MODULE - 5

SYSTEM ANALYSIS USING POLAR PLOTS: NYQUIST CRITERION

System Analysis using Polar Plots: Nyquist Criterion

Polar plots can be used to predict feed back control system stability by the application of Nyquist Criterion, and therefore are also referred as **Nyquist Plots**. It is a labor saving technique in the analysis of dynamic behaviour of control systems in which the need for finding roots of characteristic equation of the system is eliminated.

Consider a typical closed loop control system which may be represented by the simplified block diagram as shown in Figure 6.4

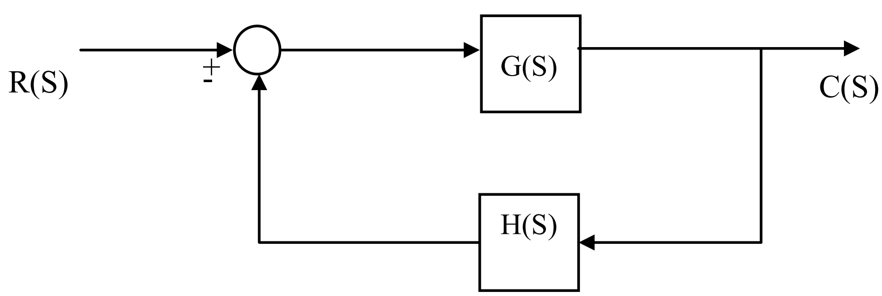


Figure 6.4 Simplified System Block Diagram

The closed-loop transfer function or the relationship between the output and input of the system is given by

$$\frac{C(S)}{R(S)} = \frac{G(S)}{1+G(S)H(S)}$$

The open-loop transfer function is G(S) H(S) (the transfer function with the feedback loop broken at the summing point).

1+ G(S) H(S) is called **Characteristic Function** which when equated to zero gives the **Characteristic Equation** of the system.

1 + G(S) H(S) = 0 **Characteristic Equation**

The characteristic function F(S) = 1 + G(S) H(S) can be expressed as the ratio of two factored polynomials.

Let $F(S) = 1 + G(S)H(S) = \dfrac{K(S+Z_1)(S+Z_2)............(S+Z_n)}{S^k(S+P_1)(S+P_2)(S+P_3)..........(S+Z_n)}$

$\therefore$ The Characteristic equation in general can be represented as

$F(S) = K(S+Z_1)(S+Z_2)(S+Z_3) \ldots\ldots\ldots (S+Z_n) = 0$

Then:

$-Z_1, -Z_2, -Z_3 \ldots. -Z_n$ are the **roots** of the characteristic equation

$\therefore$ at $S= -Z_1$, $S= -Z_2$, $S= -Z_3$, 1+ G(S) H(S) becomes zero.

These values of S are termed as **Zeros** of F(S)

Similarly:

at $S= -P_1$, $S= -P_2$, $S= -P_3$ ……. Etc. 1+ G (S) H (S) becomes infinity.

These values are called **Poles** of F (S).

Condition for Stability

For stable operation of control system all the roots of characteristic equation must be negative real numbers or complex numbers with negative real parts. Therefore, for a system to be stable all the ―Zeros" of characteristic equation (function) should be either negative real numbers or complex numbers with negative real parts. These roots can be plotted on a complex-plane or S-plane in which the imaginary axis divides the complex plane in to two parts: right half plane and left half plane. Negative real numbers or complex numbers with negative real parts lie on the left of S-plane as shown Figure 6.5.

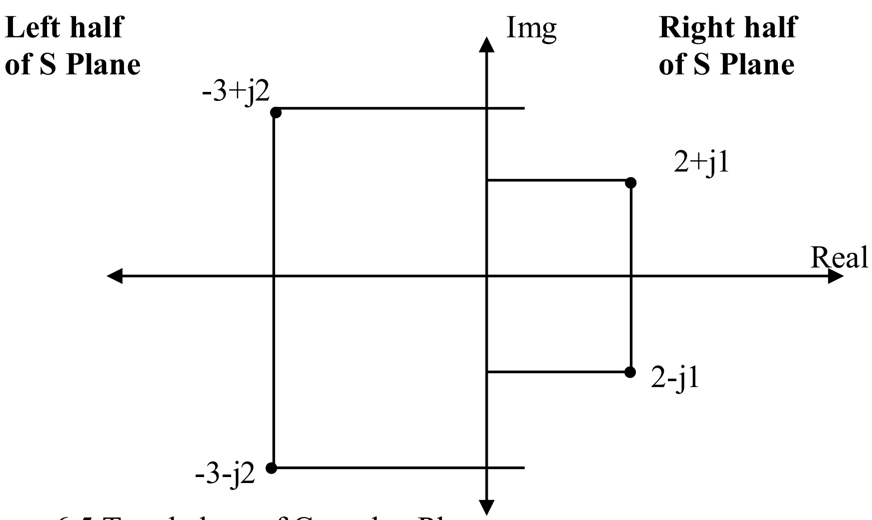


Figure 6.5 Two halves of Complex Plane

Therefore the roots which are positive real numbers or complex numbers with positive real parts lie on the right-half of S-plane.

