MODULE - 4

Root–Locus Techniques

The characteristics of the transient response of a closed loop control system are related to location of the closed loop poles. If the system has a variable loop gain, then the location of the closed loop poles depends on the value of the loop gain chosen. It is important, that the designer knows how the closed loop poles move in the s-plane as the loop gain is varied. W. R. Evans introduced a graphical method for finding the roots of the characteristic equation known as root locus method. The root locus is used to study the location of the poles of the closed loop transfer function of a given linear system as a function of its parameters, usually a loop gain, given its open loop transfer function. The roots corresponding to a particular value of the system parameter can then be located on the locus or the value of the parameter for a desired root location can be determined from the locus. It is a powerful technique, as an approximate root locus sketch can be made quickly and the designer can visualize the effects of varying system parameters on root locations or vice versa. It is applicable for single loop as well as multiple loop system.

5. ROOT LOCUS CONCEPT

To understand the concepts underlying the root locus technique, consider the second order system shown in Fig. 1.

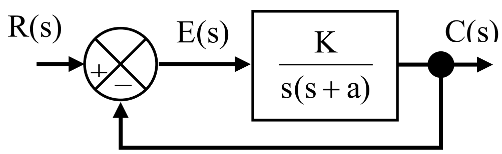


Fig. 3 Second order control system

The open loop transfer function of this system is

$$G(s) = \frac{K}{s(s+a)} \tag{1}$$

Where, K and a are constants. The open loop transfer function has two poles one at origin s = 0 and the other at s = -a. The closed loop transfer function of the system shown in Fig.1 is

$$\frac{C(s)}{R(s)} = \frac{G(s)}{1+G(s)H(s)} = \frac{K}{s^2+as+K} \tag{2}$$

The characteristic equation for the closed loop system is obtained by setting the denominator of the right hand side of Eqn.(2) equal to zero. That is,

$$1+G(s)H(s)=s^2+as+K=0 \tag{3}$$

The second order system under consideration is always stable for positive values of a and K but its dynamic behavior is controlled by the roots of Eqn.(3) and hence, in turn by the magnitudes of a and K, since the roots are given by

$$s_1, s_2 = \frac{-a}{2} \pm \sqrt{\frac{(a^2-4K)}{2a}} = \frac{-a}{2} \pm \sqrt{\left[\left(\frac{a}{2}\right)^2 - K\right]} \tag{4}$$

From Eqn.(4), it is seen that as the system parameters a or K varies, the roots change. Consider a to be constant and gain K to be variable. As K is varied from zero to infinity, the two roots s_1 and s_2 describe loci in the s-plane. Root locations for various ranges of K are:

1) K= 0, the two roots are real and coincide with open loop poles of the system $s_1 = 0$, $s_2 = -a$.
2) $0 \leq K < a^2/4$, the roots are real and distinct.
3) $K= a^2/4$, roots are real and equal.
4) $a^2/4 < K < \infty$, the rots are complex conjugates.

The root locus plot is shown in Fig.2

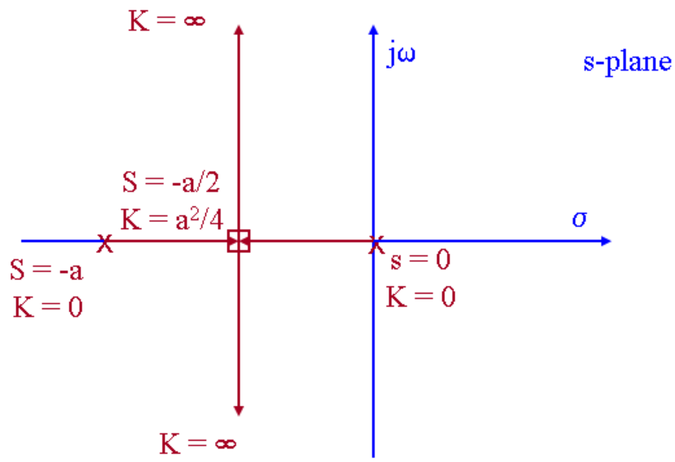


Fig. 4 Root loci of s^2+as+K as a function of K

Figure 2 has been drawn by the direct solution of the characteristic equation. This procedure becomes tedious. Evans graphical procedure helps in sketching the root locus quickly. The characteristic equation of any system is given by

$$\Delta(s)=0 \tag{5}$$

Where, $\Delta(s)$ is the determinant of the signal flow graph of the system given by Eqn.(5).

Δ=1-(sum of all individual loop gains)+(sum of gain products of all possible combinations of two nontouching loops – sum of gain products of all possible combination of three nontouching loops) + ⋯

Or

$$\Delta(s) = 1 - \sum_m P_{m1} + \sum_m P_{m2} - \sum_m P_{m3} + \cdots \qquad (6)$$

Where, P_{mr} is gain product of m^{th} possible combination of r nontouching loops of the graph.

The characteristic equation can be written in the form

$$1 + P(s) = 0$$

$$1 + \frac{KA(s)}{B(s)} = 0 \qquad (7)$$

For single loop system shown in Fig.3

$$P(s) = G(s)H(s) \qquad (8)$$

Where, G(s)H(s) is open loop transfer function in block diagram terminology or transmittance in signal flow graph terminology.

