DIGITAL COMMUNICATION

MODULE 1: BAND PASS SIGNAL TO EQUIVALENT LOW PASS

> Hilbert Transform
> Pre envelopes, complex envelopes
> Canonical representation of Bandpass signals
> Complex Low pass representation of Bandpass Systems.
> Complex representation of Bandpass signals and Systems.
> Line codes: Unipolar, Bipolar, Manchester codes and their power spectral densities.
> Overview of HDB3, B3ZS, B6ZS.

HILBERT TRANSFORM

- The Fourier Transform is particularly useful for evaluating the frequency content of an energy signal or in a limiting sense, that of a power signal.

$$X(j\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt \rightarrow \text{Analysis equation (FT)}$$

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(j\omega) e^{j\omega t} d\omega \rightarrow \text{Synthesis equation (IFT)}$$

$$x(t) \xleftrightarrow{FT} X(j\omega)$$

- Fourier Transform provides the mathematical basis for analysing and designing frequency selective filters for the separation of signals on the basis of their frequency content.

- Another method of separating signals is based on Phase Selectivity, which uses phase shift b/w the pertinent signals to achieve the desired separation.

- A phase shift of special interest in this context is $\pm 90°$.

- when the phase angles of all components of a given signal are shifted by $\pm 90°$, the resulting function of time is known as the Hilbert Transform of the signal.

- The Hilbert Transform is also called as Quadrature filter, it is called to emphasize its distinct property of providing $\pm 90°$ phase shift.

- The Hilbert Transform of $g(t)$, is denoted by $\hat{g}(t)$ is defined by

$$\boxed{\hat{g}(t) = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{g(\tau)}{(t-\tau)}\, d\tau} \longrightarrow (1)$$

* Hilbert Transform is a linear function/operation.

- The Inverse Hilbert Transform, by means of which the original signal $g(t)$ is linearly recovered from $\hat{g}(t)$, is defined by

$$\boxed{g(t) = -\frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\hat{g}(\tau)}{(t-\tau)}\, d\tau} \longrightarrow (2)$$

* The functions $\hat{g}(t)$ & $g(t)$ are said to constitute a Hilbert-Transform pair.

- The definition of the Hilbert Transform $\hat{g}(t)$ given in eqn ① may be interpreted as the convolution of $g(t)$ with the time function $\frac{1}{\pi t}$. $\left[\hat{g}(t) = g(t) * \frac{1}{\pi t}\right]$

- WKT convolution in time domain is multiplication of the 2 signals in frequency domain.

WKT

$$\mathrm{sgn}(t) \xleftrightarrow{FT} \frac{1}{j\pi f} \text{ or } \frac{2}{j\omega}$$

OR

Using Duality property.

Duality
$x(t) \xleftrightarrow{FT} X(\omega)$
$X(t) \xleftrightarrow{FT} x(-\omega)$

$$\frac{2}{jt} \longleftrightarrow 2\pi\,\mathrm{sgn}(-\omega)$$

$$\boxed{\frac{1}{\pi t} \longleftrightarrow -j\,\mathrm{sgn}(f)} \longrightarrow ③$$

where $\mathrm{Sgn}(f)$ is the Signum function, defined in the frequency domain as

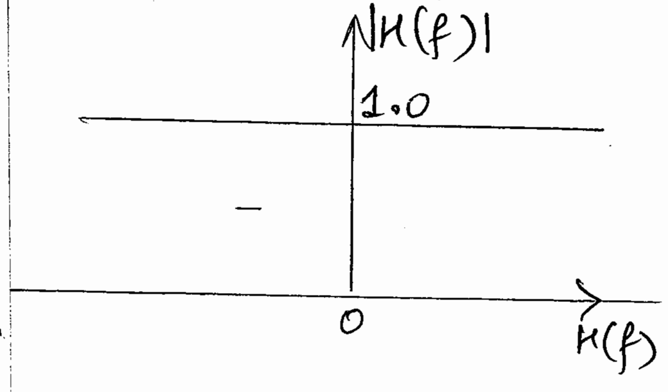
$$\mathrm{Sgn}(f) = \begin{cases} 1, & f > 0 \\ 0, & f = 0 \\ -1, & f < 0 \end{cases} \longrightarrow (3a)$$

- ∴ It follows that the FT $\hat{G}(f)$ of $\hat{g}(t)$ is given by

$$\boxed{\hat{G}(f) = -j\,\mathrm{sgn}(f)\,G(f)} \longrightarrow ④$$

Equation (4) States that given a Fourier Transformable Signal $g(t)$, we may obtain the Fourier Transform of its Hilbert Transform $\hat{g}(t)$ by passing $g(t)$ through LTI system whose frequency response is equal to $-j\,sgn(f)$.

- This System may be considered as one that produces a phase shift of -90° for all positive frequencies of the input Signal and $+90^\circ$ degrees for all negative frequencies as Shown in fig (1).
- The amplitudes of all frequency components in the Signal, are unaffected by transmission through the device. Such an ideal System is referred to as a Hilbert Transformer or Quadrature filter.



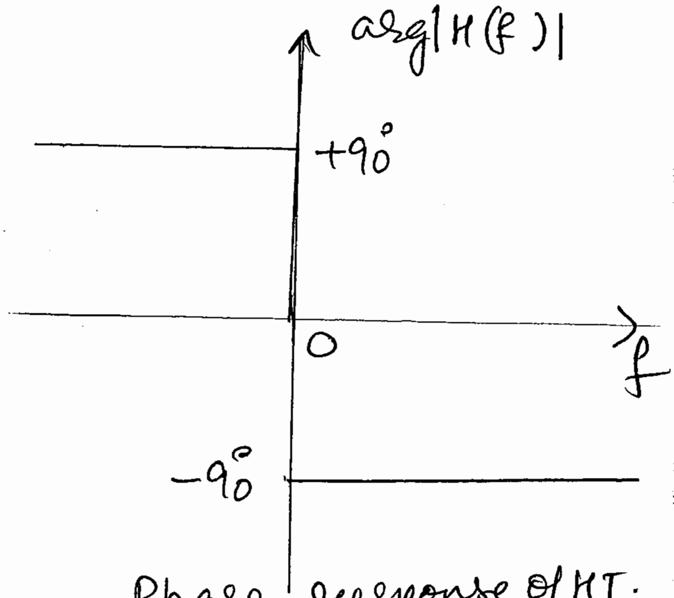


fig (1)

Magnitude Response of HT

Phase response of HT.

HILBERT TRANSFORM OF BASIC SIGNALS

	TIME FUNCTION	HILBERT TRANSFORM
1	$m(t)\cos(2\pi f_c t)$	$m(t)\sin(2\pi f_c t)$
2.	$m(t)\sin(2\pi f_c t)$	$-m(t)\cos(2\pi f_c t)$
3.	$\cos(2\pi f_c t)$	$\sin(2\pi f_c t)$
4.	$\sin(2\pi f_c t)$	$-\cos(2\pi f_c t)$
5.	$\frac{\sin t}{t}$	$\frac{1-\cos t}{t}$
6	$rect(t)$	$-\frac{1}{\pi}\ln\left[\frac{t-1/2}{t+1/2}\right]$
7.	$\delta(t)$	$\frac{1}{\pi t}$
8.	$\frac{1}{1+t^2}$	$\frac{t}{1+t^2}$
9	$\frac{1}{t}$	$-\pi\delta(t)$

PROPERTIES OF HILBERT TRANSFORM.

- The Hilbert Transform operates exclusively in the time domain. The signal $g(t)$ is assumed to be real valued, which is the usual domain of application of the Hilbert Transform.

- For this class of signals, the Hilbert Transform has the following properties.

Property 1: A signal $g(t)$ and its Hilbert Transform $\hat{g}(t)$ have the same magnitude spectrum.

$$\boxed{|G(f)| = |\hat{G}(f)|}$$

Property 2: If $\hat{g}(t)$ is the Hilbert transform of $g(t)$, then the Hilbert transform of $\hat{g}(t)$ is $-g(t)$.

$\angle G(f)$ phase

$$\boxed{\arg|G(f)| = -\arg|\hat{G}(f)|}$$

$= -\angle\hat{G}(f)$

Property 3: A signal $g(t)$ and its Hilbert transform $\hat{g}(t)$ are orthogonal over the entire time interval $(-\infty, \infty)$

$$\boxed{\int_{-\infty}^{\infty} g(t)\,\hat{g}(t)\,dt = 0}.$$

HILBERT TRANSFORM OF LOW PASS SIGNAL.

- Consider fig (2) that depicts the FT of a low pass signal $g(t)$, whose frequency content extends from $-W$ to W.

$g(t) \longleftrightarrow G(f)$

$\hat{g}(t) \longleftrightarrow \hat{G}(f) = -j\,\mathrm{sgn} f \cdot G(f)$

$G(f) = FT[g(t)]$

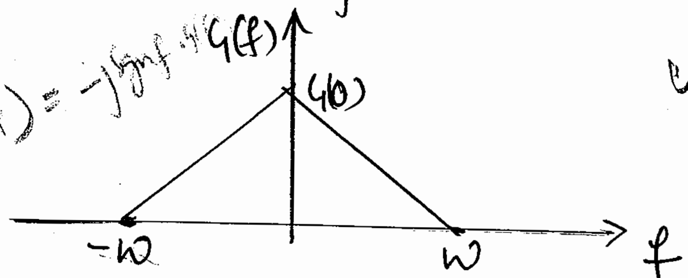


fig (2): Spectrum of Signal $g(t)$

- Applying Hilbert transform to this signal yields a new signal $\hat{g}(t)$ whose fourier transform $\hat{G}(f)$ is as shown in fig ③.

$\hat{G}(f) = \underbrace{-j\,\mathrm{sgn}(f)}_{H(f)}\, G(f)$

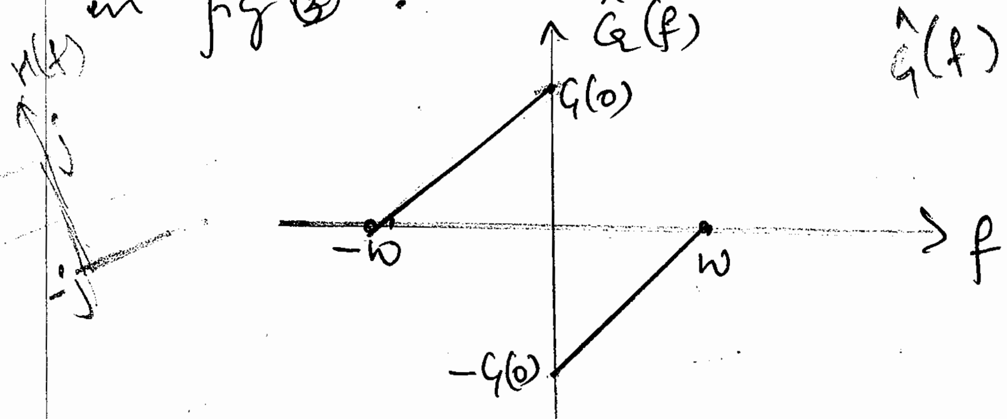


fig (3) : Spectrum of Signal $\hat{g}(t)$

- The figure illustrates that the frequency content of a Fourier Transformable Signal can be radically changed as a result of Hilbert Transformation.

PRE-ENVELOPES.

- The Hilbert transform of a signal is defined for both positive and negative frequencies.

PRE-ENVELOPE FOR +VE FREQUENCIES.

Analytic Signal

A complex valued signal called pre-envelope of g(t) is formally defined as

$$g_+(t) = g(t) + j\hat{g}(t) \longrightarrow (5)$$

where $\hat{g}(t)$ - Hilbert Transform of g(t)

- According to the definition, the given signal g(t) is the real part of the pre-envelope $g_+(t)$ and Hilbert Transform $\hat{g}(t)$ is the imaginary part of the pre envelope.

- Let $G_+(f)$ is the Fourier Transform of $g_+(t)$. Then using eqn (4) & eqn (5) we may write,

$$G_+(f) = G(f) + j\,(-j\,\mathrm{sgn}(f))\,G(f)$$

$$= G(f) + \mathrm{sgn}(f)\,G(f) \longrightarrow (6)$$

- Invoking the definition of signum function given in eqn (3a) we may rewrite eqn (6) as

$$G_+(f) = \begin{cases} 2\,G(f) & f>0 \\ G(0) & f=0 \\ 0 & f<0 \end{cases} \longrightarrow (7)$$

where $G(0)$ is the value of $G(f)$ at the origin $f=0$.

- Equation (7) clearly shows that the preenvelope of the signal $g(t)$ has no frequency content (i.e, its Fourier Transform vanishes) for all -ve frequencies.

- For a given signal $g(t)$, we may determine its pre-envelope $g_+(t)$ in one of 2 equivalent procedures.

(1) Time Domain Procedure :- Given the signal $g(t)$, we use $g_+(t) = g(t) + j\,\hat{g}(t)$ to compute the pre-envelope $g_+(t)$.