In view of this, the condition for stability can be stated as ―**For a system to be stable all the zeros of characteristic equation should lie on the left half of S-plane**".

Therefore, the procedure for investigating system stability is to search for ‗Zeros' on the right half of S-plane, which would lead the system to instability, if present. However, it is impracticable to investigate every point on S-plane as to which half of S-plane it belongs to and so it is necessary to have a short-cut method. Such a procedure for searching the right half of S-plane for the presence of Zeros and interpretation of this procedure on the Polar plot is given by the **Nyquist Criterion**.

Nyquist Criterion: Cauchy‘s Principle of Argument:

In order to investigate stability on the Polar plot, it is first necessary to correlate the region of instability on the S-plane with identification of instability on the polar plot, or 1+GH plane. The 1+GH plane is frequently the name given to the plane where 1+G(S) H(S) is plotted in complex coordinates with S replaced by jω. Likewise, the plot of G(S) H(S) with S replaced by jω is often termed as GH plane. This terminology is adopted in the remainder of this discussion.

The Nyquist Criterion is based on the Cauchy‘s principle of argument of complex variable theory. Consider [F(S) = 1+G(S) H(S)] be a single valued rational function which is analytic everywhere in a specified region except at a finite number of points in S-plane. (A function F(S) is said to be analytic if the function and all its derivatives exist). The points where the function and its derivatives does not exist are called singular points. The poles of a point are singular points.

Let C_S be a closed path chosen in S-plane as shown Figure 6.6 (a) such that the function F(S) is analytic at all points on it. For each point on C_S represented on S-plane there is a corresponding mapping point in F(S) plane. Thus when mapping is made on F(S) plane, the curve C_G mapped by the function F(S) plane is also a closed path as shown in Figure 6.6 (b). The direction of traverse of C_G in F(S) plane may be clockwise or counter clockwise, depending upon the particular function F(S).

Then the Cauchy principle of argument states that: **The mapping made on F(S) plane will encircle its origin as many number of times as the difference between the number of Zeros and Poles of F(S) enclosed by the S-plane locus C_S in the S-plane**.

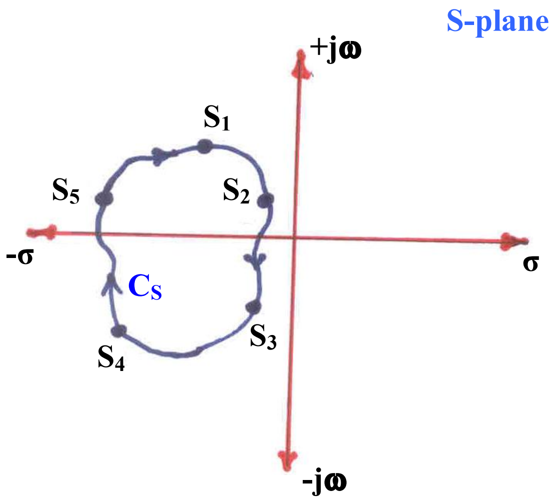


Figure 6.6 (a)

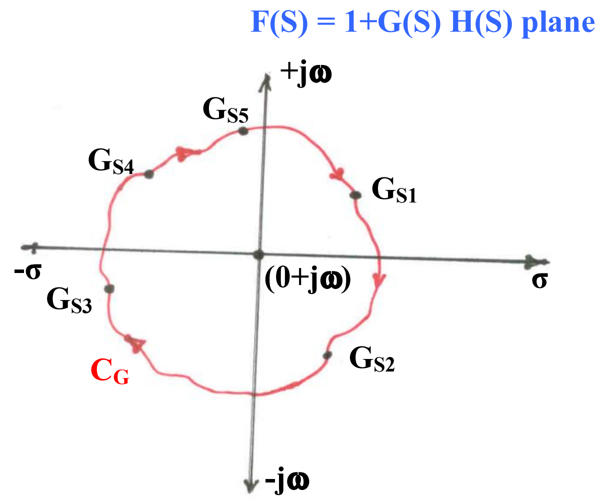


Figure 6.6 (b)

Figure 6.6 Mapping on S-plane and F(S) plane

Thus $N = Z - P$

$N_{0+j0} = Z - P$

Where N_{0+j0}: Number of encirclements made by F(S) plane plot (C_G) about its origin. Z and P: Number of Zeros and Poles of F(S) respectively enclosed by the locus C_S in the S-plane.