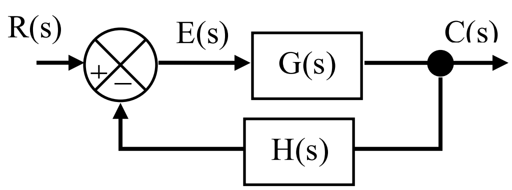


Fig. 5 Single loop feedback system

From Eqn.(7) it can be seen that the roots of the characteristic equation (closed loop poles)occur only for those values of s where

$$P(s) = -1 \qquad (9)$$

Since, s is a complex variable, Eqn.(9) can be converted into the two Evans conditions given below.

$$|P(s)| = 1 \qquad (10)$$

$$\angle P(s) = \pm 180^\circ\ (2q+1); q = 0,1,2\cdots \qquad (11)$$

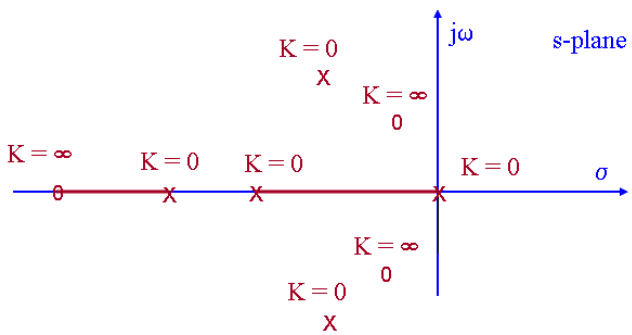
Roots of 1+P(s) = 0 are those values of s at which the magnitude and angle condition given by Eqn.(10) and Eqn.(11). A plot of points in the complex plane satisfying the angle criterion is the root locus. The value of gain corresponding to a root can be determined from the magnitude criterion.

To make the root locus sketching certain rules have been developed which helps in visualizing the effects of variation of system gain K (K > 0 corresponds to the negative feed back and K < 0 corresponds to positive feedback control system) and the effects of shifting pole-zero locations and adding in anew set of poles and zeros.

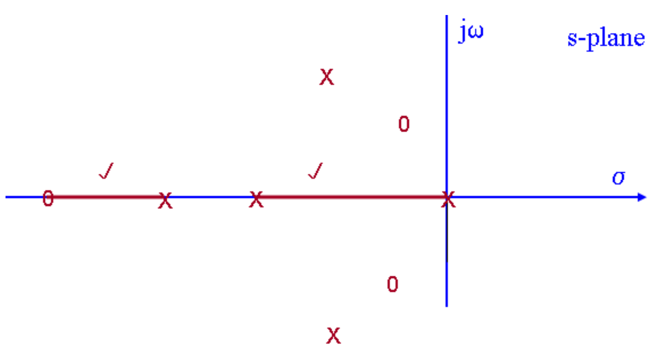
5.1 GENERAL RULES FOR CONSTRUCTING ROOT LOCUS

1) The root locus is symmetrical about real axis. The roots of the characteristic equation are either real or complex conjugate or combination of both. Therefore their locus must be symmetrical about the real axis.

2) As K increases from zero to infinity, each branch of the root locus originates from an open loop pole (n nos.) with K= 0 and terminates either on an open loop zero (m nos.) with K = ∞ along the asymptotes or on infinity (zero at ∞). The number of branches terminating on infinity is equal to (n – m).



3) Determine the root locus on the real axis. Root loci on the real axis are determined by open loop poles and zeros lying on it. In constructing the root loci on the real axis choose a test point on it. If the total number of real poles and real zeros to the right of this point is odd, then the point lies on root locus. The complex conjugate poles and zeros of the open loop transfer function have no effect on the location of the root loci on the real axis.



4) Determine the asymptotes of root loci. The root loci for very large values of s must be asymptotic to straight lines whose angles are given by

$$\text{Angle of asymptotes } \phi_A = \frac{\pm 180^\circ (2q+1)}{n-m}; \quad q = 0,1,2,\cdots n-m-1 \qquad (12)$$

5) All the asymptotes intersect on the real axis. It is denoted by σ_a, given by

$$\sigma_a = \frac{\text{sum of poles} - \text{sum of zeros}}{n-m}$$
$$= \frac{(p_1 + p_2 + \cdots p_n) - (z_1 + z_2 + \cdots z_m)}{n-m} \qquad (13)$$

6) Find breakaway and breakin points. The breakaway and breakin points either lie on the real axis or occur in complex conjugate pairs. On real axis, breakaway points exist between two adjacent poles and breakin in points exist between two adjacent zeros. To calculate these polynomial $\frac{dK}{ds} = 0$ must be solved. The resulting roots are the breakaway / breakin points. The characteristic equation given by Eqn.(7), can be rearranged as

$$B(s) + KA(s) = 0 \qquad (14)$$

where, $B(s) = (s+p_1)(s+p_2)\cdots(s+p_n)$ and

$A(s) = K(s+z_1)(s+z_2)\cdots(s+z_m)$

The breakaway and breakin points are given by

$$\frac{dK}{ds} = \left(\frac{d}{ds}A\right)B - A\left(\frac{d}{ds}B\right) = 0 \qquad (15)$$

Note that the breakaway points and breakin points must be the roots of Eqn.(15), but not all roots of Eqn.(15) are breakaway or breakin points. If the root is not on the root locus portion of the real axis, then this root neither corresponds to breakaway or breakin point. If the roots of Eqn.(15) are complex conjugate pair, to ascertain that they lie on root loci, check the corresponding K value. If K is positive, then root is a breakaway or breakin point.

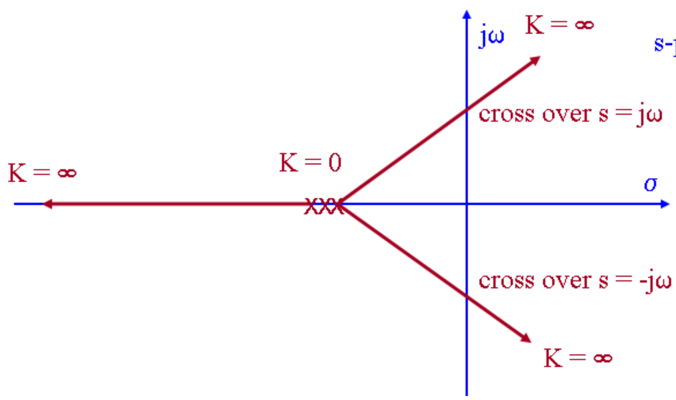
7) Determine the angle of departure of the root locus from a complex pole

$$\text{Angle of departure from a complex } p = 180^\circ$$
$$-(\text{sum of angles of vectors to a complex pole in question from other poles})$$
$$+(\text{sum of angles of vectors to a complex pole in question from other zeros}) \qquad (16)$$