(2) Frequency Domain Procedure :- We first determine FT of $g(t)$: $G(f)$ then use

$$G_+(f) = \begin{cases} 2G(f) & f>0 \\ G(0) & f=0 \\ 0 & f<0 \end{cases}$$ to determine $G_+(f)$.

and finally evaluate the inverse Fourier

Transform $G_+(f)$ to obtain

$$g_+(t) = 2\int_0^\infty G(f)\, e^{j2\pi ft}\, df \longrightarrow (8)$$

PRE-ENVELOPE FOR -VE FREQUENCIES.

- We may define the pre-envelope for negative frequencies as

$$g_-(t) = g(t) - j\hat{g}(t) \longrightarrow (9)$$

- The 2 preenvelopes $g_+(t)$ & $g_-(t)$ are simply complex conjugate of each other, as shown by

$$g_-(t) = g_+^*(t) \longrightarrow (10)$$

$*$ → complex conjugate

- The spectrum of the preenvelope $g_+(t)$ is nonzero only for +ve frequencies, hence the use of <u>+ sign</u> as the subscript.

- The use of <u>- sign</u> is to indicate that the spectrum of the pre-envelope $g_-(t)$ is nonzero only for -ve frequencies as shown below

$$G_-(f) = \begin{cases} 0 & f > 0 \\ G(0) & f = 0 \\ 2G(f) & f < 0. \end{cases} \longrightarrow (11)$$

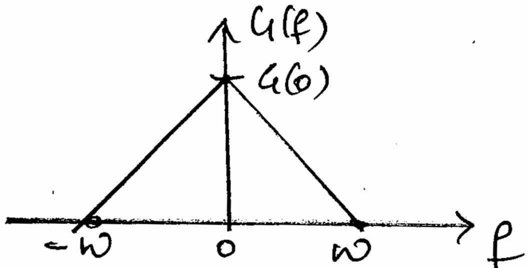
Thus the pre-envelope $g_+(t)$ and $g_-(t)$ constitute a complementary pair of complex valued signals.

The sum of $g_+(t)$ and $g_-(t)$ is exactly twice the original signal $g(t)$.

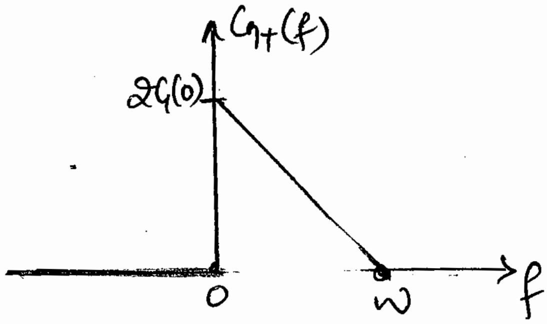
"The spectral content of a Fourier Transformable real-valued signal for positive frequencies uniquely defines that signal".

PRE-ENVELOPES OF LOW PASS SIGNAL.

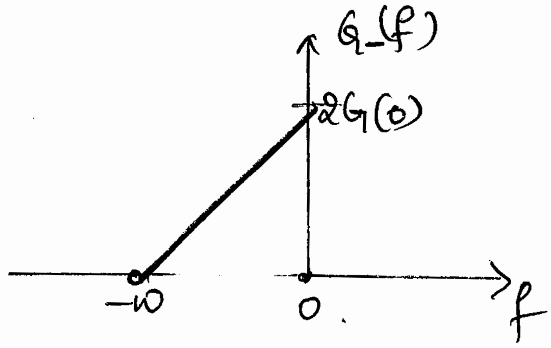
Example: consider a low pass signal $g(t)$ with its Fourier Transform $G(f)$ as shown.



Spectrum of preenvelope $g_+(t)$ and $g_-(t)$ is as shown.



Spectrum of $g_+(t)$
$(0 \leq f \leq W)$



Spectrum of $g_-(t)$
$(-W \leq f \leq 0)$

PRACTICAL IMPORTANCE OF HILBERT TRANSFORMATION.

- The pre-envelope simplifies the analysis of band pass signals and band pass systems in signal theory.
- By applying the concept of pre-envelope to a Bandpass signal, the signal is transformed into an equivalent low-pass representation.
- In a corresponding way, a bandpass filter is transformed into its own equivalent low-pass representation.
- Both transformations, rooted in the Hilbert Transform, play a key role in the formulation of modulated signals and their demodulation.

COMPLEX ENVELOPES OF BAND PASS SIGNALS.

- Pre-envelopes applies to any real valued signal - low pass or Band pass; only requirement is that the signal be Fourier Transformable.
- we will restrict attention to bandpass signals / system because it is directly applicable to analog modulation and digital modulation schemes.

Hence forth we will use $\underline{\underline{s(t)}}$ to denote a modulated signal. When such a signal is applied to the input of a Band-pass system, such as a communication channel, we will use $\underline{\underline{x(t)}}$ to denote the resulting system output (channel output). And as before we will use $h(t)$ as the impulse response of the system.

$$s(t) \longrightarrow \boxed{\quad h(t) \quad} \longrightarrow x(t)$$

Bandpass s/m & commn channel

$s(t) \rightarrow$ Modulated signal

$x(t) \rightarrow$ channel op.

- Let the bandpass signal $s(t)$ and its fourier Transform is represented by $S(f)$.
- Assume that FT $S(f)$ is essentially confined to a band of frequencies of total extent $2W$, center about some frequency $\pm f_c$ as shown in fig (1).

where $f_c \rightarrow$ Carrier frequency.

- Bandwidth $2W$ is small compared with f_c. So we may refer $s(t)$ as a narrow band signal.
- The terms narrowband and bandpass are used interchangeably.

Let the Pre-envelope of the narrow band signal $s(t)$ be expressed in the form

$$s_+(t) = \tilde{s}(t)\exp(j2\pi f_c t) \longrightarrow (12)$$

where $\tilde{s}(t)$ = complex envelope of band pass signal $s(t)$.

Eqn (12) may be viewed as the basis of a definition for the complex envelope $\tilde{s}(t)$ in terms of the preenvelope $s_+(t)$.

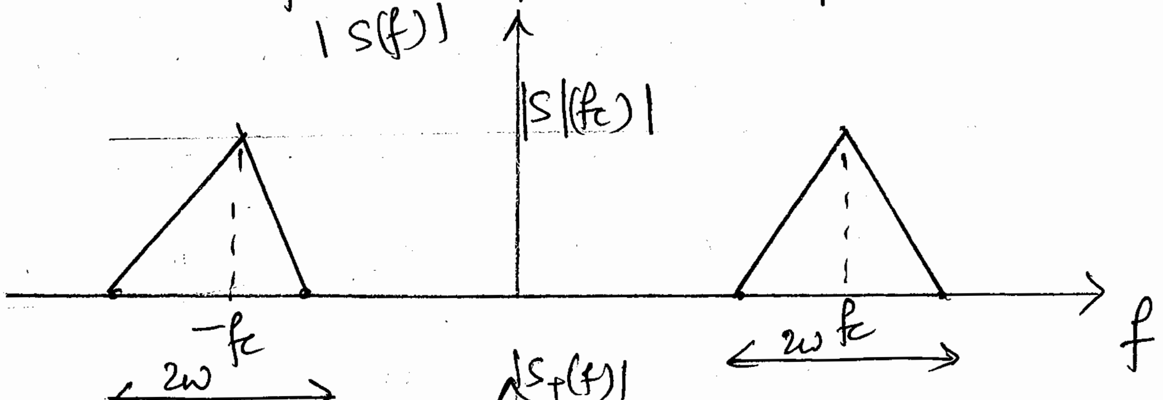


fig (1) FT of s(t)

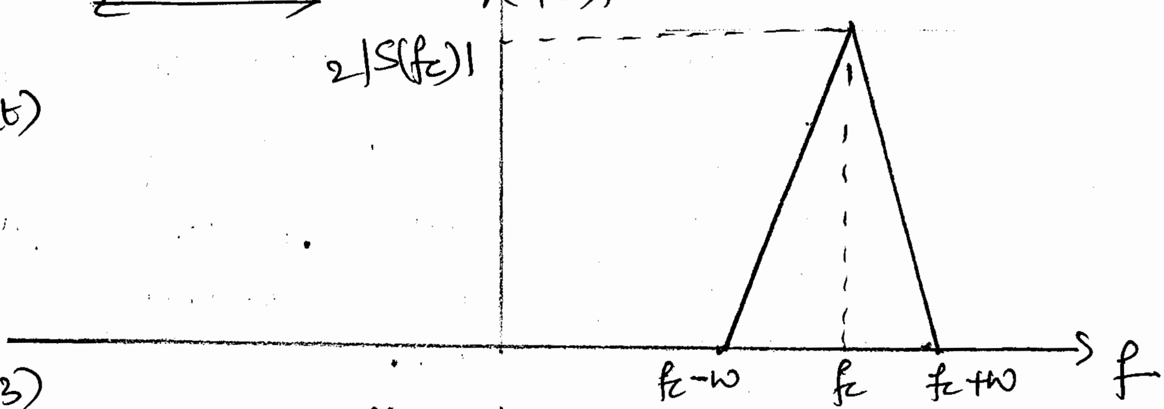


fig (2) FT of $s_+(t)$

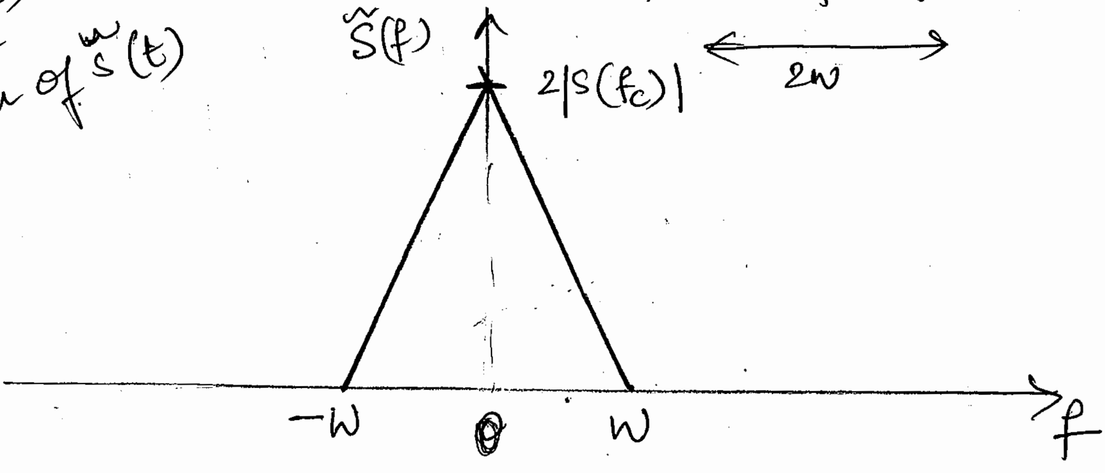


fig (3) Magnitude spectrum of $\tilde{s}(t)$

- We find that the spectrum of the pre-envelope $S_+(t)$ is limited to the positive frequency band $(f_c - W) \leq f \leq (f_c + W)$ as shown in fig (2).

- Applying the frequency shifting property of the Fourier Transform to eqn (12) we find that the spectrum of the complex envelope $\tilde{s}(t)$ is correspondingly limited to the band $(-W \leq f \leq W)$ and centered at the origin $f = 0$ as shown in fig (3).

- The complex envelope $\tilde{s}(t)$ of the bandpass signal $s(t)$ is a complex low pass signal.

- Mapping from Bandpass signal $s(t)$ to the complex low pass signal $\tilde{s}(t)$ is summarised as follows.

(a) Information content of the modulated signal $s(t)$ is fully preserved in the complex envelope $\tilde{s}(t)$.

(b) Analysis of the band pass signal $s(t)$ is complicated by the presence of f_c. In contrast the complex envelope $\tilde{s}(t)$ dispenses with f_c making its analysis simpler to deal with.

(c) The use of $\tilde{s}(t)$ requires having to handle complex notations.

CANONICAL REPRESENTATION OF BAND PASS SIGNALS

- By definition, the real part of the preenvelope $S_+(t)$ is equal to the original bandpass signal $s(t)$.
- We may therefore express the band pass signal $s(t)$ in terms of its corresponding complex envelope $\tilde{s}(t)$ as

$$s(t) = \text{Re}\left[\tilde{s}(t)\exp(j2\pi f_c t)\right] \longrightarrow (13)$$

where Re[·] denotes the real part of the quantity

- Since in general $\tilde{s}(t)$ is the complex valued quantity, we can express $\tilde{s}(t)$ in the Cartesian form as

$$\tilde{s}(t) = s_I(t) + js_Q(t) \longrightarrow (14)$$

where $s_I(t)$ & $s_Q(t)$ are both real valued low pass functions.

