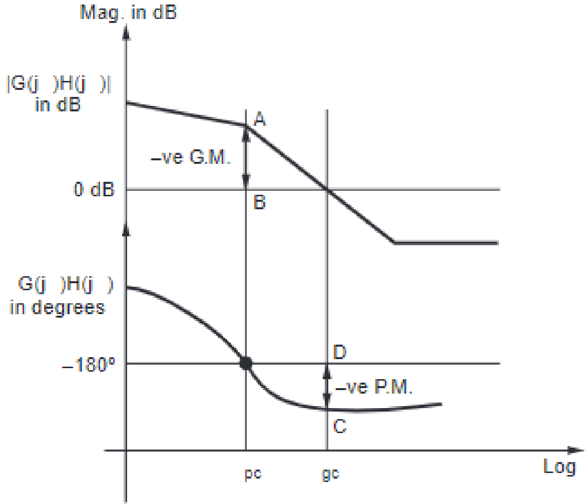


Fig. 11.7.2 $\omega_{gc} > \omega_{pc}$ G.M. and P.M. negative, unstable system

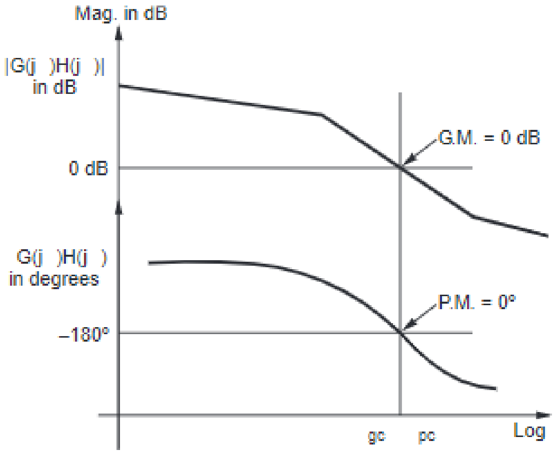


Fig. 11.7.3 $\omega_{gc} = \omega_{pc}$ G.M. and P.M. zero, marginally stable system

Assignment:

A unity feedback control system has $G(s) = \frac{80}{s(s+2)(s+20)}$; Draw the Bode plot.

Solution:

Step 1: Arrange G(s)H(s) in time constant.

$$G(s) = \frac{80}{s * 2\left(\frac{s}{2} + 1\right) 20\left(\frac{s}{20} + 1\right)} = \frac{80}{s * 40\left(\frac{s}{2} + 1\right)\left(\frac{s}{20} + 1\right)} = \frac{2}{s * \left(\frac{s}{2} + 1\right)\left(\frac{s}{20} + 1\right)}$$

Plot the Bode diagram for the following transfer and obtain the gain and phase cross over frequencies.

$$G(s) = \frac{20}{s\,(1+3s)\,(1+4s)}$$

The sinusoidal transfer function of G(jω) is obtained by replacing s by jω in the given transfer function.

$$G(j\omega) = \frac{20}{j\omega\,(1+j3\omega)\,(1+j4\omega)}$$

MAGNITUDE PLOT

The corner frequencies are, $\omega_{c1} = \frac{1}{4} = 0.25 \text{ rad / sec},$ & $\omega_{c2} = \frac{1}{3} = 0.333 \text{ rad / sec.}$

The various terms of G(jω) are listed in table-1 in the increasing order of their corner frequencies. Also the table shows the slope contributed by each term and the change in slope at the corner frequency.

$$G(j\omega) = \frac{20}{j\omega\,(1+j3\omega)\,(1+j4\omega)}$$

MAGNITUDE PLOT

The corner frequencies are, $\omega_{c1} = \frac{1}{4} = 0.25\ \text{rad/sec},$ $\omega_{c2} = \frac{1}{3} = 0.333\ \text{rad/sec}.$

The various terms of G(jω) are listed in table-1 in the increasing order of their corner frequencies. Also the table shows the slope contributed by each term and the change in slope at the corner frequency.

TABLE-1

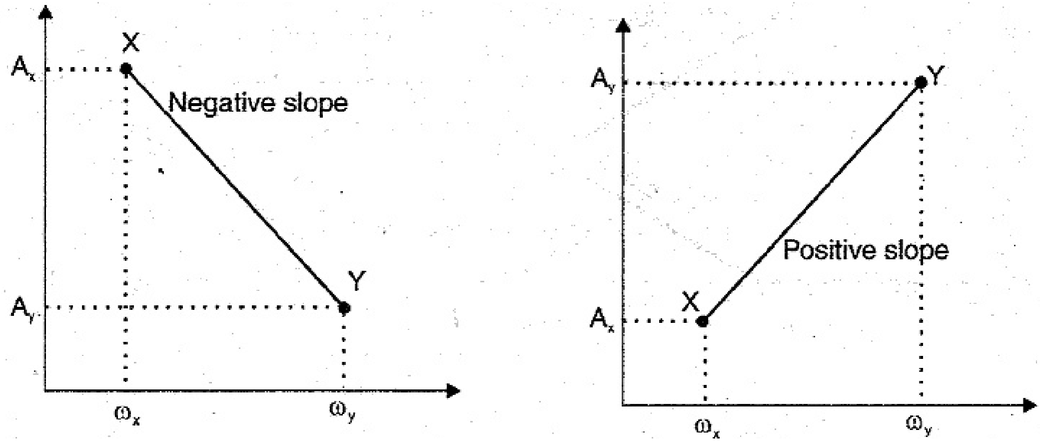
Term	Corner frequency rad/sec	Slope db/dec	Change in slope db/dec
$\frac{20}{j\omega}$	–	–20	
$\frac{1}{1+j4\omega}$	$\omega_{c1} = \frac{1}{4} = 0.25$	–20	–20 –20 = –40
$\frac{1}{1+j3\omega}$	$\omega_{c2} = \frac{1}{3} = 0.33$	–20	–40–20 = –60

Step 3: Choose an arbitrary frequency ω_L, which is lesser than the lowest corner frequency. Calculate the db magnitude of Integral Factor '20/jω' at ω_L and at the lowest corner frequency ω_{c1}