8) Determine the angle of arrival of the root locus at a complex zero

$$\text{Angle of arrival at complex zero} = 180^\circ$$
$$- (\text{sum of angles of vectors to a complex zero in question from other zeros})$$
$$+ (\text{sum of angles of vectors to a complex zero in question from other poles}) \quad (17)$$

9) Find the points where the root loci may cross the imaginary axis. The points where the root loci intersect the jω axis can be found by
   a) use of Routh's stability criterion or
   b) letting s = jω in the characteristic equation , equating both the real part and imaginary part to zero, and solving for ω and K. The values of ω thus found give the frequencies at which root loci cross the imaginary axis. The corresponding K value is the gain at each crossing frequency.



10) The value of K corresponding to any point s on a root locus can be obtained using the magnitude condition, or

$$K = \frac{\text{product of lengths between points to poles}}{\text{product of length between points to zeros}} \quad (18)$$

PHASE MARGIN AND GAIN MARGIN OF ROOT LOCUS

Gain Margin

It is a factor by which the design value of the gain can be multiplied before the closed loop system becomes unstable.

$$\text{Gain Margin} = \frac{\text{Value of K at imaginary cross over}}{\text{Design value of K}} \quad (19)$$

The Phase Margin

Find the point $j\omega_1$ on the imaginary axis for which $|G(j\omega)H(j\omega)| = 1$ for the design value of K i.e. $|B(j\omega)/A(j\omega)| = K_{design}$.

The phase margin is

$$\varphi = 180^\circ + \arg G(j\omega_1)H(j\omega_1) \qquad (20)$$

Problem No 1

Sketch the root locus of a unity negative feedback system whose forward path transfer function is $G(s) = \frac{K}{s}$.

Solution:

1) Root locus is symmetrical about real axis.

2) There are no open loop zeros(m = 0). Open loop pole is at s = 0 (n = 1). One branch of root locus starts from the open loop pole when K = 0 and goes to ∞ asymptotically when $K \to \infty$.

3) Root locus lies on the entire negative real axis as there is one pole towards right of any point on the negative real axis.

4) The asymptote angle is $\phi_A = \frac{\pm 180^\circ (2q+1)}{n-m}, \quad q = n - m - 1 = 0.$

   Angle of asymptote is $\phi_A = \pm 180°$.

5) Centroid of the asymptote is $\sigma_A = \frac{(\text{sum of poles}) - (\text{sum of zeros})}{n - m}$

   $= \frac{0}{1} = 0.0$

6) The root locus does not branch. Hence, there is no need to calculate the break points.

7) The root locus departs at an angle of -180° from the open loop pole at s = 0.

8) The root locus does not cross the imaginary axis. Hence there is no imaginary axis cross over.

The root locus plot is shown in Fig.1

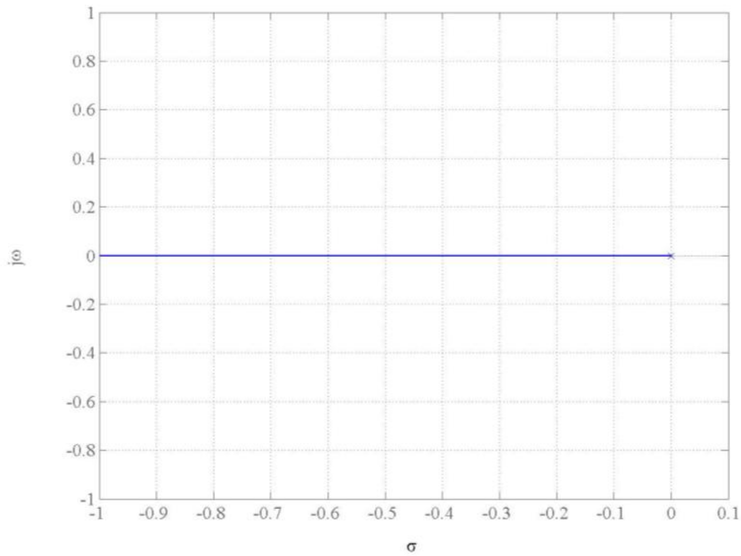


Figure 6 Root locus plot of K/s

Comments on stability:

The system is stable for all the values of K > 0. Th system is over damped.

Problem No 2

The open loop transfer function is $G(s) = \frac{K(s+2)}{(s+1)^2}$. Sketch the root locus plot

Solution:

1) Root locus is symmetrical about real axis.

2) There is one open loop zero at s=-2.0(m=1). There are two open loop poles at s=-1, -1(n=2). Two branches of root loci start from the open loop pole when K= 0. One branch goes to open loop zero at s =-2.0 when $K \to \infty$ and other goes to ∞ (open loop zero∞) asymptotically when $K \to \infty$.

3) Root locus lies on negative real axis for s ≤ -2.0 as the number of open loop poles plus number of open loop zeros to the right of s=-0.2 are odd in number.

4) The asymptote angle is $\phi_A = \frac{\pm 180^\circ (2q+1)}{n-m}$, $\quad q = n-m-1 = 0.$

   Angle of asymptote is $\phi_A = \pm 180°$.

5) Centroid of the asymptote is $\sigma_A = \frac{(\text{sum of poles}) - (\text{sum of zeros})}{n-m}$

$$= \frac{(-1-1)-(-2)}{1} = 0.0$$

6) The root locus has break points.

$$K = -\frac{(s+1)^2}{(s+2)}$$

$$\text{Break point is given by } \frac{dK}{ds} = 0$$

$$\frac{2(s+1)(s+2)-(s+1)^2}{(s+2)^2} = 0$$

$$s_1 = -1, K = 0; s_2 = -3, K = 4$$

The root loci brakesout at the open loop poles at s=-1, when K =0 and breaks in onto the real axis at s=-3, when K=4. One branch goes to open loop zero at s=-2 and other goes to ∞ along the asymptotically.