- We may therefore use eqn (13) & (14) to express the original band-pass signal $s(t)$ in the Canonical & Standard form

$$s(t) = \text{Re}\left[(s_I(t) + js_Q(t))\exp(j2\pi f_c t)\right]$$

$$= \text{Re}\left[s_I(t)e^{j2\pi f_c t} + js_Q(t)e^{j2\pi f_c t}\right]$$

$$= \text{Re}\left[s_I(t) + js_Q(t)\right]e^{j2\pi f_c t}$$

$$s(t) = \operatorname{Re}\,[S_I(t) + j\,S_Q(t)]\,[\cos 2\pi f_c t + j \sin 2\pi f_c t]$$

$$= \operatorname{Re}\left\{ S_I(t)\cos 2\pi f_c t + j\,S_I(t)\cos 2\pi f_c t + j\,S_Q(t)\sin 2\pi f_c t - S_Q(t)\sin 2\pi f_c t \right\}$$

$$\boxed{s(t) = S_I(t)\cos 2\pi f_c t - S_Q(t)\sin 2\pi f_c t} \longrightarrow (15)$$

$s(t) \longrightarrow$ PASSBAND SIGNAL WAVE FORM $[f_c >> W]$

where $S_I(t) \rightarrow$ In phase component of the bandpass signal $s(t)$

$S_Q(t) \rightarrow$ quadrature phase component or quadrature component of signal $s(t)$.

- If $\cos 2\pi f_c t * S_I(t) \rightarrow$ reference sinusoidal carrier

  $\sin 2\pi f_c t * S_Q(t) \rightarrow$ is in phase quadrature with respect to $(\cos 2\pi f_c t)$.

- According to eqn (13), the complex envelope $\tilde{s}(t)$ may be pictured as time varying phasor positioned at the origin of the (S_I, S_Q)-plane as shown in fig (4).

- Fig (5) represents the phasor representation of the complex exponential $2\pi f_c t$.

- By the eqn (13), the complex envelope $\tilde{S}(t)$ is multiplied by the complex exponential $\exp(j2\pi f_c t)$. The angles of these 2 phasors add and their lengths multiply as shown in fig (6).
- We show that (S_I, S_Q) - phase rotating with an angular velocity equal to $2\pi f_c$ radians per second.
- The complex envelope $\tilde{S}(t)$ moves in (S_I, S_Q) plane and at the same time the plane itself rotates about the origin.
- The original bandpass signal $S(t)$ is the projection of this time varying phasor on a fixed line representing the real axis as shown in fig (6).

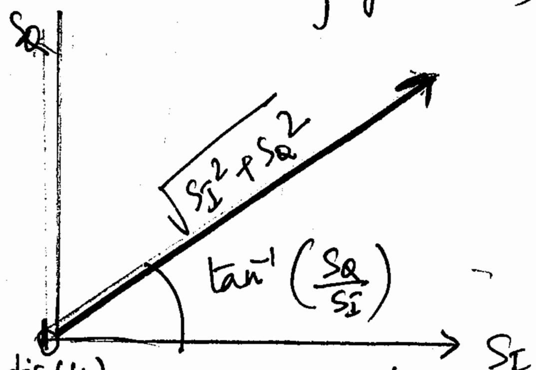


fig (4)

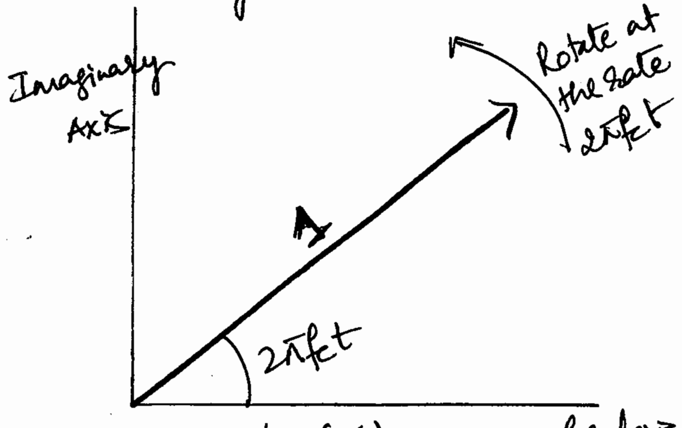


fig (5)

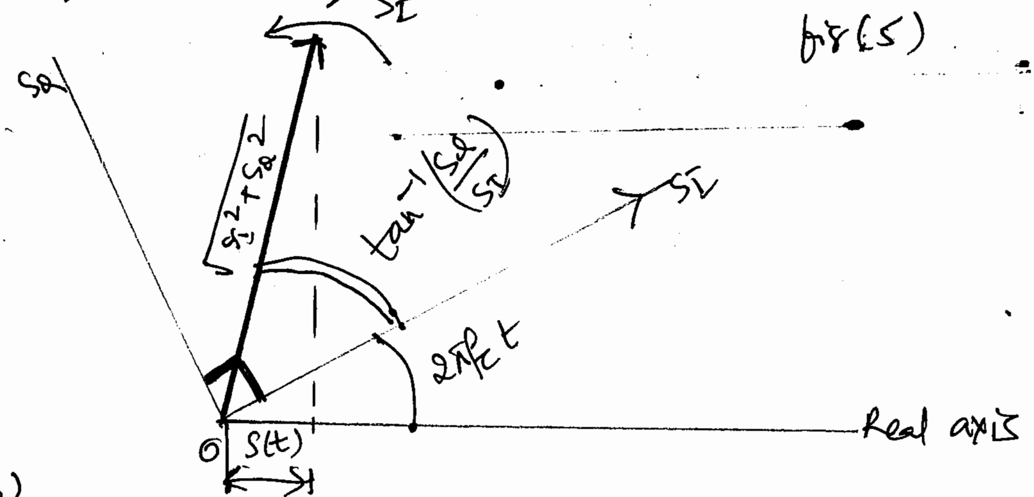
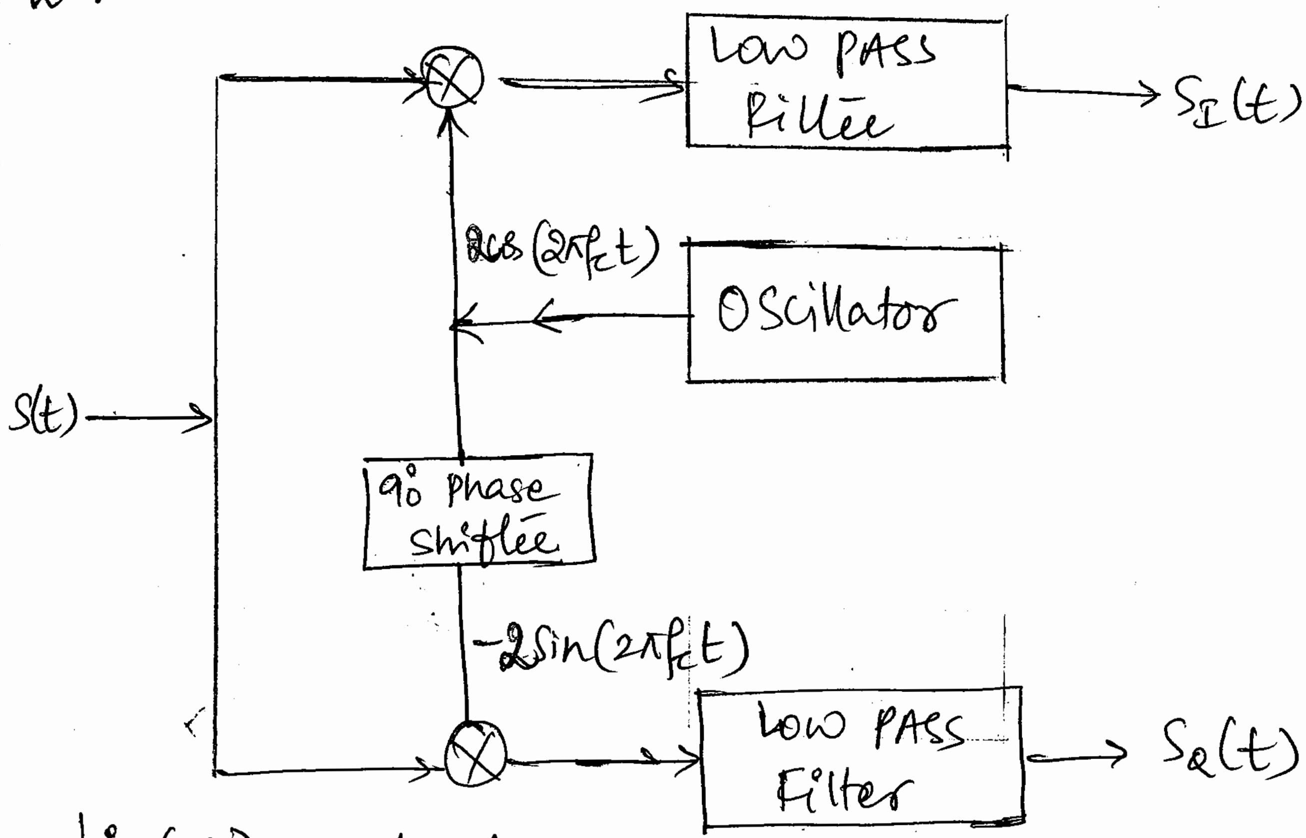


fig (6)

Deriving $S_I(t)$ & $S_Q(t)$ from $s(t)$ ANALYSER

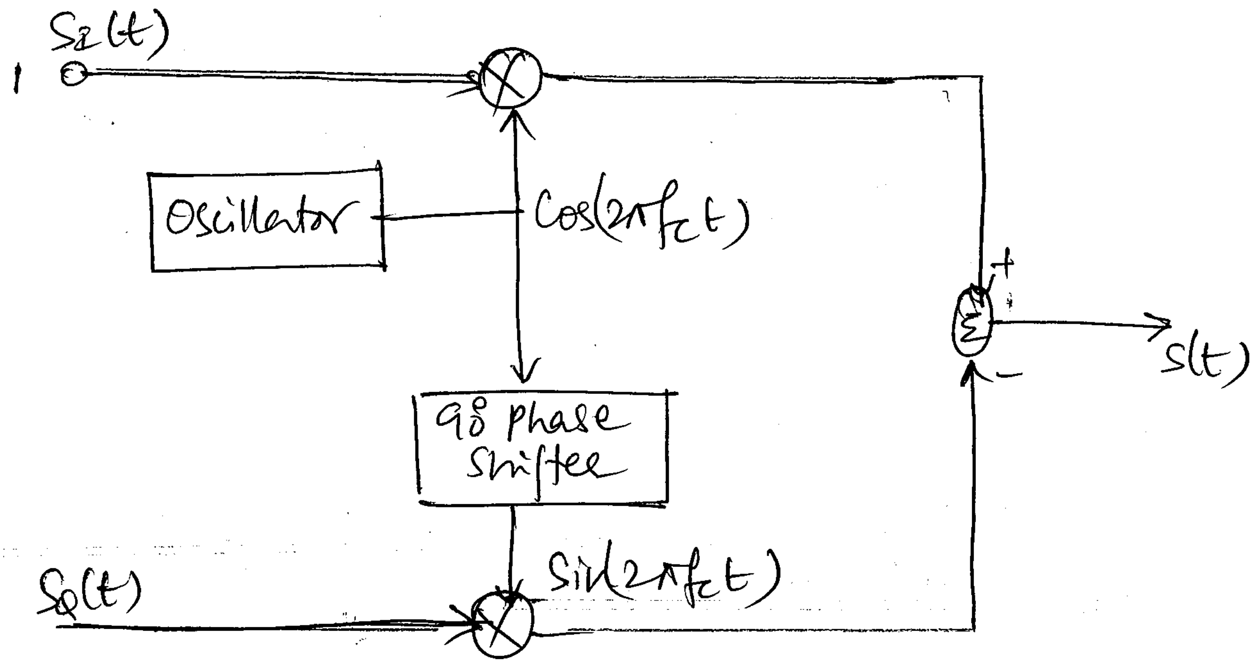
- $S_I(t)$ and $S_Q(t)$ are low-pass signals limited to the band $-W \le f \le W$, may be extracted from the band pass signal $s(t)$ using the scheme as shown in fig (7).
- Both low pass filters are designed identically each with a Bandwidth equal to W.



fig(7): Analyser.

- Reconstructing $s(t)$ from $S_I(t)$ & $S_Q(t)$ [SYNTHESISER]

- To reconstruct $s(t)$ from its inphase and quadrature components, we may use the scheme as shown in fig (8).



fig(8): SYNTESISER

- The mapping from $S_I(t)$ and $S_Q(t)$ combined into $s(t)$ is known as passband modulation.

POLAR REPRESENTATION OF BAND PASS SIGNALS

W.K.T. Cartesian form of the complex envelope $\tilde{s}(t)$ of the bandpass signal $s(t)$ is given by

$$\tilde{s}(t) = S_I(t) + j\,S_Q(t) \rightarrow (16)$$

we may write/define $\tilde{s}(t)$ in the polar form as

$$\tilde{s}(t) = a(t)\exp(j\phi(t)) \rightarrow (17)$$

where $a(t) \rightarrow$ Natural envelope of bandpass signal $s(t)$.