Step 4: Then calculate the gain (db magnitude) at every corner frequency one by one by using the below formula

$$\text{Gain at } \omega_y = \text{change in gain from } \omega_x \text{ to } \omega_y + \text{Gain at } \omega_x$$

$$= \left[\text{Slope from } \omega_x \text{ to } \omega_y \times \log\frac{\omega_y}{\omega_x}\right] + \text{Gain at } \omega_x$$



Step 5: Choose an arbitrary frequency ω_H, which is greater than the highest corner frequency. Calculate the gain ω_H by using the formula in step 4

Choose a low frequency ω_L, such that $\omega_L < \omega_{c1}$ and choose a high frequency ω_H , such that $\omega_H > \omega_{c2}$

$\omega_L = 0.15$ rad/sec, and $\omega_H = 1$rad/sec

$A = |G(j\omega)|$ in db

Let us calculate A at ω_L , ω_{c1}, ω_{c2} and ω_H

Choose a low frequency ω_L, such that $\omega_L < \omega_H$ and choose a high frequency ω_H , such that $\omega_H > \omega_{c2}$

$\omega_L = 0.15$ rad/sec and $\omega_H = 1$rad/sec

$A = |G(j\omega)|$ in db   Let us calculate A at ω_L , ω_{c1}, ω_{c2} and ω_H

$$\text{At } \omega = \omega_l,\ A = |G(j\omega)| = 20 \log \left|\frac{20}{0.15}\right| = 42.5 \text{ db}$$

$$\text{At } \omega = \omega_{c1},\ A = |G(j\omega)| = 20 \log \left|\frac{20}{0.25}\right| = 38 \text{ db}$$

$$\text{At } \omega = \omega_{c2},\ A = \left[\text{Slope from } \omega_{c1} \text{ to } \omega_{c2} \times \log\frac{\omega_{c2}}{\omega_{c1}}\right] + A_{(\text{at } \omega = \omega_{c1})}$$

$$= -40 \times \log \frac{0.33}{0.25} + 38 = 33 \text{ db}$$

$$\text{At } \omega = \omega_h,\ A = \left[\text{Slope from } \omega_{c2} \text{ to } \omega_h \times \log\frac{\omega_h}{\omega_{c2}}\right] + A_{(\text{at } \omega = \omega_{c2})}$$

$$= -60 \times \log \frac{1}{0.33} + 33 = 4 \text{ db}$$

Let the points a, b, c and d be the points corresponding to frequencies ω_L , $\omega c1$, $\omega c2$ and ω_H, respectively on the magnitude plot

In a semilog graph sheet choose a scale of 1unit = 10db on y-axis

The frequencies are marked in decades from 0.01 to 10 rad/sec on logarithmic scales in x-axis.

Fix the points a, b, c and d on the graph. Join the points by a straight line and mark the slope in the respective region.

PHASE PLOT

$$G(j\omega) = \frac{20}{j\omega\,(1+j3\omega)\,(1+j4\omega)}$$

The phase angle of G(jω) as a function of ω is given by $\phi = -90^0 - \tan^{-1} 3\omega - \tan^{-1} 4\omega$

The phase angle of G(jω) are calculated for various values of ω and listed in table-2

Table-2

ω, rad/sec	$\tan^{-1} 3\omega$, deg	$\tan^{-1} 4\omega$, deg	$\phi = \angle G(j\omega)$, deg	Points in phase plot
0.15	24.22	30.96	$-145.18 \approx -146$	e
0.2	30.96	38.66	$-159.61 \approx -160$	f
0.25	36.86	45.0	$-171.86 \approx -172$	g
0.33	44.7	52.8	$-187.5 \approx -188$	h
0.6	60.14	67.38	$-218.32 \approx -218$	i
1	71.56	75.96	$-237.56 \approx -238$	j

on the same semilog graph sheet choose a scale of 1unit = 20° on the y-axis on the right side of semilog graph sheet.

Mark the calculated phase angle on the graph sheet. Join the points by a smooth curve

From the graph,

Gain cross-over frequency = 1.1 rad/sec.

Phase cross-over frequency = rad/sec.