7) The branches of the root locus at s=-1, -1 break at K=0 and are tangential to a line s=-1+j0 hence depart at ±90°.

8) The locus arrives at open loop zero at ±180°.

9) The root locus does not cross the imaginary axis, hence there is no need to find the imaginary axis cross over.

The root locus plot is shown in Fig.2.

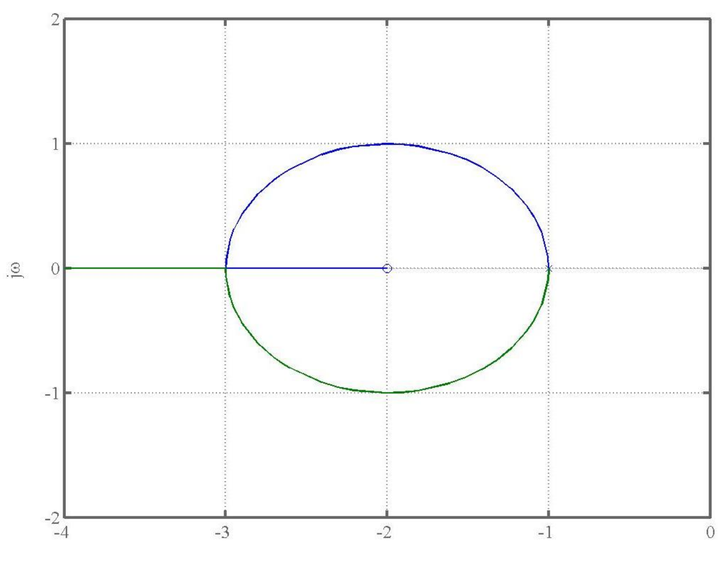


Figure 7 Root locus plot of K(s+2)/(s+1)^2

Comments on stability:

System is stable for all values of K > 0. The system is over damped for K > 4. It is critically damped at K = 0, 4.

Problem No 3

The open loop transfer function is $G(s)=\dfrac{K(s+4)}{s(s+2)}$. Sketch the root locus.

Solution:

1) Root locus is symmetrical about real axis.

2) There are is one open loop zero at s=-4(m=1). There are two open loop poles at s=0, -2(n=2). Two branches of root loci start from the open loop poles when K= 0. One branch goes to open loop zero when $K \to \infty$ and other goes to infinity asymptotically when $K \to \infty$.

3) Entire negative real axis except the segment between s=-4 to s=-2 lies on the root locus.

4) The asymptote angle is $\phi_A = \dfrac{\pm 180^\circ(2q+1)}{n-m}, \quad q = 0,1,\cdots n-m-1 = 0.$

   Angle of asymptote are $\phi_A = \pm 180°$.

5) Centroid of the asymptote is $\sigma_A = \dfrac{(\text{sum of poles})-(\text{sum of zeros})}{n-m}$

$$= \frac{(-2)-(-4)}{1} = 2.0$$

6) The brake points are given by dK/ds =0.

$$K = -\frac{s(s+2)}{(s+4)}$$

$$\frac{dK}{ds} = \frac{(2s+2)(s+4)-(s^2+2s)}{(s+4)^2} = 0$$

$$s_1 = -1.172, K = 0.343;$$

$$s_2 = -6.828, K = 11.7$$

7) Angle of departure from open loop pole at s =0 is ±180°. Angle of departure from pole at s=-2.0 is 0°.

8) The angle of arrival at open loop zero at s=-4 is ±180°

9) The root locus does not cross the imaginary axis. Hence there is no imaginary cross over.

The root locus plot is shown in fig.3.

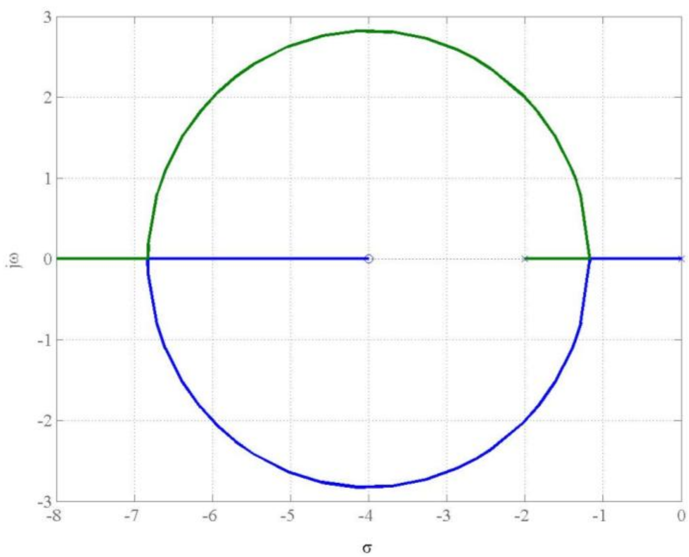


Figure 3 Root locus plot of K(s+4)/s(s+2)

Comments on stability:
System is stable for all values of K.
$0 > K > 0.343$: $\xi > 1$ over damped
$K = 0.343$: $\xi = 1$ critically damped
$0.343 > K > 11.7$: $\xi < 1$ under damped
$K = 11.7$: $\xi = 1$ critically damped
$K > 11.7$: $\xi > 1$ over damped.

Problem No 4

The open loop transfer function is $G(s) = \dfrac{K(s+0.2)}{s^2(s+3.6)}$. Sketch the root locus.

Solution:

1) Root locus is symmetrical about real axis.

2) There is one open loop zero at s = -0.2(m=1). There are three open loop poles at s = 0, 0, -3.6(n=3). Three branches of root loci start from the three open loop poles when K= 0 and one branch goes to open loop zero at s = -0.2 when $K \to \infty$ and other two go to ∞ asymptotically when $K \to \infty$.