Illustration: Consider a function F(S)

$$F(S) = \frac{K(S+1)(S+2+j2)(S+2-j2)}{S(S+3)(S+5)(S+5+j2)(S+5-j2)}$$

∴Zeros: -1, (-2-j2), (-2+j2) indicated by O (dots) in the S-plane

Poles: 0, -3, -5, (-5 –j2), (-5 +j2) indicated by X (Cross) in S-plane: As shown in Figure 6.6 (c)

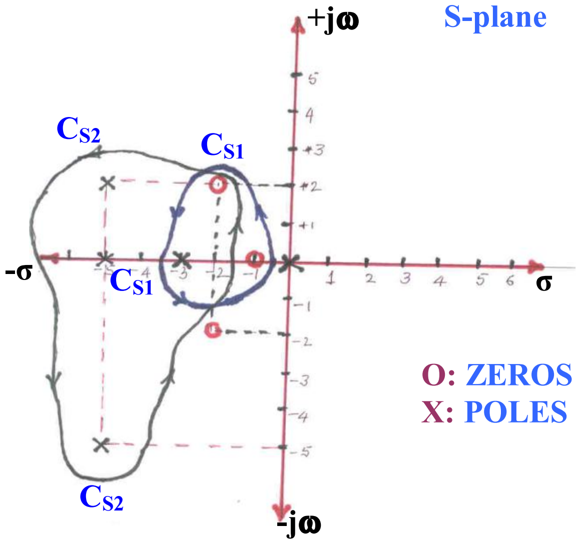


Figure 6.6 (c)

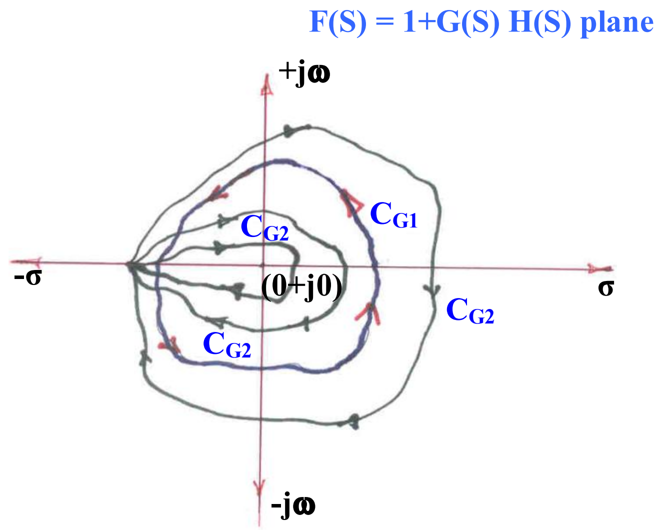


Figure 6.6 (d)

Now consider path C_{S1} (CCW) on S-plane for which:

Z: 2, P =1

Consider another path C_{S2} (CCW) in the same S plane for which:

Z = 1, P = 4

C_{G1} and C_{G2} are the corresponding paths on F(S) plane [Figure 6.6 (d)].

Considering C_{G1} [plot corresponding to C_{S1} on F (S) plane]

$\therefore N_{0+j0} = Z - P = 2\text{-}1 = +1$

C_{G1} will encircle the origin once in the same direction of C_{S1} (CCW)

Similarly for the path C_{G2}

$N_{0+j0} = Z - P = 1 - 4 = -3$

$\therefore C_{G2}$ will encircle the origin 3 times in the opposite direction of C_{S2} (CCW)

Note: The mapping on F(S) plane will encircle its origin as many number of times as the difference between the number of Zeros and Poles of F(S) enclosed by the S-plane locus.

From the above it can be observed that

In the expression

N= Z - P,

N can be positive when: Z>P

N = 0 when: Z = P

N can be negative when: Z<P

- When N is positive the map C_G encircles the origin N times in the **same direction** as that of C_S
- When N = 0, No encirclements
- N is negative the map C_G encircles the origin N times in the **opposite direction** as that of C_S

Nyquist Path and Nyquist Plot

The above Cauchy‘s principle of argument can be used to investigate the stability of control systems. We have seen that if the Zeros of characteristic function lie on the right half of S-plane it will lead to system instability. Now, to encircle the entire right half of S-plane, select a closed path as shown in Figure 6.6 (e) such that all the Zeros lying on the right-half of S-plane will lie inside this path. This path in S-plane is known as **Nyquist path**. Nyquist path is generally taken in CCW direction. This path consists of the imaginary axis of the S-plane ($S = 0+j\omega$, $-\infty < \omega < \infty$) and a closing semicircle of infinite radius. If the system being tested has poles of F(S) on the imaginary axis, it is customary to modify the contour as shown Figure 6.6 (f) excluding these poles from the path.