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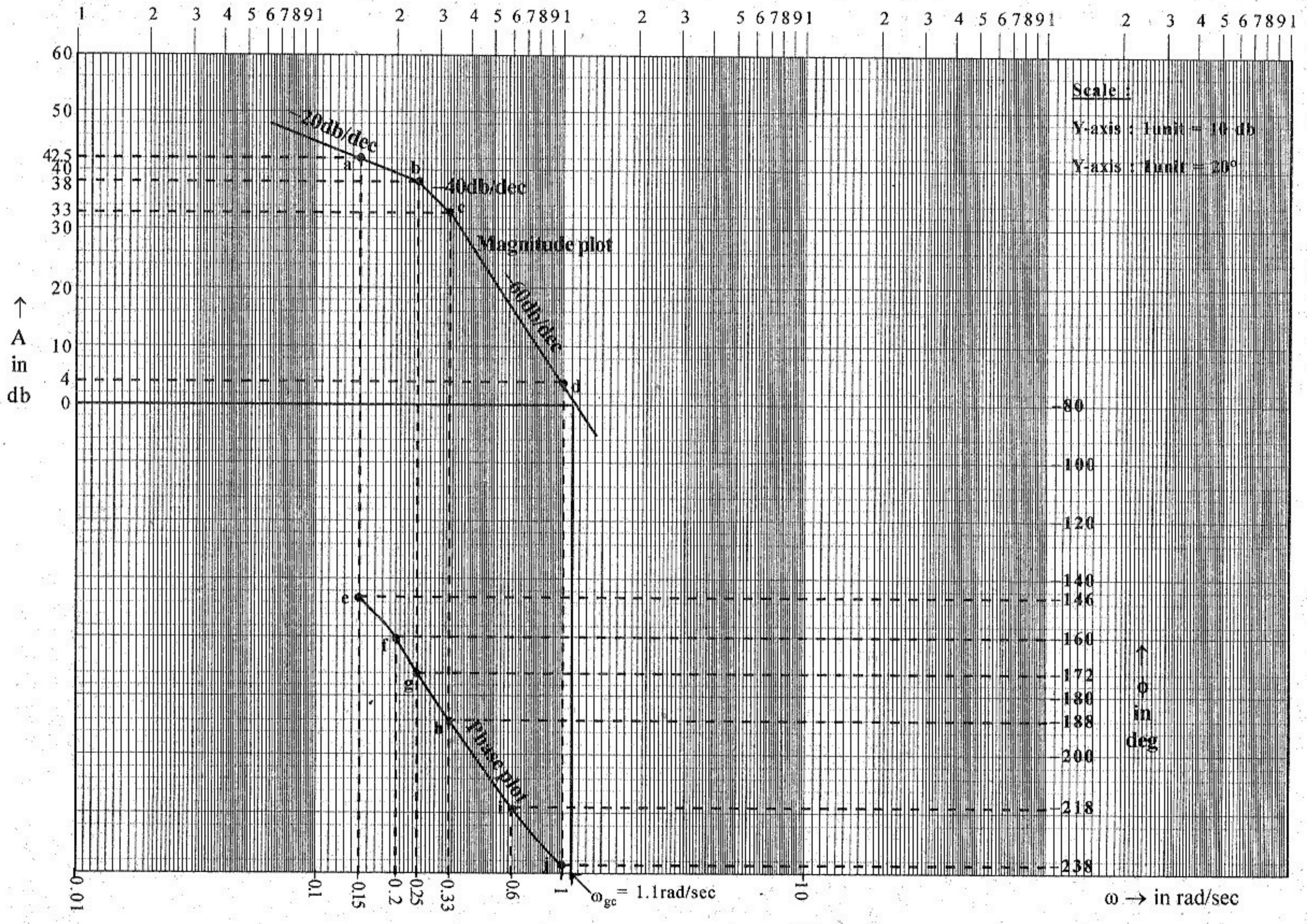
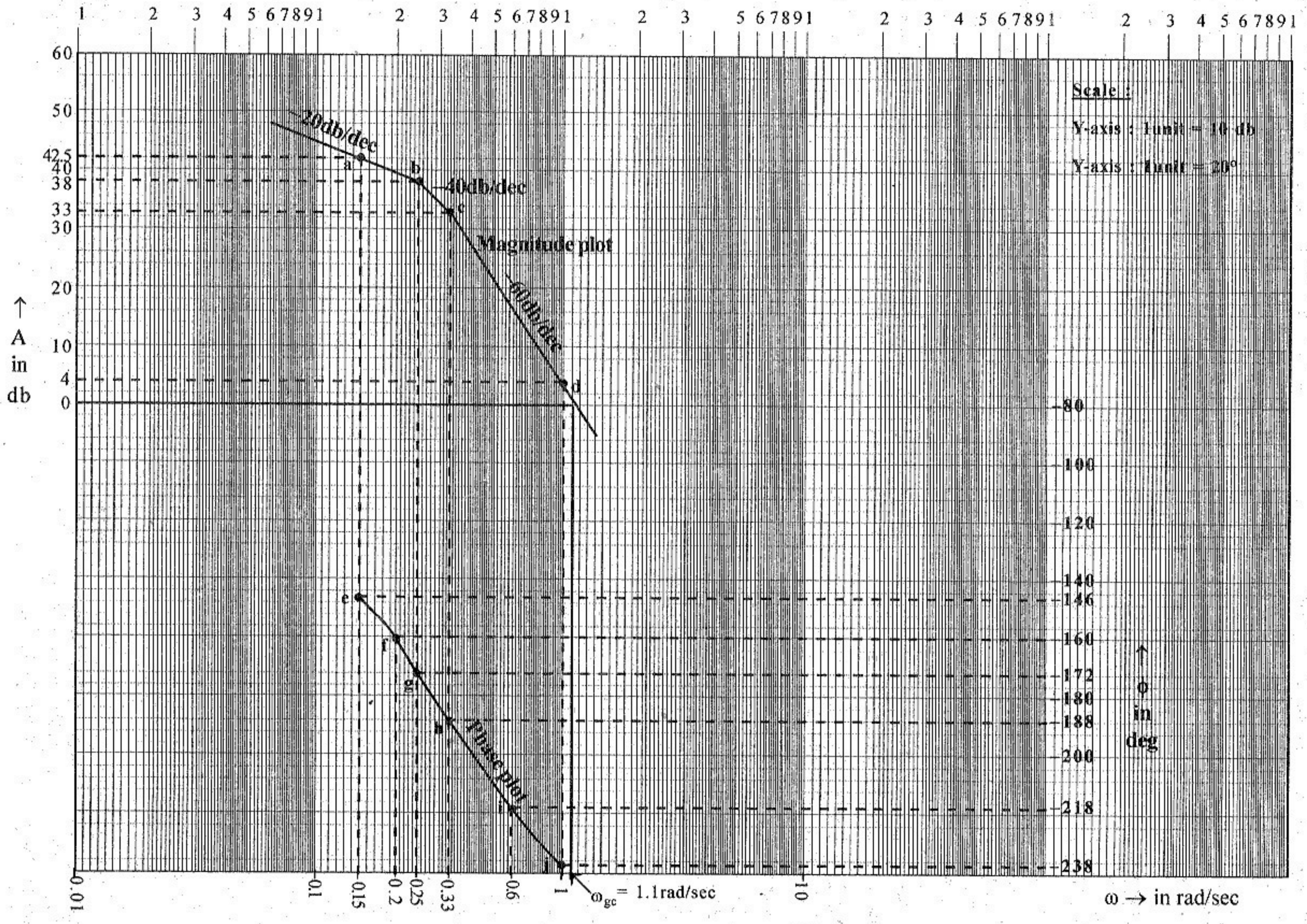


A T M E College of Engineering



Sketch the bode plot for the following transfer function and determine phase margin and gain

$$G(s) = \frac{75\,(1+0.2s)}{s\left(s^2+16s+100\right)}$$

On comparing the quadratic factor in the denominator of G(s) with standard form of quadratic factor we can estimate ζ and ωn

$$\therefore s^2+16s+100 = s^2+2\zeta\omega_n s+\omega_n^2$$

On comparing we get,

$$\omega_n^2 = 100 \quad \Rightarrow \quad \omega_n = 10$$

$$2\zeta\omega_n = 16 \quad \Rightarrow \quad \zeta = \frac{16}{2\omega_n} = \frac{16}{2\times 10} = 0.8$$

$$G(s) = \frac{75\,(1+0.2s)}{s\,(s^2+16s+100)}$$

Let us convert the given s-domain transfer function into bode form or time constant form

$$\therefore G(s) = \frac{75(1+0.2s)}{s\,(s^2+16s+100)} = \frac{75\,(1+0.2s)}{s\times 100\left(\frac{s^2}{100}+\frac{16s}{100}+1\right)} = \frac{0.75\,(1+0.2s)}{s\,(1+0.01s^2+0.16s)}$$