3) Root locus lies on negative real axis between -3.6 to -0.2 as the number of open loop poles plus open zeros to the right of any point on the real axis in this range is odd.

4) The asymptote angle is $\phi_A = \dfrac{\pm 180^\circ (2q+1)}{n-m}, \quad q = n-m-1 = 0,1$

   Angle of asymptote are $\phi_A = \pm 90°, \pm 270°$.

5) Centroid of the asymptote is $\sigma_A = \dfrac{(\text{sum of poles}) - (\text{sum of zeros})}{n-m}$

$$= \frac{(-3.6)-(-0.2)}{2} = -1.7$$

6) The root locus does branch out, which are given by dK/ds =0.

$$K = -\frac{(s^3 + 3.6s^2)}{s+0.2}$$

$$\frac{dK}{ds} = -\frac{(3s^2 + 7.2s)(s+0.2) - (s^3 + 3.6s^2)}{(s+0.2)^2}$$

$$2s^3 + 4.8s^2 + 1.44s = 0$$

$$s = 0, -0.432, -1.67 \text{ and } K = 0, \quad 2.55, \quad 3.66 \text{ respectively.}$$

The root loci brakeout at the open loop poles at s = 0, when K =0 and breakin onto the real axis at s=-0.432, when K=2.55 One branch goes to open loop zero at s=-0.2 and other goes breaksout with the another locus starting from open loop ploe at s= -3.6. The break point is at s=-1.67 with K=3.66. The loci go to infinity in the complex plane with constant real part s= -1.67.

7) The branches of the root locus at s=0,0 break at K=0 and are tangential to imaginary axis or depart at ±90°. The locus departs from open loop pole at s=-3.6 at 0°.

8) The locus arrives at open loop zero at s=-0.2 at ±180°.

9) The root locus does not cross the imaginary axis, hence there is no imaginary axis cross over.

The root locus plot is shown in Fig.4.

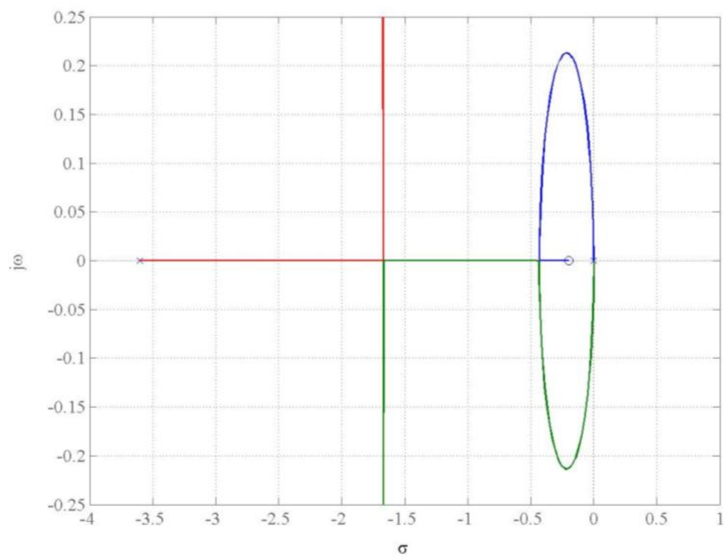


Figure 4 Root locus plot of K(s+0.2)/s^2(s+3.6)

Comments on stability:

System is stable for all values of K. System is critically damped at K= 2.55, 3.66. It is under damped for 2.55 > K > 0 and K >3.66. It is over damped for 3.66 > K >2.55.

Problem No 5

The open loop transfer function is $G(s) = \dfrac{K}{s(s+6s+25)}$. Sketch the root locus.

Solution:

1) Root locus is symmetrical about real axis.

2) There are no open loop zeros (m=0). There are three open loop poles at s=-0, -3±j4(n=3). Three branches of root loci start from the open loop poles when K= 0 and all the three branches go ∞ asymptotically when $K \to \infty$.

3) Entire negative real axis lies on the root locus as there is a single pole at s=0 on the real axis.

4) The asymptote angle is $\phi_A = \dfrac{\pm 180^\circ(2q+1)}{n-m}, \quad q = 0,1,\cdots n-m-1 = 0, 1, 2.$

   Angle of asymptote are $\phi_A = \pm\, 60°, \pm\, 180°, \pm\, 300°$.

5) Centroid of the asymptote is $\sigma_A = \dfrac{(\text{sum of poles}) - (\text{sum of zeros})}{n-m}$

$$= \frac{(-3-3)}{3} = -2.0$$

6) The brake points are given by dK/ds =0.

$K = -s(s^2 + 6s + 25) = -(s^3 + 6s^2 + 25s)$

$\frac{dK}{ds} = 3s^2 + 12s + 25 = 0$

$s_{1,2} = -2 \pm j2.0817$and

$K_{1,2} = 34 \pm j18.04$

For a point to be break point, the corresponding value of K is a real number greater than or equal to zero. Hence, $S_{1,2}$ are not break points.

7) Angle of departure from the open loop pole at s=0 is ±180°. Angle of departure from complex pole s= -3+j4 is

$\phi_p = 180^\circ$

$-(\text{sum of the angles of vectors to a complex pole in question from other poles})$

$+(\text{sum of the angles of vectors to a complex pole inquestion from zeros})$

$$\phi_p = 180^\circ - (180^\circ - \tan^{-1}\frac{4}{3} + 90^\circ) = -36.87^\circ$$

Similarly, Angle of departure from complex pole s= -3-j4 is

$$\varphi_p = 180^\circ - (233.13 + 270^\circ) = -323.13^\circ \text{ or } 36.87^\circ$$

8) The root locus does cross the imaginary axis. The cross over point and the gain at the cross over can be obtained by

5.2 Rouths criterion

The characteristic equation is $s^3 + 6s^2 + 25s + K = 0$. The Routh's array is

s^3	1	25
s^2	6	K
s^1	$\frac{150-K}{6}$	
s^0	6	

For the system to be stable K < 150. At K=150 the auxillary equation is $6s^2+150=0$.
$s = \pm j5$.

or

substitute s= jω in the characteristic equation. Equate real and imaginary parts to zero. Solve for ω and K.

$$s^3 + 6s^2 + 25s + K = 0$$
$$(j\omega)^3 + 6(j\omega)^2 + 25(j\omega) + K = 0$$
$$(-6\omega^2 + K) + j\omega(\omega^2 + 25) = 0$$
$$\omega = 0, \pm j5 \quad K = 0, 150$$

The plot of root locus is shown in Fig.5.