$\phi(t) \rightarrow$ phase of the signal.

- Both $a(t)$ & $\phi(t)$ are real valued lowpass functions.

- Based on the polar representation, the original band pass signal s(t) is itself defined by

$$\boxed{\tilde{s}(t) = a(t)\cos\left[2\pi f_c t + \phi(t)\right]} \longrightarrow (18)$$

RELATION BETWEEN CARTESIAN & POLAR REPRESENTATION OF BAND PASS SIGNAL.

- The envelope a(t) and phase $\phi(t)$ of a band pass signal s(t) are respectively related to inphase and quadrature components $s_I(t)$ & $s_Q(t)$ as follows.

$$\boxed{\begin{aligned} a(t) &= \sqrt{s_I^2(t) + s_Q^2(t)} \\ \phi(t) &= \tan^{-1}\left(\frac{s_I(t)}{s_Q(t)}\right) \end{aligned}} \longrightarrow (19)$$

Conversely

$$\boxed{\begin{aligned} s_I(t) &= a(t)\cos\left[\phi(t)\right] \\ s_Q(t) &= a(t)\sin\left[\phi(t)\right] \end{aligned}} \longrightarrow (20)$$

LINE CODES.

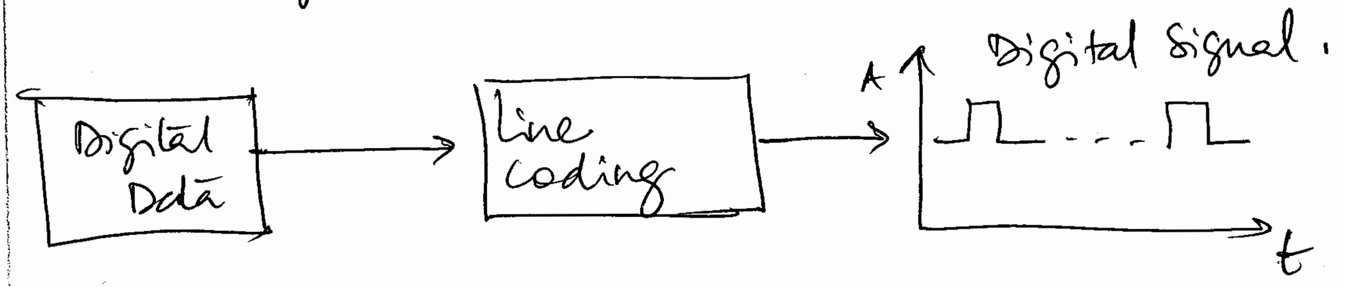
- A computer network is used for communication of data from one station to another station in the network.
- Analog or digital data traverses through a communication media in the form of a signal from the source to the destination.
- The channel bridging the transmitter and the receiver may be a guided transmission medium (wire, waveguide, space)
- Irrespective of the medium, the signal traversing the channel becomes attenuated and distorted with increasing distance.
- Hence a process is adopted to match the properties of the transmitted signal to the channel characteristics so as to efficiently communicate over the transmission media.
- There are 2 alternatives: the data can be either converted to digital or analog signal.
- For digital signalling, the data source can be either analog or signal, which is encoded into digital signal using different encoding techniques.

In Analog Signalling, data can be transmitted using a carrier signal by a process called modulation (where one or more parameters of the carrier wave i.e, amplitude, frequency, phase are being modulated by the source data). The resulting signal called modulated signal, traverses the media. The signal is demodulated at the recieving end and the original signal is extracted.

<u>Various Approaches of data into signal</u>

Data	Signal	Approach	
Digital	Digital	Encoding	1
Analog	Digital	Encoding	
Analog	Analog	Modulation	
Digital	Analog	Modulation	

The 1 approach converts digital data to digital signal known as line coding.



Line coding Techniques can be broadly classified into

1. unipolar
2. Polar — NRZ (Non return to zero), RZ (Return to zero), Manchester, Differential Manchester
3. Bipolar

① Unipolar NRZ Signalling

In this line code, Symbol 1 is represented by transmitting a pulse of Amplitude A for the duration of the symbol, and symbol 0 is represented by switching off the pulse, as in fig ①.

– The unipolar NRZ line code is also referred to as on-off signalling.

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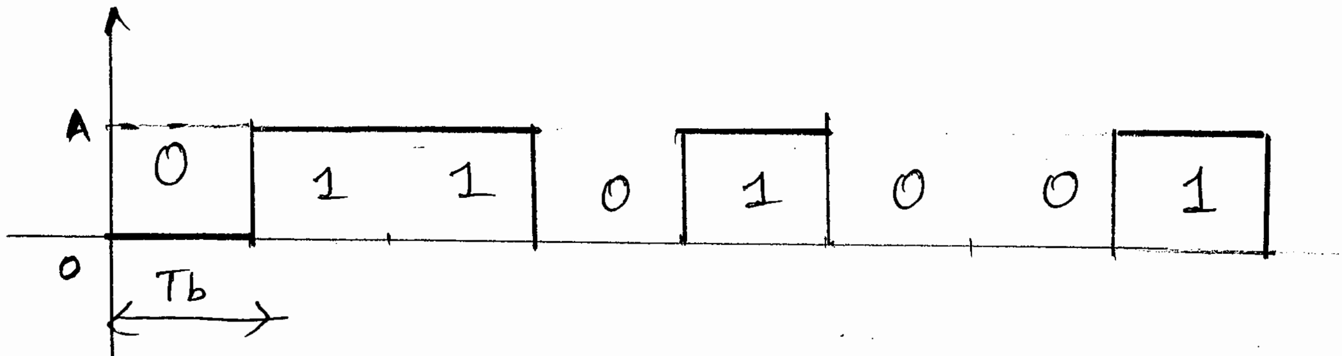
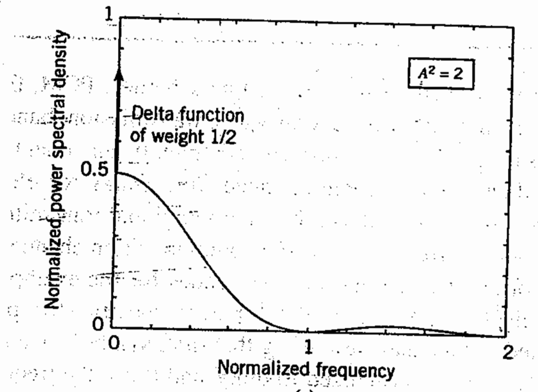


fig (1): Unipolar NRZ Signalling.

Disadvantages of on-off signalling are the waste of power due to the transmitted DC level and the power spectrum of the transmitted signal does not approach zero at zero frequency.



(a)

(2) POLAR NRZ SIGNALLING

- In this line code, symbols 1 & 0 are represented by transmitting pulses of amplitudes +A and −A respectively as shown in fig (2).

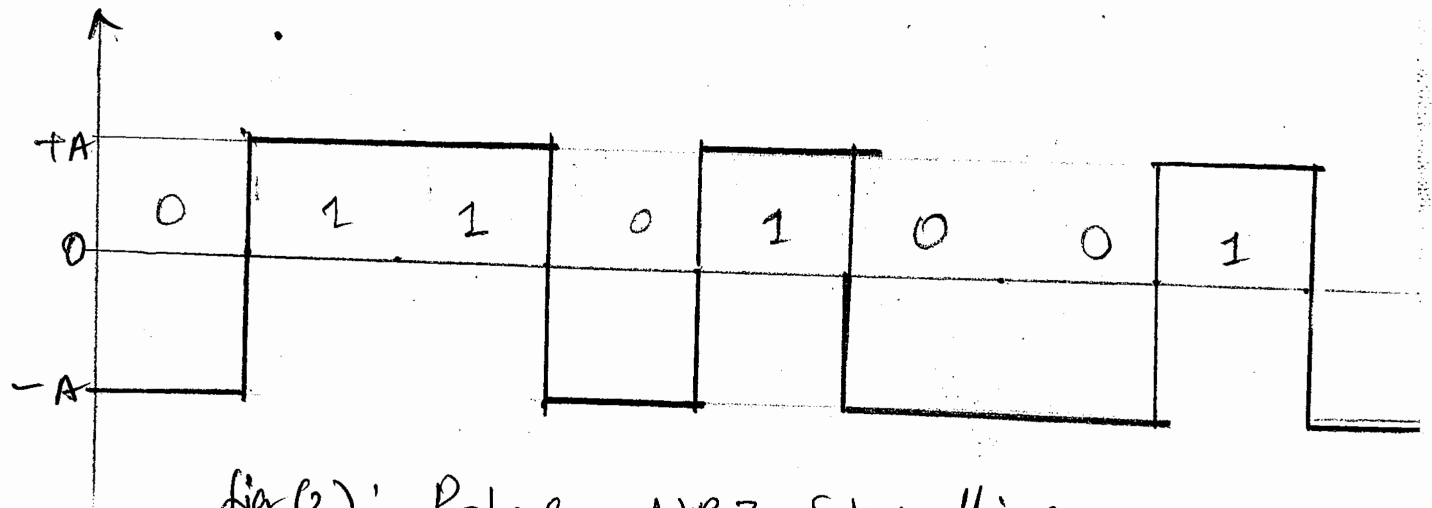
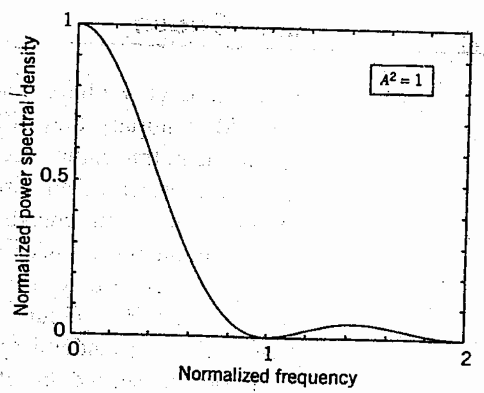


fig (2): Polar NRZ Signalling.

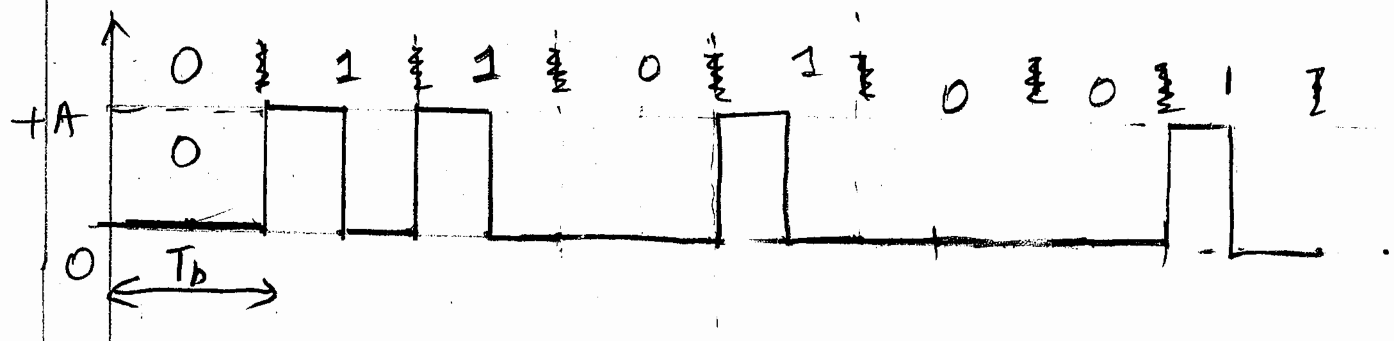
- The polar NRZ is relatively easy to generate, but the disadvantage is the power spectrum of the signal is large near zero frequency as shown in fig (5).



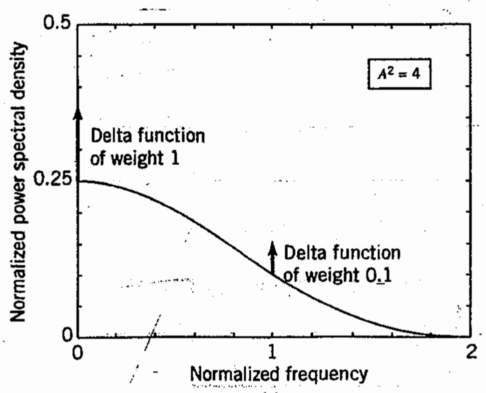
(b)

③ UNIPOLAR RZ SIGNALLING.

- In this line code, symbol 1 is represented by a rectangular pulse of Amplitude A and half symbol width.
- symbol 0 is represented by transmitting no pulse as shown in fig (3).