The sinusoidal transfer function G(jω) is obtained by replacing s by jω in G(s)

$$\therefore G(j\omega) = \frac{0.75\,(1+0.2\,j\omega)}{j\omega\,\left(1+0.01(j\omega)^2+0.16j\omega\right)} = \frac{0.75\,(1+j0.2\omega)}{j\omega\,\left(1-0.01\omega^2+j0.16\omega\right)}$$

MAGNITUDE PLOT

The corner frequencies are, $\omega_{c1} = \frac{1}{0.2} = 5 \text{ rad/sec}$ & $\omega_{c2} = \omega_n = 10 \text{ rad/sec}$

Note : For the quadratic factor the corner frequency is ω_n

The various terms of G(jω) are listed in table-1 in the increasing order of their corner frequencies. Also the table shows the slope contributed by each term and the change in slope at the corner frequency.

Term	Corner frequency rad/sec	Slope db/dec	Change in slope db/dec
$\frac{0.75}{j\omega}$	–	–20	
$1+j0.2\omega$	$\omega_{c1} = \frac{1}{0.2} = 5$	20	$-20+20=0$
$\frac{1}{1-0.01\omega^2+j0.16\omega}$	$\omega_{c2} = \omega_n = 10$	–40	$0-40=-40$

Choose an arbitrary frequency ω_L, which is lesser than the lowest corner frequency. Calculate the db magnitude of Integral Factor '75/jω' at ω_L and at the lowest corner frequency ω_{c1}

Choose an arbitrary frequency ω_H, which is greater than the highest corner frequency. Calculate the gain ω_H by using the formula in step 4

Choose a low frequency ω_L, such that $\omega_L < \omega_{c1}$ and choose a high frequency ω_H , such that $\omega_H > \omega_{c2}$

$\omega_L = 0.5$ rad/sec, and $\omega_H = 20$rad/sec

$A = |G(j\omega)|$ in db

Let us calculate A at ω_L , ω_{c1}, ω_{c2} and ω_H

$$\text{At,} \quad \omega = \omega_1, \quad A = 20\log\left|\frac{0.75}{j\omega}\right| = 20\log\frac{0.75}{0.5} = 3.5\ \text{db}$$

$$\text{At,} \quad \omega = \omega_{c1}, \quad A = 20\log\left|\frac{0.75}{j\omega}\right| = 20\log\frac{0.75}{5} = -16.5\ \text{db}$$

$$\text{At,} \quad \omega = \omega_{c2}, \quad A = \left[\text{slope from } \omega_{c1} \text{ to } \omega_{c2} \times \log\frac{\omega_{c2}}{\omega_{c1}}\right] + A_{(\text{at } \omega = \omega_{c1})}$$

$$= 0 \times \log\frac{10}{5} + (-16.5) = -16.5\ \text{db}$$

$$\text{At} \quad \omega = \omega_h, \quad A = \left[\text{slope from } \omega_{c2} \text{ to } \omega_h \times \log\frac{\omega_h}{\omega_{c2}}\right] + A_{(\text{at } \omega = \omega_{c2})}$$

$$= -40 \times \log\frac{20}{10} + (-16.5) = -28.5\ \text{db}$$

Let the points a, b, c and d be the points corresponding to frequencies ω_L, ω_{c1}, ω_{c2} and ω_H, respectively on the magnitude plot

In a semilog graph sheet choose a scale of 1unit = 5db on y-axis

The frequencies are marked in decades from 0.1 to 100 rad/sec on logarithmic scales in x-axis.

Fix the points a, b, c and d on the graph. Join the points by a straight line and mark the slope in the respective region.

PHASE PLOT

$$G(s) = \frac{75\,(1+0.2s)}{s\left(s^2+16s+100\right)}$$

The phase angle of G(jω) as a function of ω is given by $\phi = \angle G(j\omega) = \tan^{-1}0.2\omega - 90^\circ - \tan^{-1}\frac{0.16\omega}{1-0.01\omega^2}$ for $\omega \le \omega_n$

Note: In quadratic factors the phase varies from 0° to 180°. But calculator calculates tan-¹ only between 0° to 90°. Hence a correction of 180°should be added to phase after ωn

$$\phi = \angle G(j\omega) = \tan^{-1}0.2\omega - 90^\circ - \left(\tan^{-1}\frac{0.16\omega}{1-0.01\omega^2} + 180^\circ\right) \text{ for } \omega > \omega_n$$

The phase angle of G(jω) are calculated for various values of ω and listed in table-2

Table-2

ω rad/sec	$\tan^{-1}0.2\omega$ deg	$\tan^{-1}\frac{0.16\omega}{1-0.01\omega^2}$ deg	$\phi = \angle G(j\omega)$ deg	Points in phase plot
0.5	5.7	4.6	$-88.9 \approx -88$	e
1	11.3	9.2	$-87.9 \approx -88$	f
5	45	46.8	$-91.8 \approx -92$	g
10	63.4	90	$-116.6 \approx -116$	h
20	75.9	−46.8+180 =133.2	$-147.3 \approx -148$	i
50	84.3	−18.4+180 =161.6	$-167.3 \approx -168$	j
100	87.1	−92+180 =170.8	$-173.7 \approx -174$	k

The phase angle of G(jω) are calculated for various values of ω and listed in table-2

$$\phi = \angle G(j\omega) = \tan^{-1}0.2\omega - 90^\circ - \tan^{-1}\frac{0.16\omega}{1-0.01\omega^2} \quad \text{for } \omega \le \omega_n$$

$$\phi = \angle G(j\omega) = \tan^{-1}0.2\omega - 90^\circ - \left(\tan^{-1}\frac{0.16\omega}{1-0.01\omega^2} + 180^\circ\right) \quad \text{for } \omega > \omega_n$$