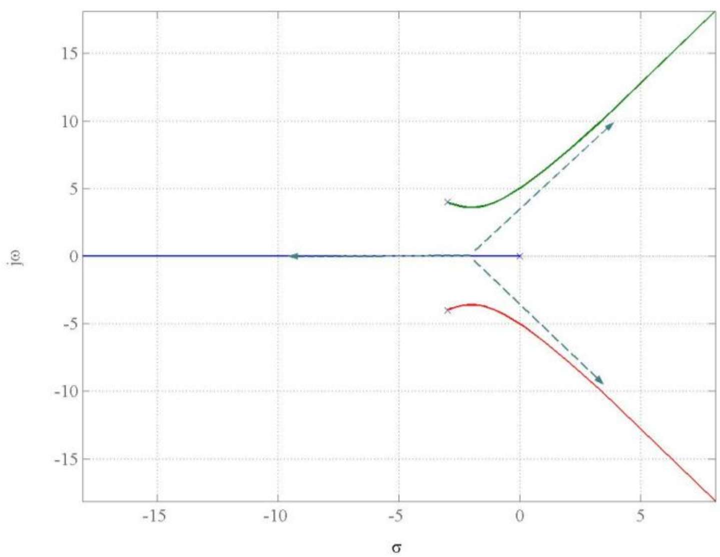


Figure 5 Root locus plot of K/s(s^2+6s+25)

Comments on stability:

System is stable for all values of 150 > K > 0. At K=150, it has sustained oscillation of 5rad/sec. The system is unstable for K >150.

Problem No 1

Sketch the root locus of a unity negative feedback system whose forward path transfer function is $G(s)H(s) = \dfrac{K(s+2)}{(s+1)(s+3+j)(s+3-j)}$. Comment on the stability of the system.

Solution:

9) Root locus is symmetrical about real axis.

10) There is one open loop zero at s = -2 (m = 1). There are three open loop poles at s = -1, -3 ± j (n=3). All the three branches of root locus start from the open loop poles when K = 0. One locus starting from s = -1 goes to zero at s = -2 when $K \to \infty$, and other two branches go to ∞ asymptotically (zeros at ∞) when $K \to \infty$.

11) Root locus lies on the negative real axis in the range s=-1 to s= -2 as there is one pole to the right of any point s on the real axis in this range.

12) The asymptote angle is $\phi_A = \dfrac{\pm 180^\circ(2q+1)}{n-m}$, $q = n-m-1 = 0,1.$

Angle of asymptote is $\phi_A = \pm 90°, 270°$.

13) Centroid of the asymptote is $\sigma_A = \dfrac{(\text{sum of poles}) - (\text{sum of zeros})}{n - m}$

$$= \frac{(-1-3-3)-(-2)}{1} = -2.5$$

14) The root locus does not branch. Hence, there is no need to calculate break points.

15) The angle of departure at real pole at s=-1 is 180°. The angle of departure at the complex pole at s=-3+j is 71.57°.

$$\phi_p = 180^\circ$$
$$-(\text{sum of the angles of vectors to a complex pole in question from other poles})$$
$$+(\text{sum of the angles of vectors to a complex pole in question from zeros})$$

$$\theta_1 = \tan^{-1}\frac{1}{-2} = -26.57^\circ \text{ or } 153.43^\circ$$
$$\theta_1 = \text{atan2}(-2,1) = 153.43^\circ$$
$$\phi = \tan^{-1}\frac{1}{-1} = -45^\circ \text{ or } 135^\circ, \quad \theta_3 = \tan^{-1}\frac{2}{0} = 90^\circ$$
$$\phi_p = 180^\circ - (153.43^\circ + 90^\circ) + 135^\circ = 71.57^\circ$$

The angle of departure at the complex pole at s=-3-j is -71.57°.

$$\phi_p = 180^\circ - (206.57^\circ + 270^\circ) + 225^\circ = -71.57^\circ$$

16) The root locus does not cross the imaginary axis. Hence there is no imaginary axis cross over.

The root locus plot is shown in Fig.1

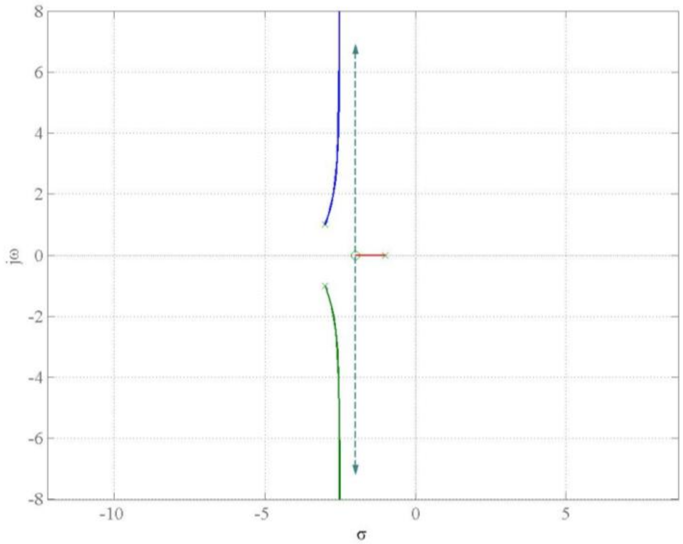


Figure 1 Root locus plot of K(s+2)/(s+1)(s+3+j)(s+3-j)

Comments on stability:

The system is stable for all the values of $K > 0$.