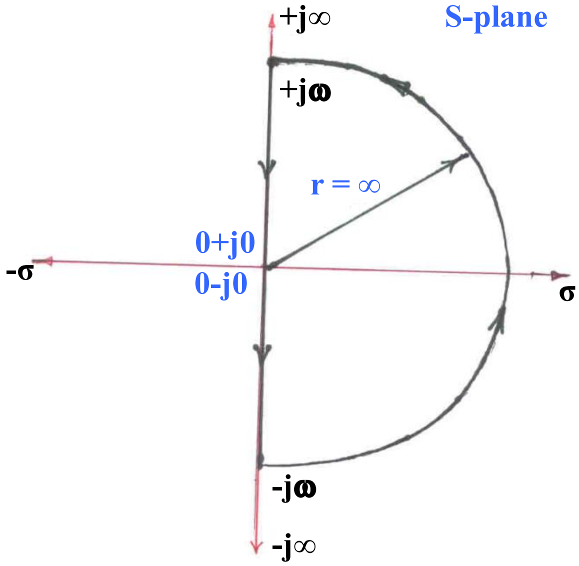


Figure 6.6 (e): Nyquist Path

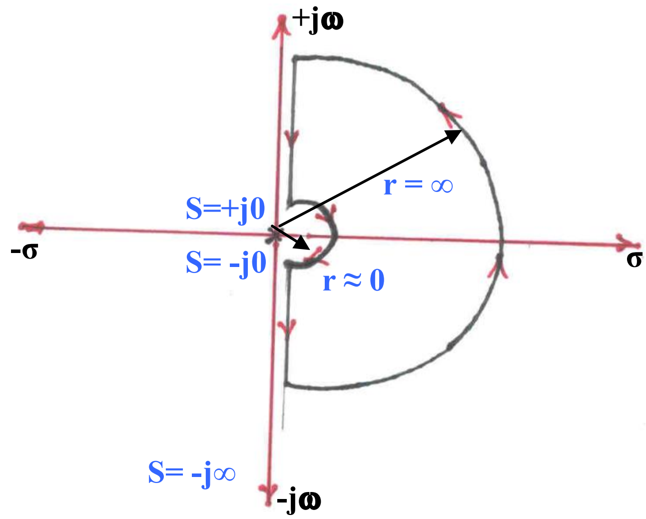


Figure 6.6 (f)

Corresponding to the Nyquist path a plot can be mapped on F(S) = 1+G(S) H(S) plane as shown in Figure 6.6 (g) and the number of encirclements made by this F(S) plot about its origin can be counted.

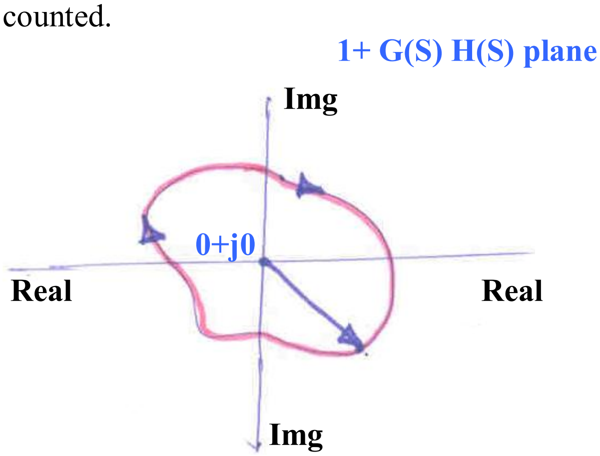


Figure 6.6 (g)

Now from the principle of argument

N_{0+j0} = Z-P

N_{0+j0} = number of encirclements made by F(S) plane plot

Z, P: Zeros and Poles lying on right half of S-plane

For the system to be stable: Z = 0

$\therefore N_{0+j0} = -P$ Condition for Stability

Apart from this, the Nyquist path can also be mapped on G(S) H(S) plane (Open-loop transfer function plane) as shown in Figure 6.6 (h).

Now consider

F(S) = 1+ G(S) H(S) for which the origin is (0+j0) as shown in Figure 6.6 (g).