Table-2

ω rad/sec	$\tan^{-1}0.2\omega$ deg	$\tan^{-1}\frac{0.16\omega}{1-0.01\omega^2}$ deg	$\phi = \angle G(j\omega)$ deg	Points in phase plot
0.5	5.7	4.6	$-88.9 \approx -88$	e
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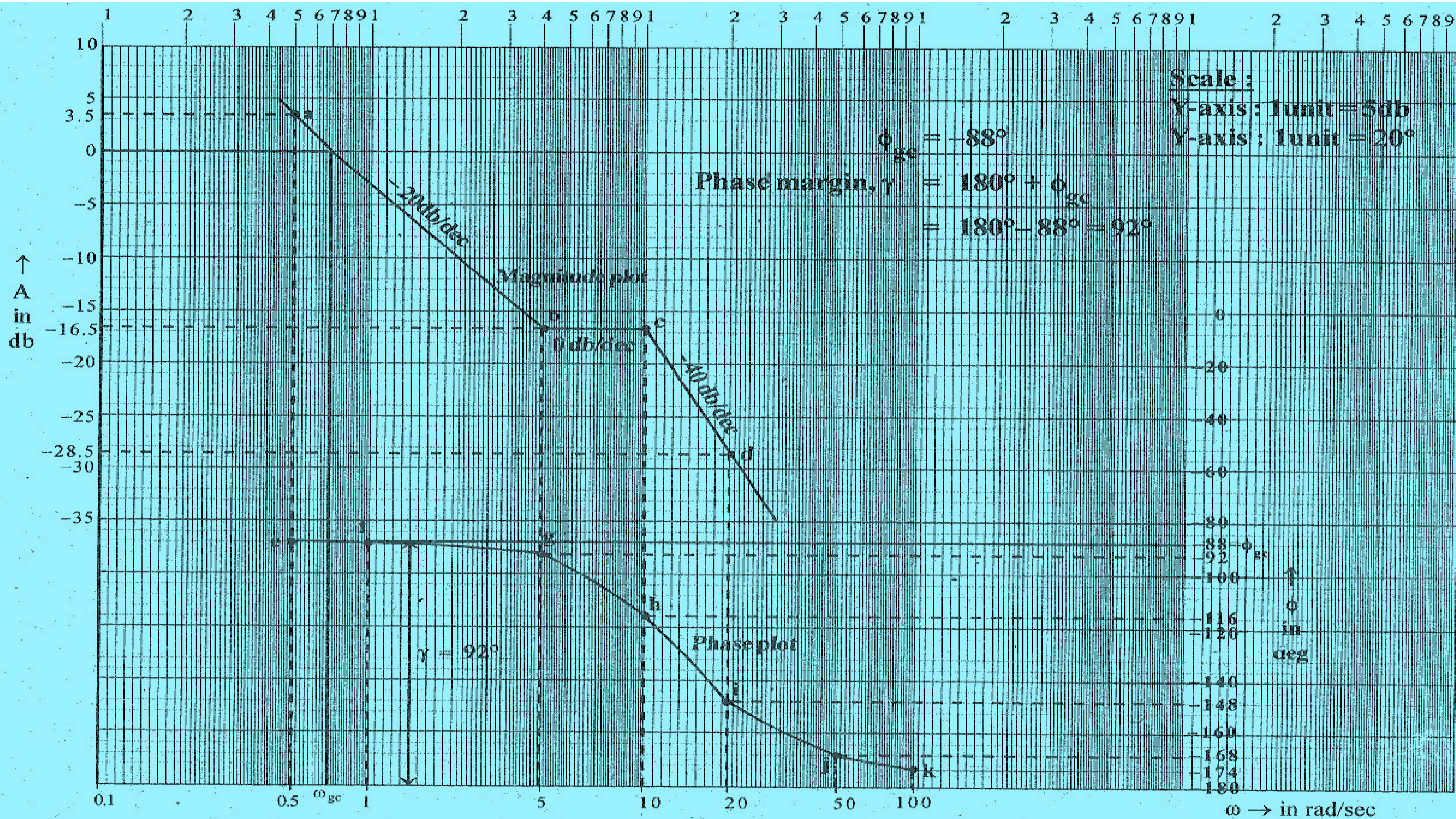
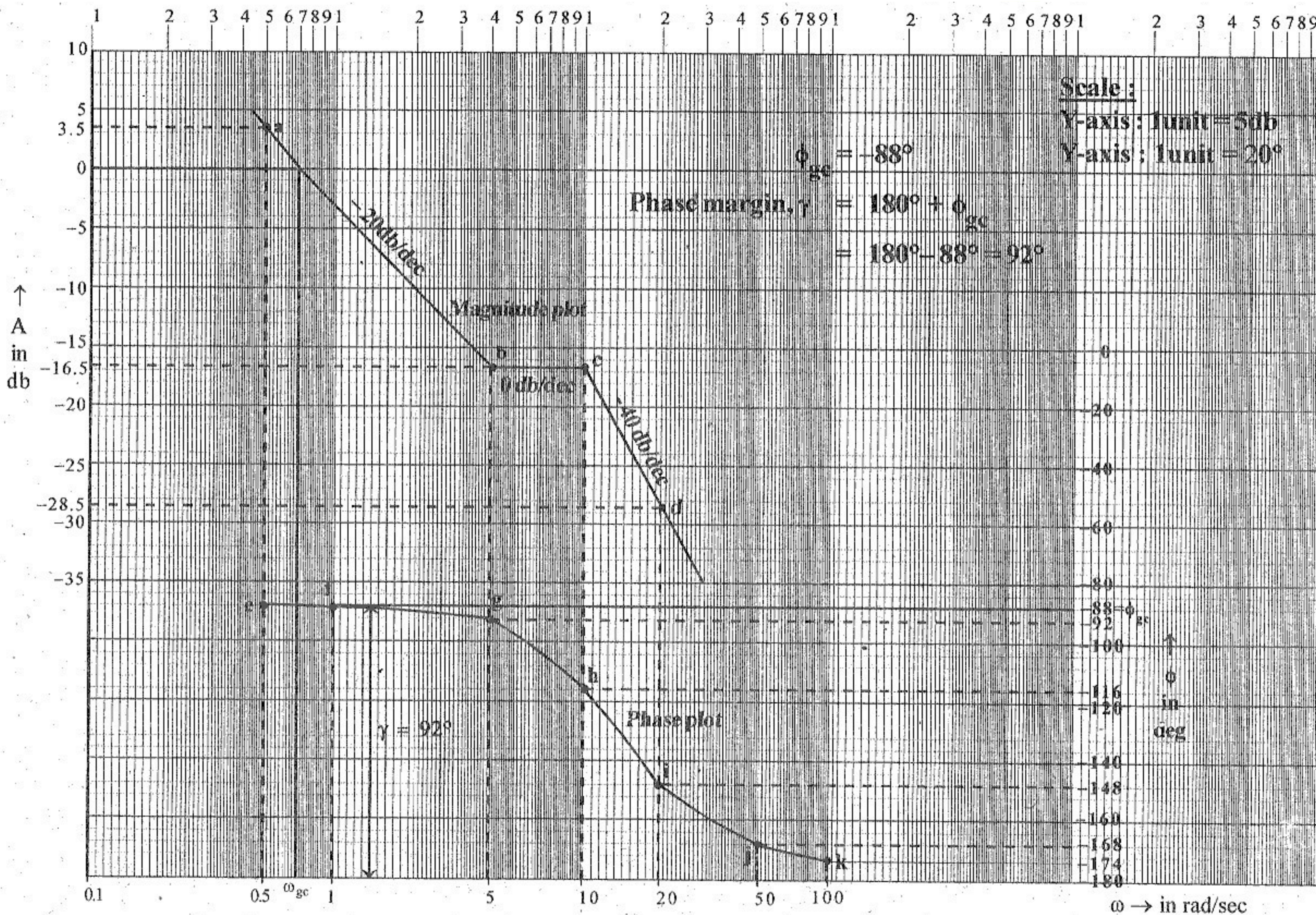
20	75.9	$-46.8+180=133.2$	$-147.3 \approx -148$	i
50	84.3	$-18.4+180=161.6$	$-167.3 \approx -168$	j
100	87.1	$-92+180=170.8$	$-173.7 \approx -174$	k