Problem No 2

The open loop transfer function is $G(s)H(s) = \dfrac{K}{s(s+0.5)(s^2+0.6s+10)}$ Sketch the root locus plot. Comment on the stability of the system.

Solution:

10) Root locus is symmetrical about real axis.

11) There are no open loop zeros (m=0). There are four open loop poles (n=4) at s=0, -0.5, -0.3 ± j3.1480. Four branches of root loci start from the four open loop poles when K= 0 and go to ∞ (open loop zero at infinity) asymptotically when $K \to \infty$.

12) Root locus lies on negative real axis between s = 0 to s = -0.5 as there is one pole to the right of any point s on the real axis in this range.

13) The asymptote angle is $\phi_A = \dfrac{\pm 180^\circ(2q+1)}{n-m}, \quad q = n-m-1 = 0,1,2,3.$

Angle of asymptote is $\phi_A = \pm 45°, \pm 135°, \pm 225°, \pm 315°$.

14) Centroid of the asymptote is $\sigma_A = \dfrac{(\text{sum of poles}) - (\text{sum of zeros})}{n - m}$

$$= \frac{(-0.5 - 0.3 - 0.3)}{4} = -0.275$$

The value of K at s=-0.275 is 0.6137.

15) The root locus has break points.

$K = -s(s+0.5)(s^2+0.6s+10) = -(s^4+1.1s^3+10.3s^2+5s)$

Break points are given by dK/ds = 0

$$\frac{dK}{ds} = 4s^3 + 3.3s^2 + 20.6s + 5 = 0$$

s= -0.2497, -0.2877± j 2.2189

There is only one break point at -0.2497. Value of K at s = -0.2497 is 0.6195.

16) The angle of departure at real pole at s=0 is ±180° and at s=-0.5 is 0°. The angle of departure at the complex pole at s = -0.3 + j3.148 is -91.8°

$\phi_p = 180^\circ$

$-$(sum of the angles of vectors to a complex pole in question from other poles)

$+$(sum of the angles of vectors to a complex pole inquestion from zeros)

$$\theta_1 = \tan^{-1}\frac{3.148}{-0.3} = -84.6^\circ \text{ or } 95.4^\circ$$

$$\theta_2 = \tan^{-1}\frac{3.148}{0.2} = 86.4^\circ, \quad \theta_3 = \tan^{-1}\frac{6.296}{0} = 90^\circ$$

$$\phi_p = 180^\circ - (95.4^\circ + 86.4^\circ + 90^\circ) = -91.8^\circ$$

The angle of departure at the complex pole at s = -0.3 - j3.148 is 91.8°

$$\phi_p = 180^\circ - (264.6^\circ + 273.6^\circ + 270^\circ) = 91.8^\circ$$

17) The root locus does cross the imaginary axis, The cross over frequency and gain is obtained from Routh's criterion.

The characteristic equation is

s(s+0.5)(s2+0.6s+10)+K =0 or s4+1.1s3+10.3s2+5s+K=0

The Routh's array is

s^4	1	10.3	K
s^3	1.1	5	
s^2	5.75	K	
s^1	$\frac{28.75-1.1K}{5.75}$		
s^0	K		

The system is stable if $0 < K < 26.13$

The auxiliary equation at K 26.13 is $5.75s^2+26.13 = 0$ which gives $s = \pm j2.13$ at imaginary axis crossover.

The root locus plot is shown in Fig.2.

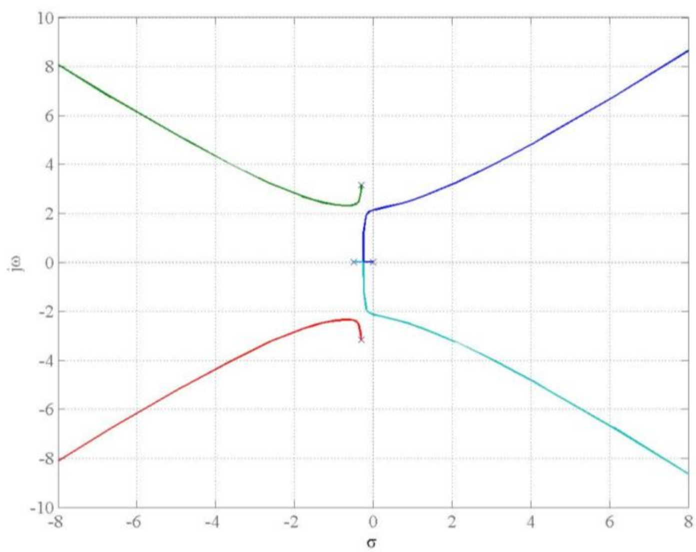


Figure 8 Root locus plot of $K/s(s+0.5)(s^2+0.6s+10)$

Comments on stability:

System is stable for all values of $26.13 > K > 0$. The system has sustained oscillation at $\omega = 2.13$ rad/sec at K=26.13. The system is unstable for $K > 26.13$.

Problem No 3

The open loop transfer function is $G(s) = \dfrac{K}{s(s+4)(s^2+4s+20)}$. Sketch the root locus.

Solution:

10) Root locus is symmetrical about real axis.

11) There are no open loop zeros (m=0). There are three open loop poles (n=3) at s = -0, -4, -2 ± j4. Three branches of root loci start from the three open loop poles when K= 0 and to infinity asymptotically when $K \to \infty$.

12) Root locus lies on negative real axis between s = 0 to s = -4.0 as there is one pole to the right of any point s on the real axis in this range.