Therefore G(S) H(S) = F(S) – 1

= (0+j0) – 1 = (-1+j0) Coordinates for origin on G(S) H(S) plane as shown in Figure 6.6(h)

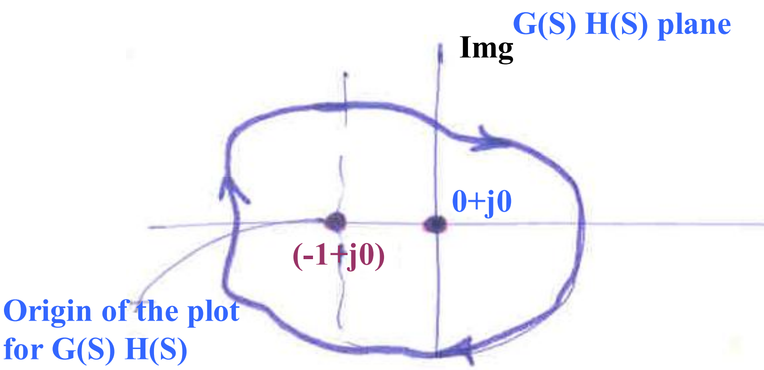


Figure 6.6 (h)

Thus a path on 1+ G(S) H(S) plane can be easily converted to a path on G(S) H(S) plane or open loop transfer function plane. This path will be identical to that of 1+G(S) H(S) path except that the origin is now shifted to the left by one as shown in Figure 6.6 (h).

This concept can be made use of by making the plot in G(S) H(S) plane instead of 1+ G(S) H(S) plane. The plot made on G(S) H(S) plane is termed as the **Niquist Plot** and its net encirclements about (-1+j0) (known as critical point) will be the same as the number of net encirclements made by F(S) plot in the F(S) = 1+G(S) H(S) plane about the origin.

Now, the principle of argument now can be re-written as

N_{-1+j0} = Z-P

Where N_{-1+j0} = Number of **net encirclements** made by the G(S) H(S) plot (Nyquist Plot) in the G(S) H(S) plane about -1+j0

For a system to be stable Z = 0

$\therefore N_{-1+j0} = -P$

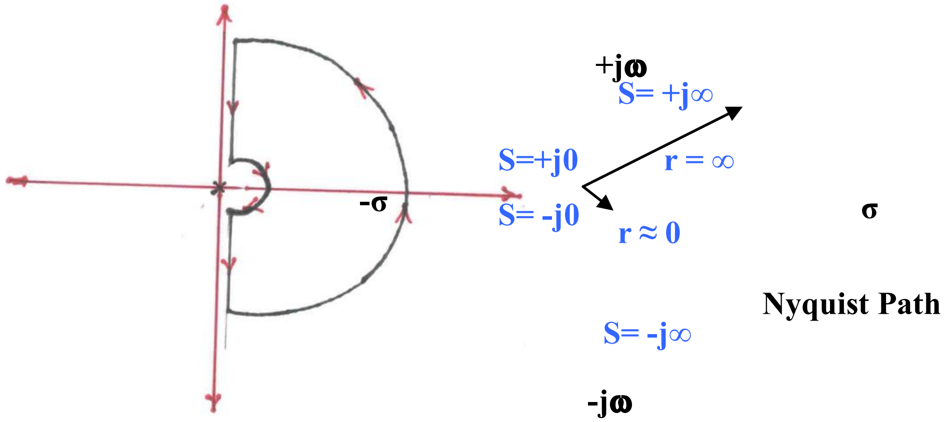
Thus the Nyquist Criterion for a stable system can be stated as **The number of net encirclements made by the Nyquist plot in the G(S) H(S) plane about the critical point (-1+j0) is equal to the number of poles of F(S) lying in right half of S-plane. [Encirclements if any will be in the opposite direction. Poles of F(S) are the same as the poles of G(S) H(S)]**. Thus the stability of closed-loop control system is determined from its open-loop transfer function.

System Analysis using Nyquist Criterion: Illustrations

Illustration 1: Sketch the Nyquist plot for the system represented by the open loop transfer function and comment on its stability.

$G(S)H(S)\frac{K}{S(S+a)} K>0, a>0$ Poles: S = 0 (on imaginary axis) and S = -a

Step 1: Define Nyquist path. Let the Nyquist path be defined as given below.



Section I: S = +j∞ to S = +j0; Section II: S = +j0 to S = -j0

Section III: S = -j0 to S = -j∞; Section IV: S = -j∞ to S = +j∞

2. Corresponding to different sections namely I, II, III, and IV Obtain polar plots on G(S) H(S) plane, which are nothing but Nyquist Plots.