Scale :
Y-axis : 1unit = 5db
Y-axis : 1unit = 20°
$\phi_{gc} = -88°$
Phase margin, $\gamma = 180° + \phi_{gc}$
$= 180° - 88° = 92°$
-20db/dec
Magnitude plot
0 db/dec
-40 db/dec
Phase plot
$\gamma = 92°$
A in db
ϕ in deg
$\omega \rightarrow$ in rad/sec
ω_{gc}
10
5
3.5
0
-5
-10
-15
-16.5
-20
-25
-28.5
-30
-35
0
-20
-40
-60
-80
$-88 = \phi_{gc}$
-92
-100
-116
-120
-140
-148
-160
-168
-174
-180
0.1
0.5
1
5
10
20
50
100


Scale :
Y-axis : 1unit = 5db
Y-axis : 1unit = 20°
$\phi_{gc} = -88°$
Phase margin, $\gamma = 180° + \phi_{gc}$
$= 180° - 88° = 92°$
-20db/dec
Magnitude plot
0 db/dec
-40 db/dec
Phase plot
$\gamma = 92°$
A in db
ϕ in deg
$\omega \rightarrow$ in rad/sec
ω_{gc}
$-88 = \phi_{gc}$

on the same semilog graph sheet choose a scale of 1unit = 20° on the y-axis on the right side of semilog graph sheet.

Mark the calculated phase angle on the graph sheet. Join the points by a smooth curve

Let **Φ_{gc}**, be the phase of $G(j\omega)$ at gain cross-over frequency, ω_{gc}.

From the fig, we get, **Φ_{gc}**= -88°

:: Phase margin, g=180° + **Φ_{gc}** = 180° – 88° = 92°

The phase plot crosses – 180° only at infinity. The G|(jω)| at infinity is -∞db.

Hence gain margin is +∞.

Program-Bode Plot

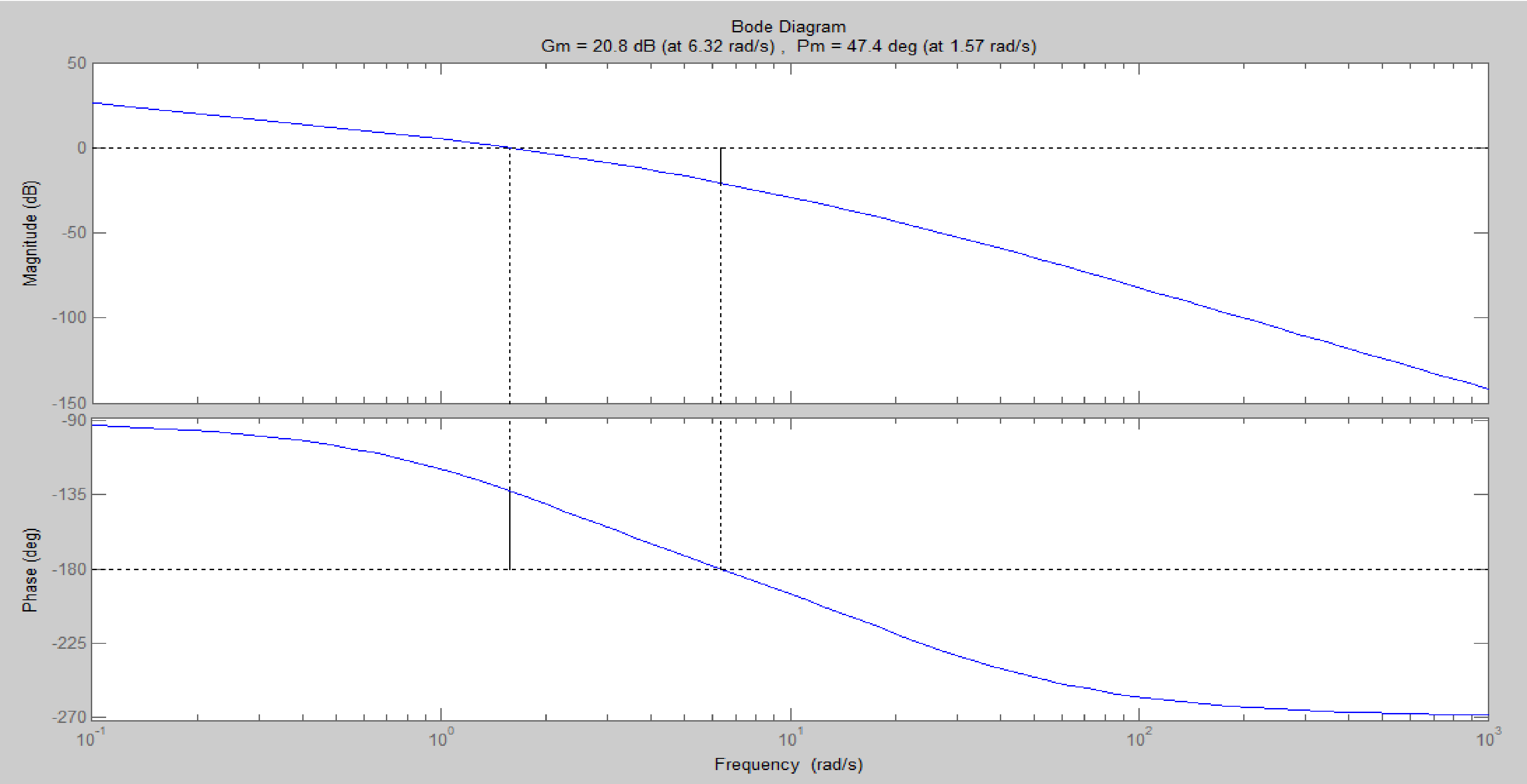
Program:

$$G(s) = \frac{80}{s(s+2)(s+20)} = \frac{80}{s^3 + 22s^2 + 40s}$$

```
num = [80];
den =[1 22 40 0];
G = tf(num,den);
margin(G)
%grid
```

clc #clears all the text from the Command Window, resulting in a clear screen
num = [80]; #Coefficients of the numerator
den =[1 22 40 0]; #Coefficients of the denominator
G=tf(num, den);#creates a continuous-time transfer function with numerator(s) and denominator(s) specified by num and den
margin(sys)# margin(sys) plots the Bode response of sys on the screen and indicates the gain and phase margins on the plot. Gain margins are expressed in dB on the plot
[Gm,Pm,Wcg,Wcp] = margin(sys) #Compute the gain margin, phase margin and frequencies.

Bode Plot



Bode Diagram
Gm = 20.8 dB (at 6.32 rad/s) , Pm = 47.4 deg (at 1.57 rad/s)
Magnitude (dB)
50
0
-50
-100
-150
Phase (deg)
-90
-135
-180
-225
-270
10^{-1}
10^{0}
10^{1}
10^{2}
10^{3}
Frequency (rad/s)

From the plot note down the gain margin (GM), phase margin (PM) and the corresponding cross over frequencies.

For unstable systems, GM and PM will not be displayed correctly but can be obtained by clicking on the plot at suitable points

Click on the phase angle curve and find the frequency at which the curve crosses the 180^0 line. This gives the phase cross over frequency ω_p

Click on the magnitude curve and find the magnitude at $\omega=\omega_p$. The gain margin is calculated as, GM = 0 - magnitude at phase cross over frequency

Click on the magnitude curve and find the frequency at which the curve crosses 0dB line. This will give the gain cross over frequency ω_g.

Click on the phase angle curve and find the phase angle at $\omega=\omega_g$. The phase margin is calculated as PM = phase angle at the gain cross over frequency + 180° at $\omega=\omega_g$.

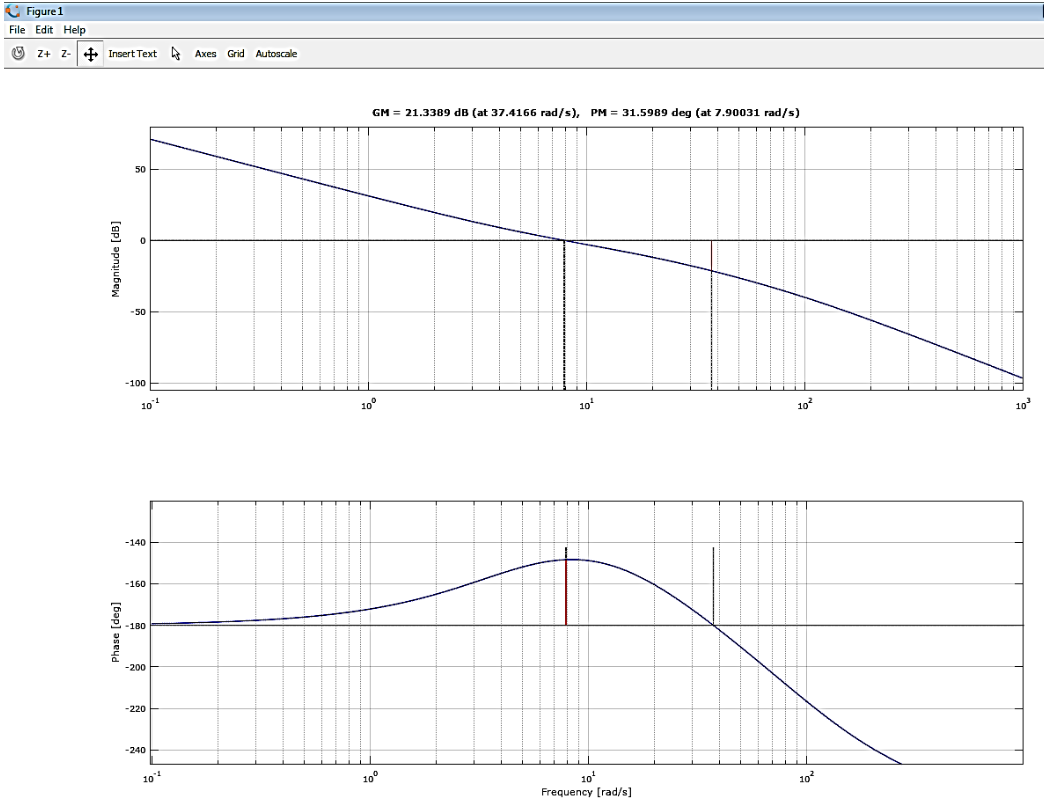
The GM, PM and the cross over frequencies can also be obtained using the following function. [Gm,Pm,Wcg,Wcp] = margin(sys)