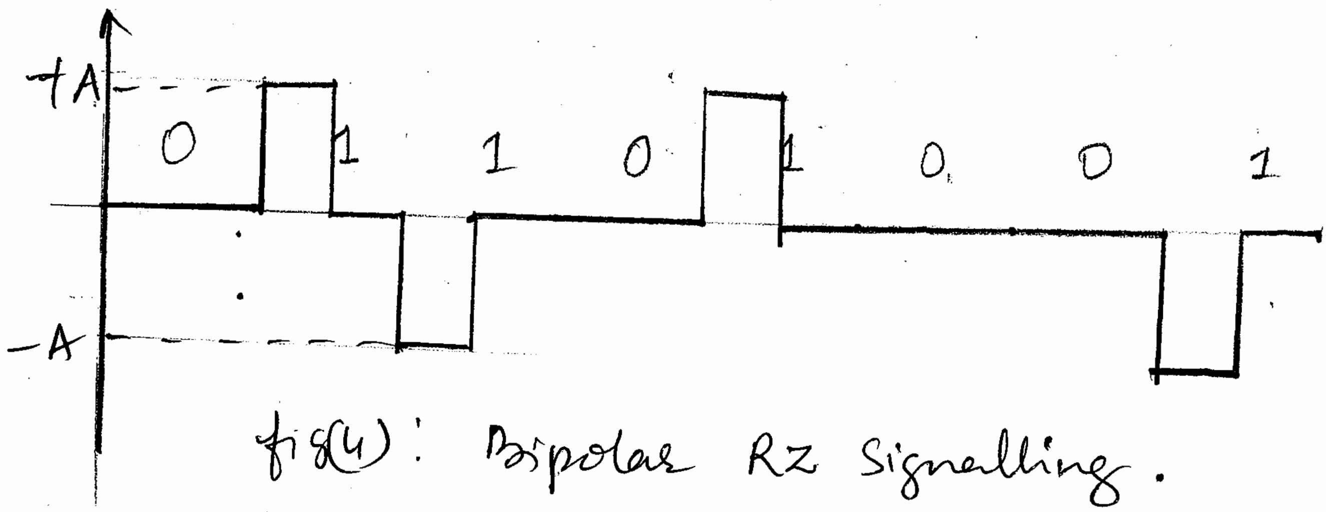
- In unipolar RZ line code, a delta function at $f=0$, $\pm 1/T_b$ is present in the power spectrum of the transmitted signal. The delta function can be used for bit timing recovery at the receiver (fig(C)). Its disadvantage is that it requires 3dB more power than polar RZ signalling for the same probability of symbol error.



(c)

(4) BIPOLAR RZ SIGNALLING

- This line code uses 3 amplitude levels as shown in fig (4).
- Specifically positive and negative pulses of equal amplitude (+A and −A) are used alternately for symbol 1, with each pulse having a half symbol width.
- No pulse is used for symbol 0.



fig(4): Bipolar RZ Signalling.

- Power spectrum of the transmitted signal has no DC component and relatively insignificant low-frequency components for the case when symbols 1 & 0 occur with equal probability (fig (d) next page)
- The bipolar RZ line code is also called alternate mark inversion (AMI) signaling.

(5) MANCHESTER (SPLIT PHASE) SIGNALING.

- In this method of Signaling, Symbol 1 is represented by a positive pulse of Amplitude A followed by a negative pulse of Amplitude -A, with both pulses being half-symbol wide.

- For Symbol 0, the polarities of these two pulses are reversed. fig (5)

- A unique property of the Manchester code is that it suppresses the DC component and has relatively insignificant low-frequency components, regardless of the Signal statistics. This property is essential in some applications.

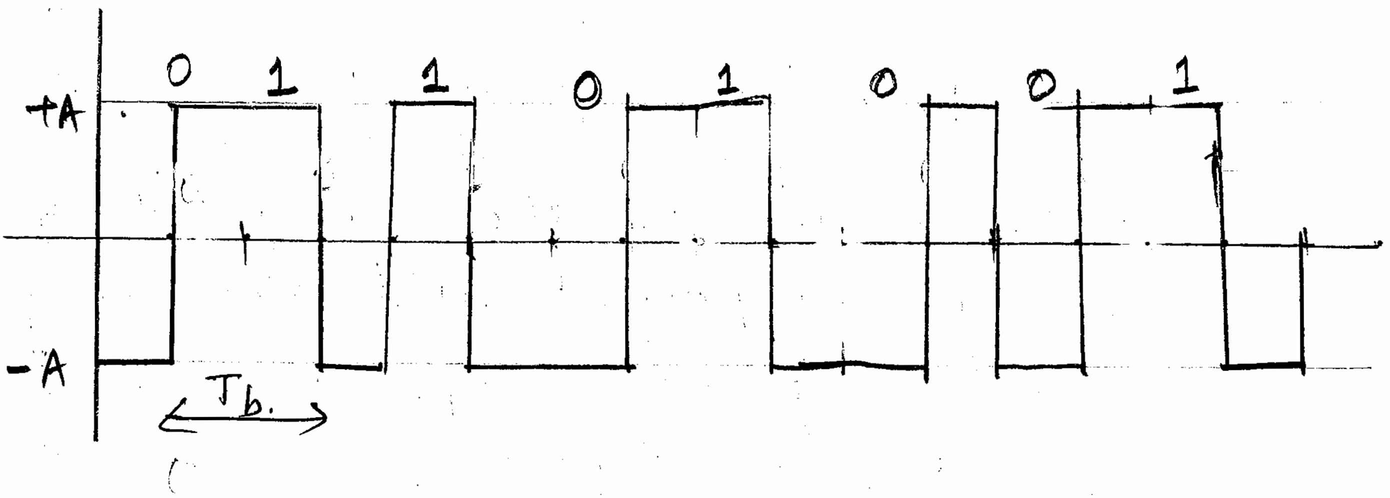
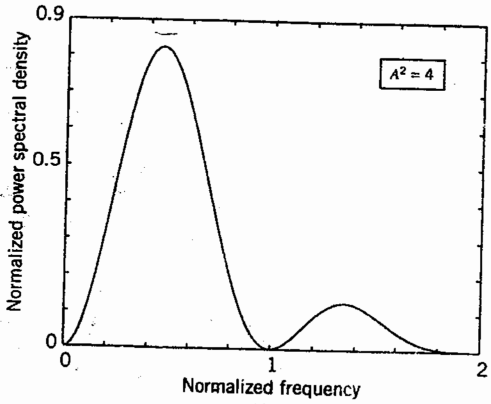
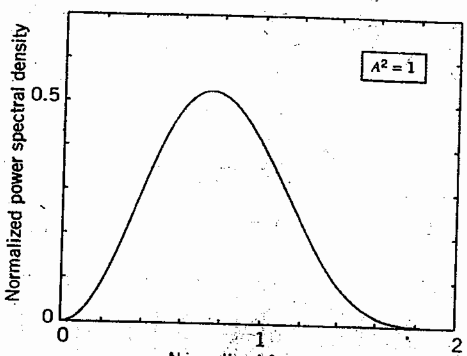


fig (5): Manchester encoded signal.



(d)



(e)

fig(d) : Bipolar RZ Signalling.

fig(e) : Manchester Signaling

Problems on Hilbert Transform

① $x(t) = \cos 2\pi ft + \sin 2\pi ft$

Solⁿ $\hat{x}(t) = \cos(2\pi ft - \pi/2) + \sin(2\pi ft - \pi/2)$

$= \cos(\pi/2 - 2\pi ft) + \sin(\pi/2 - 2\pi ft)$

$= \sin 2\pi ft - \cos 2\pi ft$

② $x(t) = e^{-j2\pi ft}$

$x(t) = \cos 2\pi ft - j\sin 2\pi ft$

$\hat{x}(t) = \cos(2\pi ft - \pi/2) - j\sin(2\pi ft - \pi/2)$

$= \cos(\pi/2 - 2\pi ft) + j\sin(\pi/2 - 2\pi ft)$

$= \sin 2\pi ft + j\cos 2\pi ft$

$= j(-j\sin 2\pi ft + \cos 2\pi ft)$

$= j\, e^{-j2\pi ft}$

③ $x(t) = \delta(t)$

$\hat{x}(t) = x(t) * \frac{1}{\pi(t)} = \delta(t) * \frac{1}{\pi(t)}$

$\hat{x}(t) = \frac{1}{\pi t}$ [convolution of $\delta(t)$ is identity operator

④ $x(t) = \frac{1}{1+t^2}$

$\hat{x}(t) = x(t) * \frac{1}{\pi t}$

$= \frac{1}{1+t^2} * \frac{1}{\pi t}$

2

$$\hat{x}(t) = \int_{-\infty}^{\infty} \frac{1}{1+\tau^2}\cdot\frac{1}{\pi(t-\tau)}\,d\tau$$

$$= \frac{1}{\pi}\int_{-\infty}^{\infty}\frac{1}{1+\tau^2}\cdot\frac{1}{t-\tau}\,d\tau \rightarrow 1$$

let $\dfrac{1}{(1+\tau^2)(t-\tau)} = \dfrac{-1}{(1+\tau^2)(\tau-t)}$

$$= \frac{A\tau+B}{1+\tau^2} + \frac{C}{\tau-t}$$

$$A = \frac{1}{1+t^2}, \quad B = \frac{t}{1+t^2} \quad C = \frac{-1}{1+t^2}$$

$$1 \Rightarrow \hat{x}(t) = \frac{1}{\pi}\int_{-\infty}^{\infty}\left(\frac{\tau/1+t^2 + t/1+t^2}{1+\tau^2} - \frac{1/1+t^2}{\tau-t}\right)d\tau$$

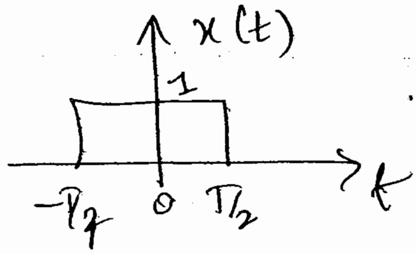
$$= \frac{1}{\pi}\left(\frac{1}{1+t^2}\right)\overbrace{\cancel{\int_{-\infty}^{\infty}\frac{\tau}{1+\tau^2}\,d\tau}}^{=0} + \frac{1}{\pi(1+t^2)}\overbrace{\int_{-\infty}^{\infty}\frac{t}{1+\tau^2}\,d\tau}^{=\pi}$$

$$- \frac{1}{\pi(1+t^2)}\overbrace{\cancel{\int_{-\infty}^{\infty}\frac{1}{\tau-t}\,d\tau}}^{=0} \rightarrow 2$$

$$= \frac{1\ \pi}{\pi(1+t^2)} = \underline{\underline{\frac{1}{1+t^2}}}$$

(3)

(3)



Sol^n

$$\hat{x}(t) = x(t) * \frac{1}{\pi t}$$

$$= \int_{-\infty}^{\infty} x(\tau) \frac{1}{\pi(t-\tau)} \, d\tau$$

$$= -\frac{1}{\pi} \int_{-T/2}^{T/2} \frac{1}{(\tau - t)} \, d\tau .$$

$$= \frac{-1}{\pi} \ln(\tau - t) \Big|_{-T/2}^{T/2}$$

$$= \frac{-1}{\pi} \left[\ln\left(T/2 - t\right) - \ln\left(-T/2 - T\right) \right]$$

$$= \frac{-1}{\pi} \left[\ln \frac{(T/2 - t)}{(-T/2 - t)} \right]$$

$$= \frac{1}{\pi} \left[\ln \left(\frac{-T/2 - t}{T/2 + t} \right) \right]$$

$$\hat{x}(t) = \frac{1}{\pi} \ln \left(\frac{T/2 + t}{t - T/2} \right)$$

b) For the AM Signal $s(t) = m(t)[\cos(2\pi f_c t + \phi)]$

find, the following:

(a) Pre-envelope (b) complex envelope

(c) Inphase & quadrature components

(d) natural envelope.

Sol^n: (a) Preenvelope of $s(t)$ is

$$s_+(t) = s(t) + j\,\hat{s}(t)$$

$$= m(t)\cos(2\pi f_c t + \phi) + j\,m(t)\sin(2\pi f_c t + \phi)$$

$$= m(t)\left[\cos(2\pi f_c + \phi) + j\sin(2\pi f_c t + \phi)\right]$$

$$= m(t)\,e^{j(2\pi f_c t + \phi)}$$

(b) complex envelope

$$\tilde{s}(t) = s_+(t)\,e^{-j2\pi f_c t}$$

$$= m(t)\cdot e^{j2\pi f_c t}\cdot e^{j\phi}\cdot e^{-j2\pi f_c t}$$

$$= m(t)\,e^{j\phi}.$$

(c) $\tilde{s}(t) = s_I(t) + s_Q(t)$

$$m(t)(\cos\phi + j\sin\phi) = s_I(t) + s_Q(t)$$

$$s_I(t) = m(t)\cos\phi \qquad s_Q(t) = m(t)\sin\phi.$$

(d) Natural Envelope

$$a(t) = |s_+(t)| = m(t)$$

(7) $s(t) = \cos[2\pi f_c t + \phi(t)]$ find

a) pre envelope b) complex envelope

c) Natural envelope d) Inphase and quadrature component

Sol^n a) $s_+(t) = s(t) + j\hat{s}(t)$

$$= [\cos 2\pi f_c t + \phi(t)] + j\left[\cos(2\pi f_c t + \phi(t) - \pi/2\right.$$

$$= \cos[2\pi f_c t + \phi(t)] + j[\sin(2\pi f_c t + \phi(t))]]$$

$$= e^{j[2\pi f_c t + \phi(t)]}$$

b) $\tilde{s}(t) = s_+(t) e^{-j2\pi f_c t}$

$$= e^{j\phi(t)}$$

c) $a(t) = |s_+(t)| = |\tilde{s}(t)| = 1.$

d) $\tilde{s}(t) = s_I(t) + js_Q(t)$

$\cos\phi(t) + j\sin\phi(t) = s_I(t) + j\, s_Q(t)$

$\therefore$ $s_I(t) = \cos\phi(t)$

$s_Q(t) = \sin\phi(t)$

(6)

(8) $x(t) = \cos 2\pi f_1(t) \sin 2\pi f_2(t)$ $f_1 >> f_2$

Find the Hilbert Transform of the Signal.