Nyquist Plot for Section I: In S-plane section I runs from S= + j∞ to S = +j0

To obtain polar plot in G(S) H(S) plane:

$$G(S)H(S)\frac{K}{S(S+a)}\,K>0, a>0$$

(i) $G(S)H(S)\big|_{S\to j\infty} = \dfrac{K}{S^2}$,

$$|G(S)H(S)| = \left|\frac{K}{S^2}\right| = 0$$

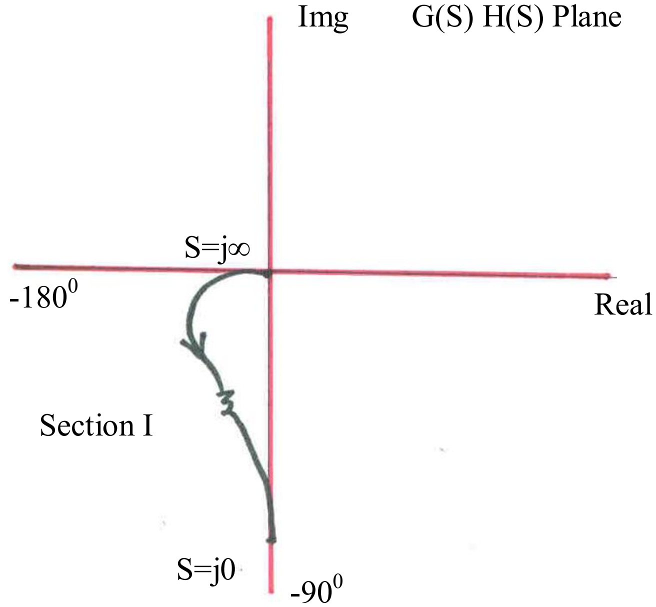
$$\angle G(S)H(S) = \angle\frac{K}{S^2} = \angle K - \angle S^2$$

$= 0 – 2*90^0 = - 180^0$ S= -j∞

(ii) $G(S)H(S)\Big|_{S\to j0} = \frac{K}{S.a} = \frac{K/a}{S} = \left|\frac{K^1}{S}\right| = \infty,$

$\angle G(S)\,H(S) = \angle K/a - \angle S = 0 - 90^0 = -90^0$

S	M(ω)	ϕ(ω)
$S\to\infty$	0	-180^0
$S\to 0$	∞	-90^0



Nyquist Plot for Section II: In S-plane section II runs from S= + j0 to S = -j0

In this region $S\to 0$

In S-plane section II is a semicircle from S = + j0 to S = -j0 of radius $r \approx 0^0$, covering an angle of 180^0 in clockwise direction

$G(S)H(S) = \frac{K}{S(a)} = \frac{K^1}{S}$ where $K^1 = K/a$

But $\mathbf{S = re^{+j\theta}}$ equation of a circle in exponential form

$\therefore G(S)H(S) = \frac{K^1}{re^{+j\theta}} = \mathrm{Re}^{-j\theta}, where \quad R = \frac{K^1}{r}$

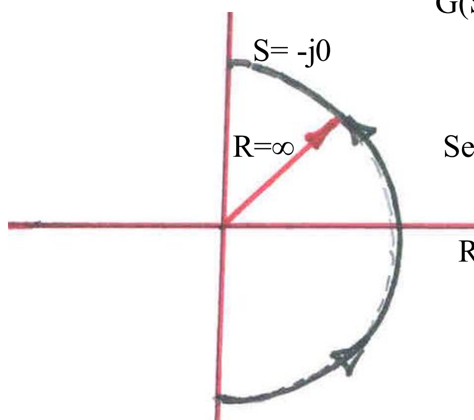
$G(S)\,H(S) = R.e^{-j\theta} \quad R = \frac{K^1}{r} \Rightarrow \infty$ (as r is very small)

This shows that G(S) H(S) plot of section II of Nyquist path is a circle of radius = ∞ starting from S = +j0 and ending at a point S = -j0 covering an angle of 180^0 in opposite direction of section II of Nyquist path (CCW direction i.e., negative sign)

In general if G (S) H (S) = $\frac{K^1}{S^n}$

$$G(S)H(S) = \frac{K^1}{r^n e^{+jn\theta}} = R.e^{-jn\theta}$$

Where R = $\frac{K^1}{r^n} \Rightarrow \infty$



The G (S) H (S) plot will be a portion circle (part) of radius $R \to \infty$, starting at a point S = +jθ and ending at a point S = -jθ covering an angle of ($n*180^0$) in the opposite direction (CCW) (since sign is negative)