Where GM = gain margin in abs unit (20logGM is the GM in db)

Outcome

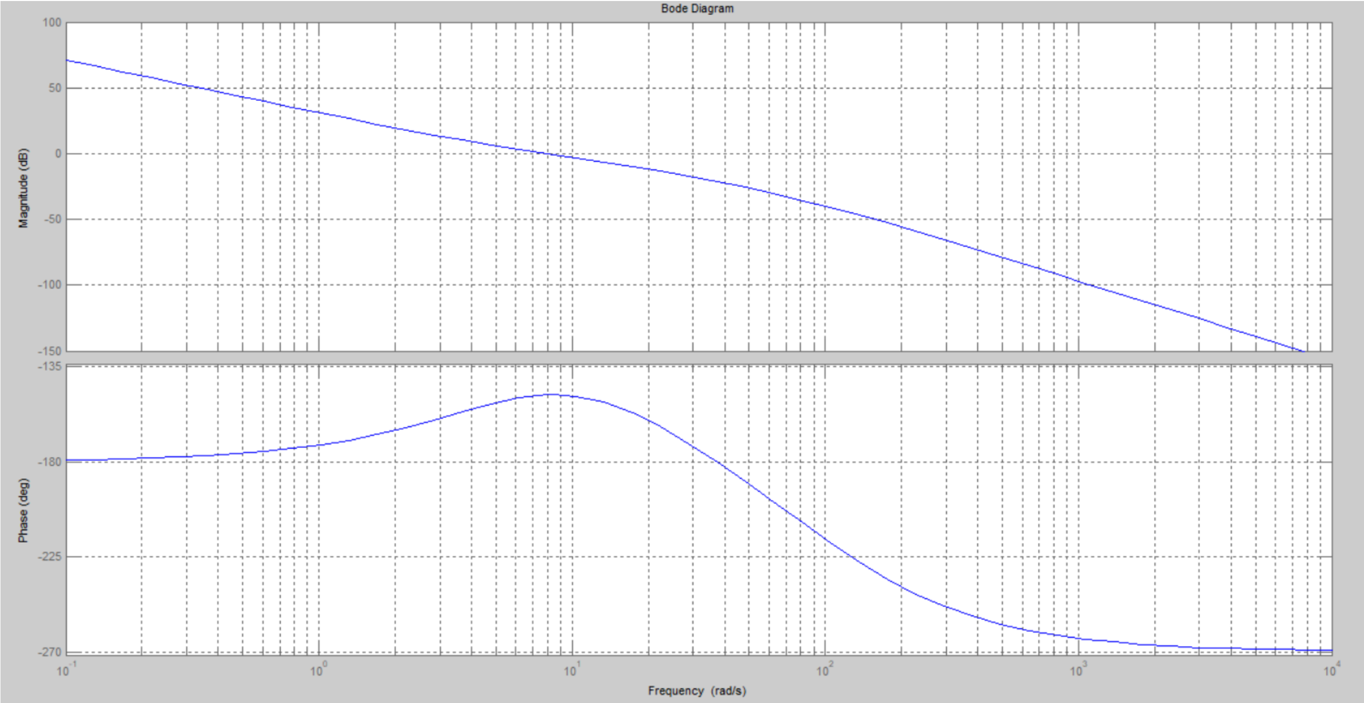
- **The Gain Margin = 20.8 dB at phase cross over frequency of 6.32 rad/sec.**
- **The phase Margin = 47.4^{o} at gain cross over frequency of 1.57 rad/sec.**
- **For the given system both GM and PM are positive hence, the system is stable.**

```
num = [7.2 36]; \\NUMERATOR
den = [0.0005 0.06 1 0 0]; \\DENOMINATOR
sys = tf(num, den);   \\Transfer Function
bode(sys); \\Frequency response
margin(sys);
```



```
clc
clear all;
close all;
num=[7.2 36]
den=[0.0005 0.06 1 0 0]
G=tf(num,den);
[gm,pm wep weg]=margin(G)
bode(G),grid
```

```
num =    7.2000   36.0000
den =    0.0005    0.0600    1.0000         0         0
gm =   21.3376
pm =   31.5989
wep =   37.4091
weg =    7.9003
```



BODE DIAGRAMS USING MATLAB

Two functions exist that assist in Bode diagrams:

1. "bode" returns/plots the Bode response of a system.
2. "margins" the gain and phase margins and their associated frequencies

Valid syntax for the "bode" utility, for transfer functions, is:

1. [mag,phase,w] = bode(num,den)
2. [mag,phase,w] = bode(num,den,w)
3. [mag,phase] = bode(num,den,w)
4. bode(num,den,w)
5. bode(num,den)

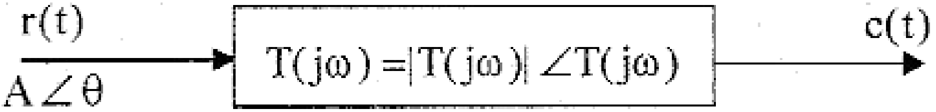
where "num" and "den" contain the polynomial coefficients,

NYQUIST DIAGRAMS USING MATLAB

[re,im] = nyquist(num,den,w)
where
re = real part on the Nyquist diagram
im = imaginary part on the Nyquist Diagram
num = row matrix format representing the numerator of the polynomial
den = row matrix format representing the denominator of the polynomial
w = frequency in rad/sec

Consider a linear time invariant system with frequency domain transfer function, T(jω) shown in fig

$r(t) = A\sin(\omega t + \theta) = A\angle\theta$



$c(t) = B\angle\phi$

where, $B = A \times |T(j\omega)|$

$\phi = \theta + \angle T(j\omega)$

System with sinusoidal transfer function T(jω).

Let S=jω in G(s)H(s) & calculate magnitude and phase

ωpc" Phase cross over frequency

$|G(j\omega)H(j\omega)|$

$$M\angle\emptyset = |\mathbf{G(j\omega)H(j\omega)}|\ \angle\mathbf{G(j\omega)H(j\omega)}$$

Magnitude M **Phase Ø** $\angle\mathbf{G(j\omega)H(j\omega)}$