Soln

$$m(t)\widehat{c(t)} = m(t)\hat{c}(t)$$

$$\hat{x}(t) = \underbrace{\cos(2\pi f_1 t - \pi/2)}_{\text{carrier}} \underbrace{\sin(2\pi f_2 t)}_{\text{message}}$$

$$\hat{x}(t) = \sin(2\pi f_1 t) \cdot \sin(2\pi f_2(t))$$

DIGITAL COMMUNICATION

MODULE 2

SIGNALING OVER AWGN CHANNELS

AWGN: Additive White Gaussian Noise

AWGN is a basic noise model used in information theory. It is a generally accepted model with following assumptions.

- The noise is additive
  i.e. the received signal equals the transmit signal plus some noise, where the noise is statiscally independent of the signal.

- The noise is white
  i.e. the power spectral density is flat, so the autocorrelation of the noise in time domain is zero. White refers to the idea that it has uniform power across the frequency band for the information system.

- The noise samples have a Gaussian distribution

AWGN Model of a channel.

Let $s_1(t)$ and $s_2(t)$ be the two distinct signals ~~denoted by~~ suitable for transmission over the analog signal.

The energy of the signal $s_i(t)$ of duration T_b seconds is given by

$$E_i = \int_0^{T_b} s_i^2(t)\, dt, \quad i = 1, 2 \qquad \text{—— (1)}$$

Let $x(t)$ be the received signal and $w(t)$ be the channel noise.

An AWGN model of a channel can be depicted as follows

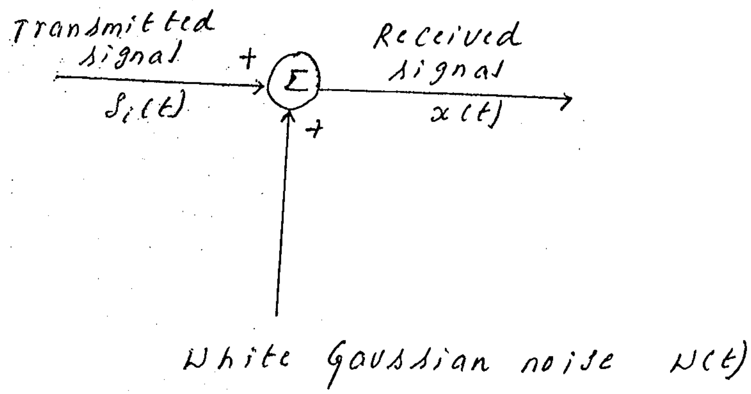


Fig: AWGN model of a channel

From the above figure, the received signal can be defined as follows

$$x(t) = S_i(t) + w(t) \quad \begin{cases} 0 \le t \le T_b \\ i = 1, 2 \end{cases}$$

The receiver observes the received signal $x(t)$ for a duration of T_b seconds and then makes an estimate of the transmitted signal $S_i(t)$. However, due to the presence of channel noise, the receiver will make occasional errors. Therefore, the requirement is to design the receiver to minimize the average probability of symbol error.

Geometric representation of signals

Let us consider M symbols from the input information sequence.

Any set of M energy signals $\{S_i(t)\}$, $1 \le i \le M$ can be expressed as linear combination of N orthonormal basis function, where $N \le M$.

If $S_1(t), S_2(t), \ldots, S_M(t)$ are real valued energy signals, each of duration 'T' sec, then

$$S_i(t) = \sum_{j=1}^{N} S_{ij}\, \phi_j(t), \quad \begin{cases} 0 \le t \le T \\ i = 1, 2, \ldots, M \end{cases} \longrightarrow (2)$$

where

S_{ij} are scalar coefficients

and $\phi_j(t)$ are basis function.

Scalar coefficient S_{ij} is given by

$$S_{ij} = \int_0^T S_i(t)\, \phi_j(t)\, dt \quad \begin{cases} i = 1, 2, \ldots, M \\ j = 1, 2, \ldots, N \end{cases} \longrightarrow (3)$$

Let $\phi_1(t), \phi_2(t), \ldots, \phi_N(t)$ be the real valued basis function, which form an orthonormal set.

$$\int_0^T \phi_i(t)\,\phi_j(t)\,dt = \delta_{ij} = \begin{cases} 1 & \text{if } i=j \\ 0 & \text{if } i\neq j \end{cases} \longrightarrow (4)$$

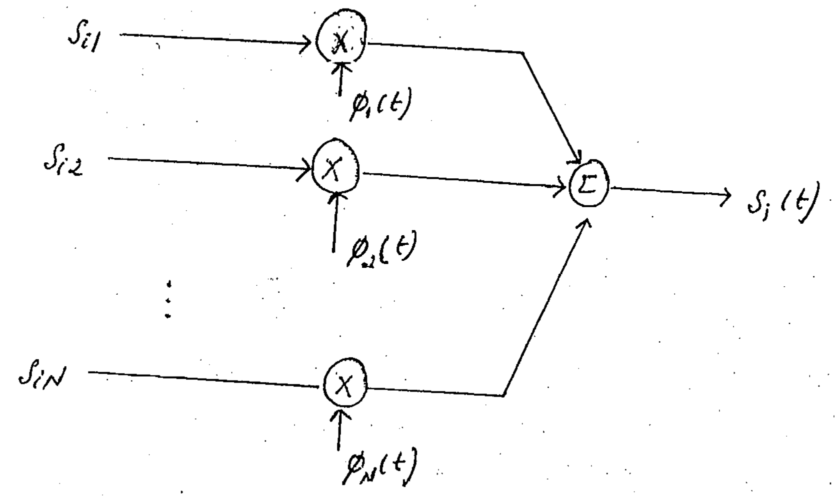
where δ_{ij} is the Kronecker delta function.

Note

1. Each basis function has unit energy over the symbol duration 'T'.
2. The basis functions $\phi_1(t), \phi_2(t), \ldots, \phi_N(t)$ are orthogonal with respect to each other over the interval $0 \le t \le T$

<u>Synthesizer for generating the signal $s_i(t)$</u>

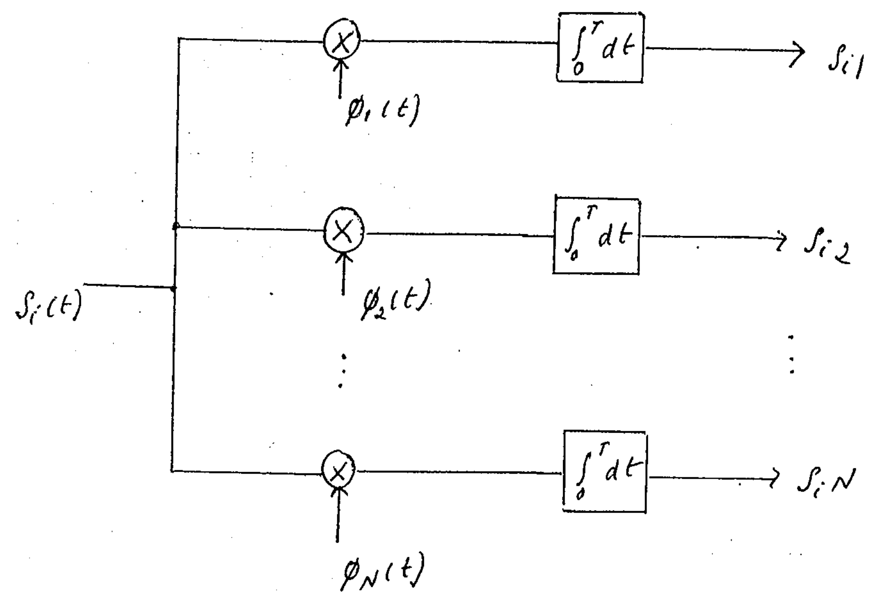
Given the N elements of a vector s_i operating as input, and if the basis functions are known, the energy signal $s_i(t)$ can be generated using the following figure.



The figure consists of a bank of N multipliers with each multiplier having its own basis function followed by a summer. The above scheme is also called as synthesizer.

Analyzer for reconstructing the signal:

Given the signals $s_i(t)$, $i = 1, 2, \ldots, M$, operating as input and if the basis functions are known, the coefficients $s_{i1}, s_{i2}, \ldots, s_{iN}$ can be calculated using the following figure.



The scheme consists of a bank of N product integrators with a common input, and with each one of them supplied with its own basis function. The above scheme is also known as analyzer.

* Prove that the energy of an energy signal $s_i(t)$ is equal to the squared length of the corresponding signal vector $s_i(t)$.

Each signal in the set $\{s_i(t)\}$ is completely determined by the signal vector.

$$s_i = \begin{bmatrix} s_{i1} \\ s_{i2} \\ \vdots \\ s_{iN} \end{bmatrix} \qquad i = 1, 2, \ldots M \qquad \text{--- (5)}$$

Let $\|s_i\|$ be the absolute value of the signal vector s_i.

The squared length of any signal vector s_i is defined to be the inner product or dot product of s_i with itself

i.e. $\|s_i\|^2 = s_i^T s_i$ where T denotes matrix transposition

$$= \sum_{j=1}^{N} s_{ij}^2 \qquad i = 1, 2, \ldots, M \quad \text{— (6)}$$

where s_{ij} is the j^{th} element of s_i.

We know that energy of a signal is given by

$$E_i = \int_0^T s_i^2(t)\,dt \quad \text{— (7)}$$

where T is duration

Substituting equation (2) in eqn (7)

$$E_i = \int_0^T \left[\sum_{j=1}^{N} s_{ij}\,\phi_j(t)\right]\left[\sum_{k=1}^{N} s_{ik}\,\phi_k(t)\right] dt$$

Interchanging the order of summation and integration

$$E_i = \sum_{j=1}^{N} \sum_{k=1}^{N} s_{ij}\, s_{ik} \int_0^T \phi_j(t)\phi_k(t)\,dt$$

from eqn (4)

$$E_i = \sum_{j=1}^{N} s_{ij}^2$$

$$= \|s_i\|^2 \quad \rightarrow (8)$$

Above equation indicates that the energy signal $s_i(t)$ is equal to squared length of the corresponding signal vector $s_i(t)$.

Formulas to remember

1. Let $s_i(t)$ and $s_k(t)$ represent the pair of signals. Then

$$\|s_i - s_k\|^2 = \sum_{j=1}^{N} (s_{ij} - s_{kj})^2 \quad \rightarrow (9)$$

$$= \int_0^T (s_i(t) - s_k(t))^2\,dt$$

where $\|s_i - s_k\|$ is the Euclidean distance between points represented by signal vectors s_i and s_k.

2. Let θ_{ik} be the angle subtended between two signal vectors s_i and s_k.

$$\cos(\theta_{ik}) = \frac{s_i^T s_k}{\|s_i\|\,\|s_k\|} \quad \rightarrow (10)$$

3. $\int_0^T s_i(t)\, s_k(t)\, dt = s_i^T s_k \longrightarrow (11)$

The Schwarz Inequality

Consider any pair of energy signals $s_1(t)$ and $s_2(t)$. The Schwarz inequality states that

$$\left(\int_{-\infty}^{\infty} s_1(t)\, s_2(t)\, dt\right)^2 \leq \left(\int_{-\infty}^{\infty} s_1^2(t)\, dt\right)\left(\int_{-\infty}^{\infty} s_2^2(t)\, dt\right)$$

The equality holds if and only if, $s_2(t) = c\, s_1(t)$, where c is any constant.