Nyquist Plot for Section III: In S-plane section III runs from S= -j0 to S = j∞

$$G(S)H(S)\big|_{S\to -j0} = \frac{K}{S(S+a)} \quad = \frac{K}{S.a} = K^1/S \qquad \text{where } K^1 = (K/a)$$

$$\therefore |G(S)H(S)| = K^1/S = \infty$$

∠G(S) H(S) = ∠K^1 – ∠S, K^1 is negative

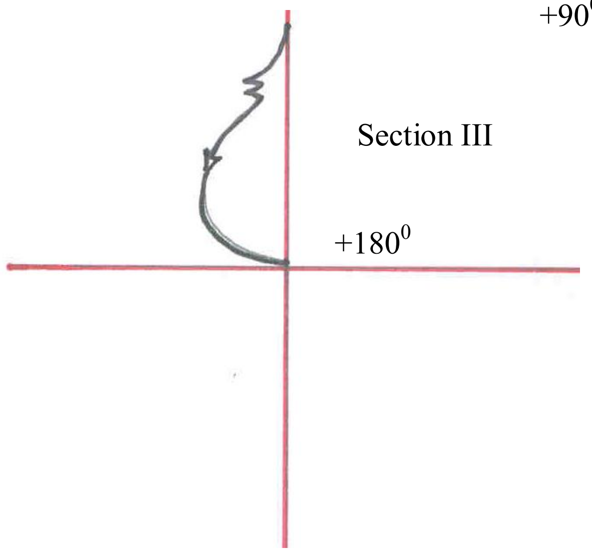
= ∠-K^1 – ∠S

∠G(S) H(S) = $180^0 - 90^0 = 90^0$

$$G(S)H(S)\big|_{S=-j\infty} = \frac{K}{S(S+a)}\Big|_{S\to -j\infty} \quad = K/S^2$$

$$|G(S)H(S)| = \left|\frac{K}{S^2}\right| = 0$$

$\angle$G(S) H(S) = $\angle$K - $\angle S^2$ = 0 – 2 (-90^0) = 180^0 ; (‗S' is negative)



This Section III is the mirror image of section of the Section I.

Nyquist Plot for Section IV: In S-plane section IV runs from S= -j∞ to S = j∞

In this region $S \to \infty$

In the S-plane it is a semicircle of radius $R \to \infty$ from S = - j ∞ to S = + j ∞ covering an angle of 180^0 in the counter clockwise direction.

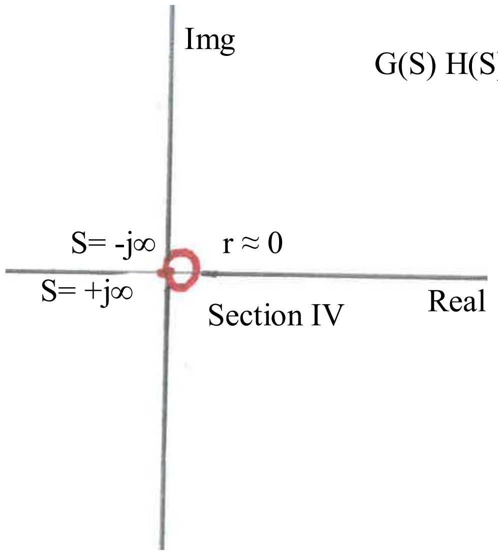
In the G(S) H(S) plane

$$G(S)H(S)\big|_{S\to\infty} = \frac{K}{S^2}$$

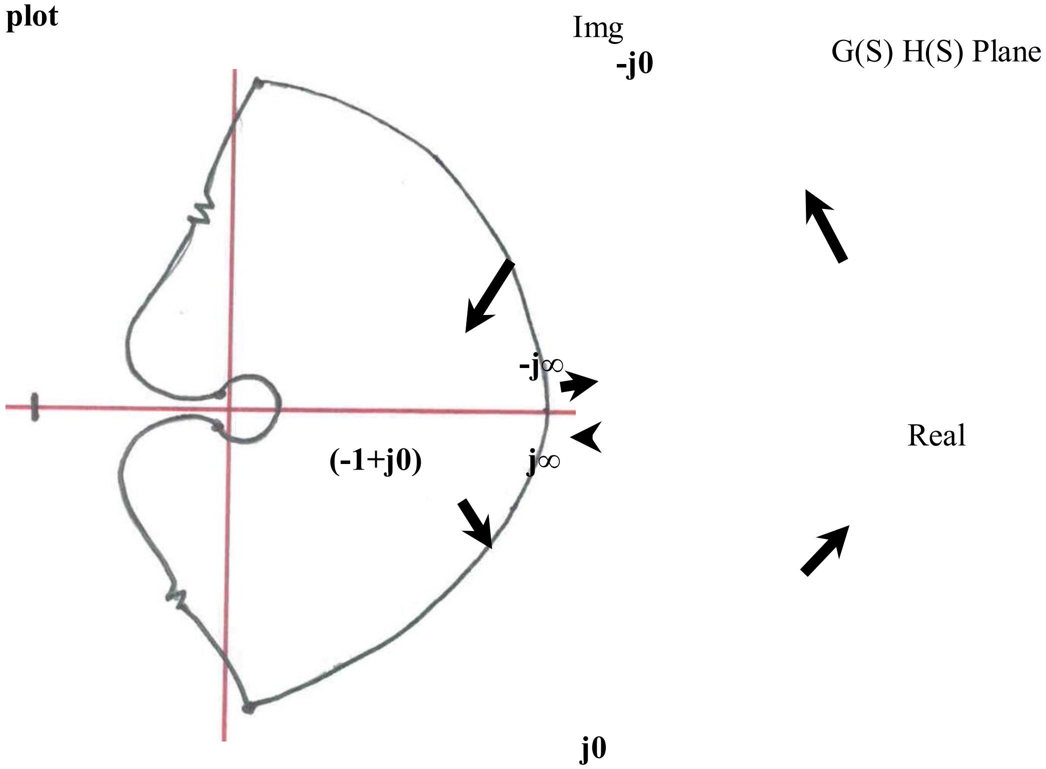
But S = R $e^{+j\phi}$ Equation in circle in exponential form