13) The asymptote angle is $\phi_A = \dfrac{\pm 180^\circ (2q+1)}{n-m}, \quad q = n-m-1 = 0, 1, 2, 3$

Angle of asymptote are $\phi_A = \pm 45°, \pm 135°, \pm 225°, \pm 315°$.

14) Centroid of the asymptote is $\sigma_A = \dfrac{(\text{sum of poles}) - (\text{sum of zeros})}{n-m}$

$$= \frac{(-2.0 - 2.0 - 4.0)}{4} = -2.0$$

15) The root locus does branch out, which are given by dK/ds =0.

$$K = -s(s+4)(s^2+4s+20)$$
$$= -(s^4+8s^3+36s^2+80s)$$
$$\text{Break point is given by } \frac{dK}{ds} = 0$$
$$4s^3+24s^2+72s+80 = 0$$
$$4s^3+8s^2+16s^2+32s+40s+80 = 0$$
$$(s+2)(4s^2+16s+40)$$
$$s_1 = -2.0, K = 64;$$
$$s_2 = -2.0 \pm j2.45, K = 100$$

The root loci brakeout at the open loop poles at s = -2.0, when K = 64 and breakin and breakout at s=-2+j2.45, when K=100

16) The angle of departure at real pole at s=0 is ±180° and at s=-4 is 0°. The angle of departure at the complex pole at s = -2 + j4 is -90°.

$\phi_p = 180^\circ$
$-$(sum of the angles of vectors to a complex pole in question from other poles)
$+$(sum of the angles of vectors to a complex pole in question from zeros)

$$\theta_1 = \tan^{-1}\frac{4}{-2} = 63.4^\circ \text{ or } 116.6^\circ$$
$$\theta_1 = \text{atan2}(4,\text{-}2) = 116.6^\circ$$
$$\theta_2 = \tan^{-1}\frac{4}{2} = 63.4^\circ, \quad \theta_3 = \tan^{-1}\frac{8}{0} = 90^\circ$$
$$\phi_p = 180^\circ - (116.6^\circ + 63.4^\circ + 90^\circ) = -90^\circ$$

The angle of departure at the complex pole at s = -2 – j4 is 90°

$$\phi_p = 180^\circ - (243.4^\circ + 296.6^\circ + 270^\circ)$$
$$= -270^\circ = 90^\circ$$

17) The root locus does cross the imaginary axis, The cross over point and gain at cross over is obtained by either Routh's array or substitute s= jω in the characteristic equation and solve for ω and gain K by equating the real and imaginary parts to zero.

Routh's array

The characteristic equation is $s^4 + 8s^3 + 36s^2 + 80s + K = 0$

The Rouths array is

s^4	1	36	K
s^3	8	80	
s^2	26	K	
s^1	$\frac{2080 - 8K}{26}$		
s^0	K		

For the system to be stable K > 0 and 2080-8K > 0. The imaginary crossover is given by 2080-8K=0 or K = 260.

At K = 260, the auxiliary equation is $26s^2+260 = 0$. The imaginary cross over occurs at s= ±j√10.

or

$$s^4 + 8s^3 + 36s^2 + 80s + K = 0$$
$$\text{put} \quad s = j\omega$$
$$(j\omega)^4 + 8(j\omega)^3 + 36(j\omega)^2 + 80(j\omega) + K = 0$$
$$\left(\omega^4 - 36\omega^2 + K\right) + j\left(-8\omega^3 + 80\omega\right) = 0$$
$$\text{Equate real and imaginary parts to zero}$$
$$-8\omega^3 + 80\omega = 0 \Rightarrow \omega = 0, \pm j\sqrt{10}; s = j\sqrt{10}$$
$$\omega^4 - 36\omega^2 + K = 0 \Rightarrow K = 260$$

The root locus plot is shown in Fig.3.

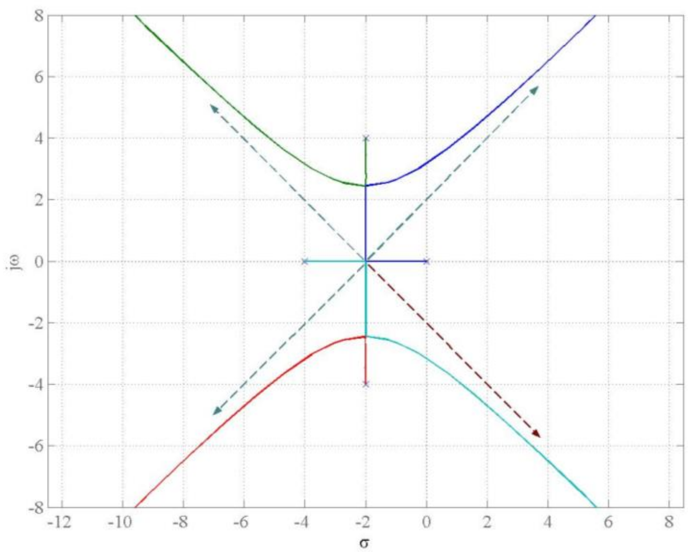


Figure 9 Root locus plot of K/s(s+4)(s^2+4s+20)

Comments on stability:
For 260 > K > 0 system is stable
K = 260 system has stained oscillations of √10 rad/sec.
K > 260 system is unstable.

Recommended Questions:

1. Give the general rules for constructing root locus.

2. Define Phase margin and Gain margin of root locus.

3. Sketch the root locus of a unity negative feedback system whose forward path transfer function is $G(s)=\frac{K}{s}$.

4. The open loop transfer function is $G(s)=\frac{K(s+2)}{(s+1)^2}$. Sketch the root locus plot.

5. The open loop transfer function is $G(s)=\frac{K(s+4)}{s(s+2)}$. Sketch the root locus.

6. The open loop transfer function is $G(s)=\frac{K}{s(s+6s+25)}$. Sketch the root locus.

7. The open loop transfer function is $G(s)=\frac{K}{s(s+4)(s^2+4s+20)}$. Sketch the root