Proof:

Let $s_1(t)$ and $s_2(t)$ be expressed in terms of the pair of orthonormal basis functions $\phi_1(t)$ and $\phi_2(t)$ as follows.

$$s_1(t) = s_{11}\, \phi_1(t) + s_{12}\, \phi_2(t)$$
$$s_2(t) = s_{21}\, \phi_1(t) + s_{22}\, \phi_2(t)$$

where $\phi_1(t)$ and $\phi_2(t)$ satisfy the orthonormality conditions over the time interval $(-\infty, \infty)$

$$\int_{-\infty}^{\infty} \phi_i(t)\, \phi_j(t)\, dt = \delta_{ij} = \begin{cases} 1 & \text{for } i=j \\ 0 & \text{otherwise} \end{cases}$$

Signals $s_1(t)$ and $s_2(t)$ can be represented using pair of vectors as follows shown in below fig.

$$s_1 = \begin{bmatrix} s_{11} \\ s_{12} \end{bmatrix} \qquad s_2 = \begin{bmatrix} s_{21} \\ s_{22} \end{bmatrix}$$

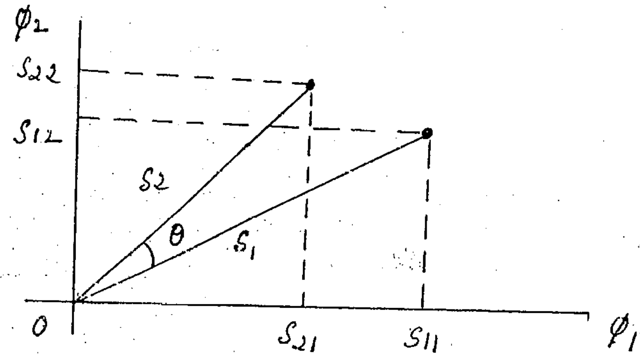


Fig: Vector representation of signals $s_1(t)$ and $s_2(t)$

Using equation (10) we can write

$$\cos\theta = \frac{s_1^T s_2}{\|s_1\| \|s_2\|}$$

Using equation (11), we can write

$$= \frac{\int_{-\infty}^{\infty} s_1(t) s_2(t)\, dt}{\left(\int_{-\infty}^{\infty} s_1^2(t) dt\right)^{1/2} \left(\int_{-\infty}^{\infty} s_2^2(t) dt\right)^{1/2}}$$

Since $|\cos\theta| \leq 1$, above equation can be written as

$$\left(\int_{-\infty}^{\infty} s_1(t)\, s_2(t)\, dt\right)^2 \leq \left(\int_{-\infty}^{\infty} s_1^2(t) dt\right)\left(\int_{-\infty}^{\infty} s_2^2(t)\, dt\right)$$

Gram - Schmidt orthogonalization procedure

Gram - Schmidt orthogonalization is a procedure which takes non orthogonal set of linearly independent functions and constructs an orthogonal basis over an arbitrary interval.

consider a set of M energy signals denoted by $s_1(t), s_2(t), \ldots, s_M(t)$.

Let $\phi_1(t)$ be the first basis function, which is given by

$$\phi_1(t) = \frac{s_1(t)}{\sqrt{E_1}}$$

where E_1 is the energy of the signal $s_1(t)$

Above equation can be rewritten as

$$s_1(t) = \sqrt{E_1}\, \phi_1(t) \longrightarrow (12)$$

Let $s_2(t)$ be the second signal. From eqn (3), scalar coefficient s_{21} is given by

$$s_{21} = \int_0^T s_2(t)\, \phi_1(t)\, dt$$

Let $g_2(t)$ be the new intermediate function which is orthogonal to $\phi_1(t)$ over the interval $0 \le t \le T$.

The second basis function $\phi_2(t)$ can be defined as follows

$$\phi_2(t) = \frac{g_2(t)}{\sqrt{\int_0^T g_2^2(t)\,dt}} \rightarrow (13)$$

from eqn (1) for denominator

where $\phi_1(t)$ and $\phi_2(t)$ satisfy the orthonormality conditions.

i.e. $$\int_0^T \phi_2^2(t)\,dt = 1$$

and $$\int_0^T \phi_1(t)\,\phi_2(t)\,dt = 0$$

In general, given $g_i(t)$, equation (13) can be written as follows

$$\phi_i(t) = \frac{g_i(t)}{\sqrt{\int_0^T g_i^2(t)\,dt}}, \quad j = 1, 2, \ldots, N$$

2B1Q code

Extra Information

Two binary, one quaternary code

In 2B1Q line coding scheme, a block of two binary bits can represent four different values: 00, 01, 10 and 11. These four values are mapped to one quaternary value, which is encoded using four different voltages, as shown below

Table: Amplitude levels of the 2B1Q code

Signal	Amplitude	Gray code
$s_1(t)$	−3	00
$s_2(t)$	−1	01
$s_3(t)$	+1	11
$s_4(t)$	+3	10

The 2B1Q code is used in North America for a special class of modems called digital subscriber lines.

The signal space representation of 2B1Q code can be represented as shown in below figure.

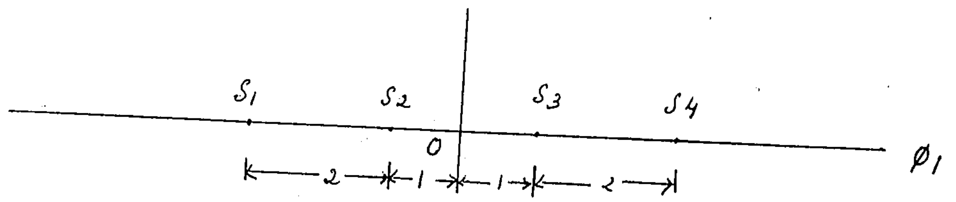
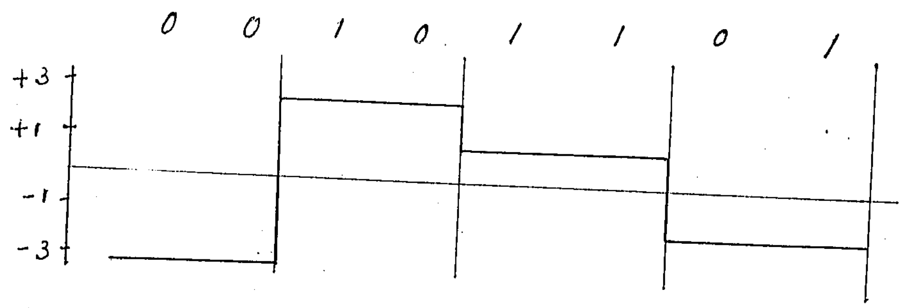


Fig: Signal-space representation of the 2B1Q code.

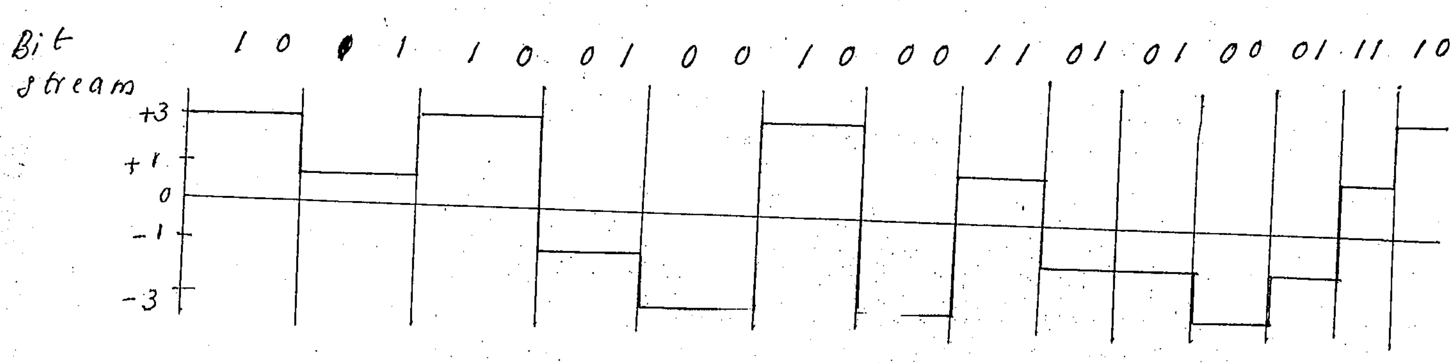
Example 2.1

For the given bit stream 00 10 11 01, write the corresponding 2B1Q code.



Example 2.2

Represent the bit stream 10 11 10 01 00 10 00 11 01 01 00 01 11 10 using 2B1Q code.



Example 2.3

Write the corresponding 2B1Q code for the bit stream 00 01 10 11 01 10 00 01

Extra Information ends

Conversion of the continuous AWGN channel into a vector channel.

Let $s_i(t)$ be the transmitted signal, $w(t)$ be the channel noise and $x(t)$ be the received signal.

From the AWGN model of a channel

$$x(t) = s_i(t) + w(t), \quad \begin{cases} 0 \le t \le T \\ i = 1, 2, \ldots, M \end{cases} \longrightarrow (14)$$

Considering the output of correlator j, its sample value is given by

$$x_j = \int_0^T x(t)\, \phi_j(t)\, dt$$

$$= s_{ij} + w_j \qquad j = 1, 2, \ldots, N \longrightarrow (15)$$

The first component, s_{ij} is the deterministic quantity contributed by the transmitted signal $s_i(t)$

From eqn (13) $s_{ij} = \int_0^T s_i(t)\, \phi_j(t)\, dt \longrightarrow (16)$

The second component, w_j is a random quantity due to channel noise $w(t)$

$$w_j = \int_0^T w(t)\, \phi_j(t)\, dt \longrightarrow (17)$$

Consider a new random process $X'(t)$ whose sample function $x'(t)$ is related to the received signal $x(t)$ as follows

$$x'(t) = x(t) - \sum_{j=1}^{N} x_j\, \phi_j(t) \longrightarrow (18)$$

Substituting eqn (14) & eqn (15) into eqn (18), we get

$$x'(t) = s_i(t) + w(t) - \sum_{j=1}^{N} (s_{ij} + w_j)\, \phi_j(t)$$

$$= s_i(t) + w(t) - \sum_{j=1}^{N} s_{ij}\, \phi_j(t) - \sum_{j=1}^{N} w_j\, \phi_j(t)$$

From equation (2)

$$x'(t) = s_i(t) + w(t) - s_i(t) - \sum_{j=1}^{N} w_j\, \phi_j(t)$$

$$= w(t) - \sum_{j=1}^{N} w_j \phi_j(t)$$

$$x'(t) = w'(t) \longrightarrow (19)$$

Above equation indicates, the sample function $x'(t)$ depends entirely on channel noise $w(t)$.

Rewriting equation (18) and substituting eqn (19), we get

$$x(t) = \sum_{j=1}^{N} x_j \phi_j(t) + x'(t)$$

$$x(t) = \sum_{j=1}^{N} x_j \phi_j(t) + w'(t)$$

Here $w'(t)$ is viewed as a remainder term that must be included on the RHS, to preserve equality. This entire signal is deterministic and is random due to the channel noise at the receiver input.

Extra Information to understand the concepts.

Statistical Characteristics of Correlator outputs.

Let $X(t)$ denote random function of the received signal $x(t)$. Correspondingly, let X_j denote random variable whose samples are represented by correlator o/p x_j, $j = 1, 2, \ldots N$.

According to AWGN model, $X(t)$ is Gaussian process. Hence X_j is characterized completely by its mean, variance, co-variance and finaly probability density function.

1. Mean :- Let denote W_j as Noise random variable having zero mean.

Consequently, then mean of X_j is given as

$$\mu_{X_j} = E[X_j] = E[S_{ij} + W_j] = E[S_{ij}] + E[W_j]$$

$$\Rightarrow \mu_{X_j} = S_{ij}.$$

2. Variance: By defination, $\sigma_{x_j}^2 = \text{var}[x_j]$

$$= E[(x_j - s_{ij})^2]$$

$$= E[W_j^2] \quad \text{since } x_j = s_{ij} + w_j$$

Here $w_j = \int_0^T w(t)\, \phi_j(t)\, dt$.

$$\therefore \sigma_{x_j}^2 = E\left[\int_0^T w(t)\, \phi_j(t)\, dt \int_0^T w(u)\, \phi_j(u)\, du\right]$$

$$= E\left[\int_0^T \int_0^T \phi_j(t)\, \phi_j(u)\, w(t)\, w(u)\, dt\, du\right]$$

$$= \int_0^T \int_0^T \phi_j(t)\, \phi_j(u)\, E[w(t)\, w(u)]\, dt\, du$$

$$= \int_0^T \int_0^T \phi_j(t)\, \phi_j(u)\, R_W(t, u)\, dt\, du.$$

where $R_W(t, u)$ is autocorrelation func^{tn}. & expressed as $R_W(t, u) = \frac{N_0}{2}\, \delta(t - u)$.

Substituting in above eq^n., $\sigma_{x_j}^2$ is simplified as

$$\therefore \sigma_{x_j}^2 = \frac{N_0}{2} \int_0^T \int_0^T \phi_j(t)\, \phi_j(u)\, \delta(t-u)\, dt\, du$$

$$= \frac{N_0}{2} \int_0^T \phi_j^2(t)\, dt \qquad \text{if } t = u,\ \delta(t-u) = 1.$$

Since $\phi_j(t)$ has unit energy, then

$$\sigma_{x_j}^2 = \frac{N_0}{2} \quad \text{for all } j.$$

i.e x_j have a variance equal to power spectral density $N_0/2$ of the noise process $W(t)$.