$$G(S)H(S) = \frac{K}{R^2.e^{j2\phi}} = r.e^{-j2\phi} \qquad \text{where } r = \frac{K}{R^2} \to 0$$

Thus G(S) H(S) plot for section IV is also a circle of radius $r \to 0$ starting at S = - j ∞ and ending at S = + j ∞ covering an angle of 2* 180^0 (2ϕ) in the opposite direction (CW).



Now assemble the Nyquist plots of all the sections as given below to get the overall Nyquist plot



From Nyquist Criterion:

No. of encirclements made by Nyquist plot about $(-1+j0) = N_{-1+j0} = Z - P$

P = No. of poles lying in the right half of S plane

For the function $G(S)H(S) = \dfrac{K}{S(S+a)}$, the Poles are: S = 0, S = - a = 0, which lie on the left half of S-plane

Therefore P = 0: Number of poles on the right half of S-plane

$\therefore N_{-1+j=0} = 0$ as counted from the Nyquist plot

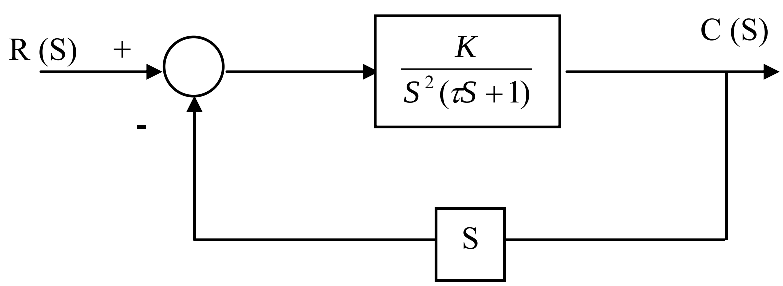
$N_{-1+j0} = Z - P$

0 = Z – 0

Therefore Z = 0

Number of zeros lying on the right half of S-plane is 0 and hence the system is stable

Illustration 2: Obtain the Nyquist diagram for the system represented by the block diagram given below and comment on its stability



$$G(S) = \frac{K}{S^2(\tau S+1)}$$

$$H(S) = S$$

$\therefore G(S)H(S) = \dfrac{K}{S^2(\tau S+1)} * S = \dfrac{K}{S(\tau S+1)}$, Poles are S = 0, -1/τ $\therefore$ P = 0

Recommended Questions:

1. Explain briefly the different Graphical Methods to Represent Frequency Response Data .

2. A second order system has a natural frequency of 10 rad/sec and a damping ratio of 0.5. Sketch the polar plot for the system.

3. Obtain the polar plot for the transfer function

4. Sketch the polar plots for the system represented by the following open loop transfer function.

$$G(S)H(S) = \frac{K}{S^2(S+5)}$$

5. Sketch the polar plots for the system represented by the following open loop transfer function.

$$G(S)H(S) = \frac{10}{S^2(S+5)(S+8)}$$

6. Sketch the polar plots for the system represented by the following open loop transfer function.

$$G(S)H(S) = \frac{10}{S(S-2)(S+4)}$$

7. Draw Nyquist path for the function F(s)

$$F(S) = \frac{K(S+1)(S+2+j2)(S+2-j2)}{S(S+3)(S+5)(S+5+j2)(S+5-j2)}$$