3. Co-variance :- By defn $\mathrm{cov}[X_j X_k] = E[(X_j - \mu_{X_j})(X_k - \mu_{X_k})]$

$$\therefore \mathrm{cov}[X_j X_k] = E[(X_j - S_{ij})(X_k - S_{ik})]$$

$$= E[W_j W_k]$$

$$= E\left[\int_0^T W(t)\,\phi_j(t)\,dt \int_0^T W(u)\,\phi_k(u)\,du\right]$$

$$= \int_0^T \int_0^T \phi_j(t)\,\phi_k(u)\,R_W(t,u)\,dt\,du$$

$$= \frac{N_0}{2}\int_0^T \int_0^T \phi_j(t)\,\phi_k(u)\,\delta(t-u)\,dt\,du$$

$$= \frac{N_0}{2}[0] \quad \text{if } j \neq k.$$

$$\therefore \mathrm{cov}[X_j X_k] = 0.$$

i.e The Gaussian random variables X_j implies that they are also statistically independent.

Hence can define the vector of N random variables,

$$X = \begin{bmatrix} X_1 \\ X_2 \\ X_3 \\ \vdots \\ X_N \end{bmatrix}$$

whose mean is S_{ij} and variance equal to $\frac{N_0}{2}$.

4) Probability Density function :- Since vector X is statistically independent, conditional pdf of vector X can be expressed assuming m_i symbol was sent, as the product of the conditional pdf of its individual elements.

i.e $f_X(x|m_i) = \prod_{j=1}^{N} f_{X_j}(x_j|m_i)$ $i = 1, 2, \ldots M$.

where x is sample value of X &
x_j is sample value of X_j ..
and x is called observation vector &
x_j is called element of observation vector x.

If the channel satisfies above eqn, then is called memoryless channel.

Since X_j is Gaussian random variable, with mean - s_{ij} and variance - $\frac{N_0}{2}$, The pdf is expressed as

$$f_{X_j}(x_j|m_i) = \frac{1}{\sqrt{2\pi\sigma_{x_j}^2}} \exp\left[\frac{-(x_j - \mu_{x_j})^2}{2\sigma_{x_j}^2}\right]$$

On substituting $\mu_{x_j} = s_{ij}$ & $\sigma_{x_j}^2 = \frac{N_0}{2}$, we get

$$f_{X_j}(x_j|m_i) = \frac{1}{\sqrt{\pi N_0}} \exp\left[\frac{(x_j - s_{ij})^2}{N_0}\right] \quad \text{where } \begin{array}{l} i = 1, 2, \ldots M \\ j = 1, 2, \ldots N \end{array}$$

$$\therefore f_X(x|m_i) = [\pi N_0]^{-\frac{N}{2}} \exp\left[-\frac{1}{N_0}\sum_{j=1}^{N}(x_j - s_{ij})^2\right]$$

where $i = 1, 2, 3, \ldots M$

This eqn. completely characterises the received signal.

Likelihood function: $f_X(x|m_i)$ provides characterization of an AWGN channel, which is dependent on observation vector x assuming that m_i is transmitted. But at the receiver, the situation is complitely opposite, i.e for a given observation vector x, we need to estimate msg symbol m_i, i.e responsible for generating x. This can be achived by likelihood function denoted by $l(m_i)$ & defined as $l(m_i) = f_X(x|m_i)$, $i = 1, 2, \ldots M$.

For convience log-likelihood funtion denoted by $L(m_i)$ & defined by $L(m_i) = \ln l(m_i)$, $i = 1, 2, \ldots M$.

$$\Rightarrow L(m_i) = \ln \left[f_X(x|m_i) \right]$$

$$= \ln \left[(\pi N_0)^{-N/2} \exp\left[-\frac{1}{N_0} \sum_{j=1}^{N} (x_j - s_{ij})^2 \right] \right]$$

Here log-likelihood f^n bears one-to-one relation with likelihood f^n for 2 reasons

1) By def^n. PDF is nonnegative always. Therefore, likelihood f^n is likewise a nonnegative quantity
2) The logarithmic f^n is monotanically increasing f^n of its argument.

Therefore $L(m_i) = -\frac{1}{N_0} \sum_{j=1}^{N} (x_j - s_{ij})^2$, $i = 1, 2, \ldots M$.

Extra Information Ends here

Maximum - Likelihood Detector

→ When the received signal $x(t)$ is applied to the bank of N correlators & its output defines a new vector, x called observation or received vector.

→ For the given observation vector x, we have to perform a mapping from x to an estimate $\hat{m}$ of the transmitted symbol m_i, in order to minimize the average probability of symbol error in the detection ckt.

This procedure is called maximum likelihood detector.

→ Here, the average probability of symbol error is denoted as $P_e(m_i, x)$ or $1 - P_e(x / m_i \text{ sent})$ — (1)

→ Our criterion is to minimize the probability of error in mapping each given observation vector x into a decision, Thus optimum decision rule is as follows:

$$\text{set } \hat{m} = m_i \text{ if}$$

$$P(m_i \text{ sent} / x) \geq P(m_k \text{ sent} / x) \quad \text{for all } k \neq i$$

$$\text{where } k = 1, 2, \ldots M. \quad \text{— (2)}$$

This rule is known as maximum a posteriori probability

→ Eq^n (2) can be expressed in terms of a priori probabilities of the tx-ed signal & in terms of likelihood f^n.

Applying bayes rule; the decision rule is restated as:

$$\text{set } \hat{m} = m_i \text{ if} \quad \text{— (3)}$$

$$\frac{p_k f_x(x / m_k)}{f_x(x)} \text{ is maximum for } k = i$$

where P_k is the priori probability of occurance of symbol m_k,

$f_X(x/m_k)$ is likelihood fn. that results when symbol m_k is transmitted

$f_X(x) \rightarrow$ PDF of random vector x.

→ Suppose, if $f_X(x)$ is independent of m_i & priori probabilities are equiprobable, then rule is restated that as follows:

$$\text{set } \hat{m} = m_i \text{ if} \quad (4)$$
$$f_X(x/m_k) \text{ is maximum for } k = i$$

→ We known that, likelihood fn. is nonnegative & since if $A > B > 0$ then $\ln A > \ln B$. Considering this decision rule is restated as follows:

$$\text{set } \hat{m} = m_i \text{ if} \quad (5)$$
$$\ln[f_X(x/m_k)] \text{ is maximum for } k=i$$

→ This decision rule is called as maximum likelihood and device for its implementation is referred as the maximum-likelihood detector.

→ According to eq.n (5), a maximum-likelihood detector computes the metric for each transmitted message, compares them, and the decides in favor of the maximum.

→ The graphical representation of the maximum-liklihood decision rule is as follows:

Let Z denote the N-dimensional space of all possibly observed vector x. & space is called observation space.

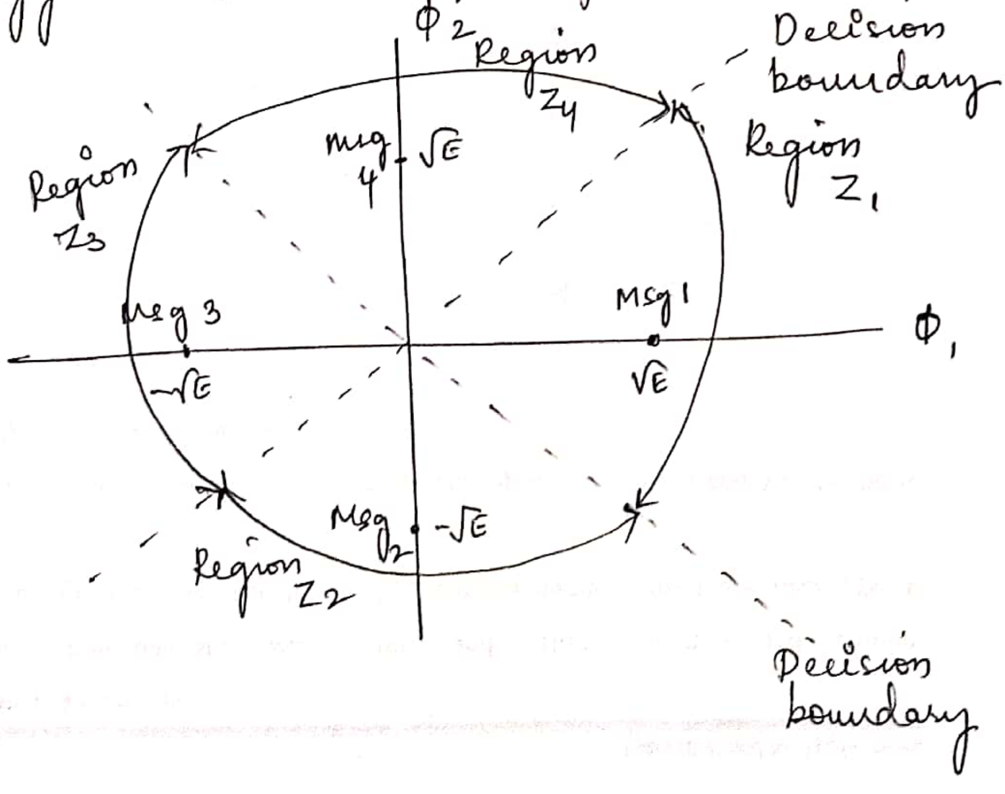
As per the decision rule we say $\hat{m} = m_i$; where $i = 1, 2, \ldots M$

The total observation space Z is correspondingly partitioned into M decision regions denoted as $Z_1, Z_2, \ldots Z_M$.

Now again restating the decision rule eqn ⑤ as follows : vector x lies inside region Z_i if $\ln[f_X(x|m_k)]$ is maximum for $k = i$.

Ex: for an AWGN channel, the decision regions are regions of the N-dimensional observation space Z bounded by linear (N-1)-dimensional hyperplane boundaries.

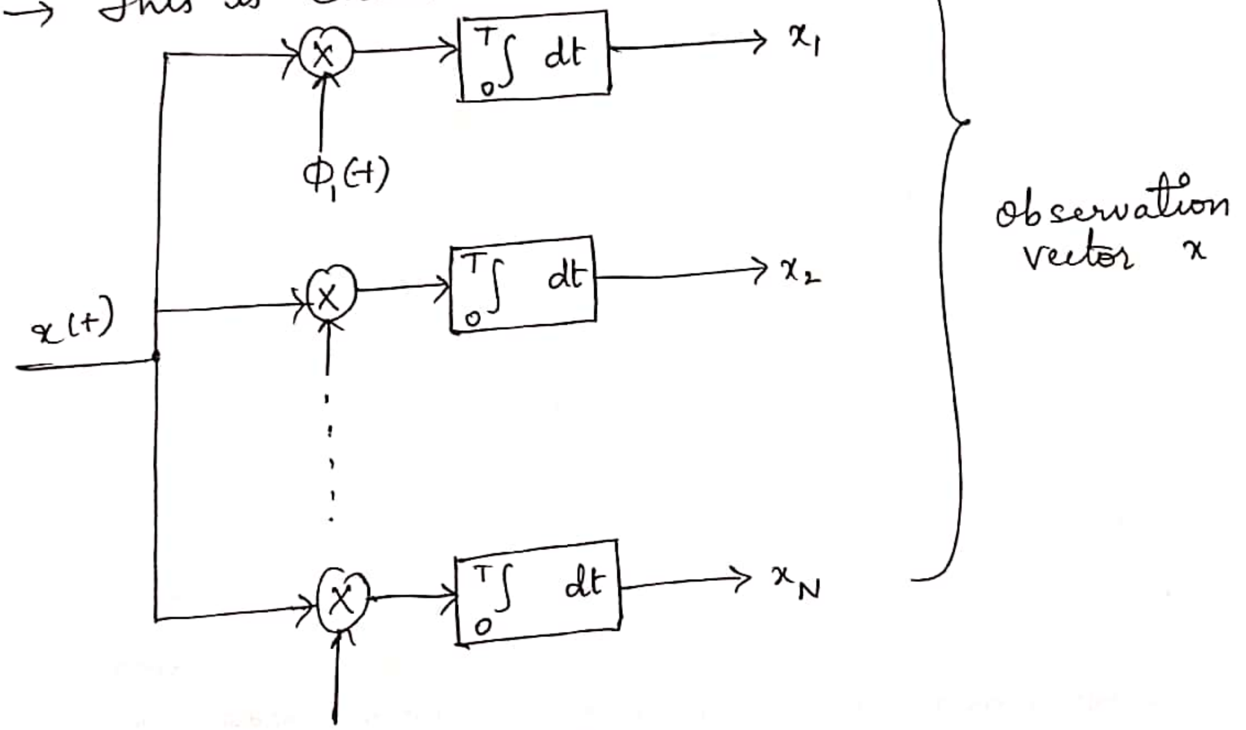
The fig below shows the example of decision regions for M = 4 signals & N = 2 dimensions assuming signals are transmitted with equal energy E & with equal probability.



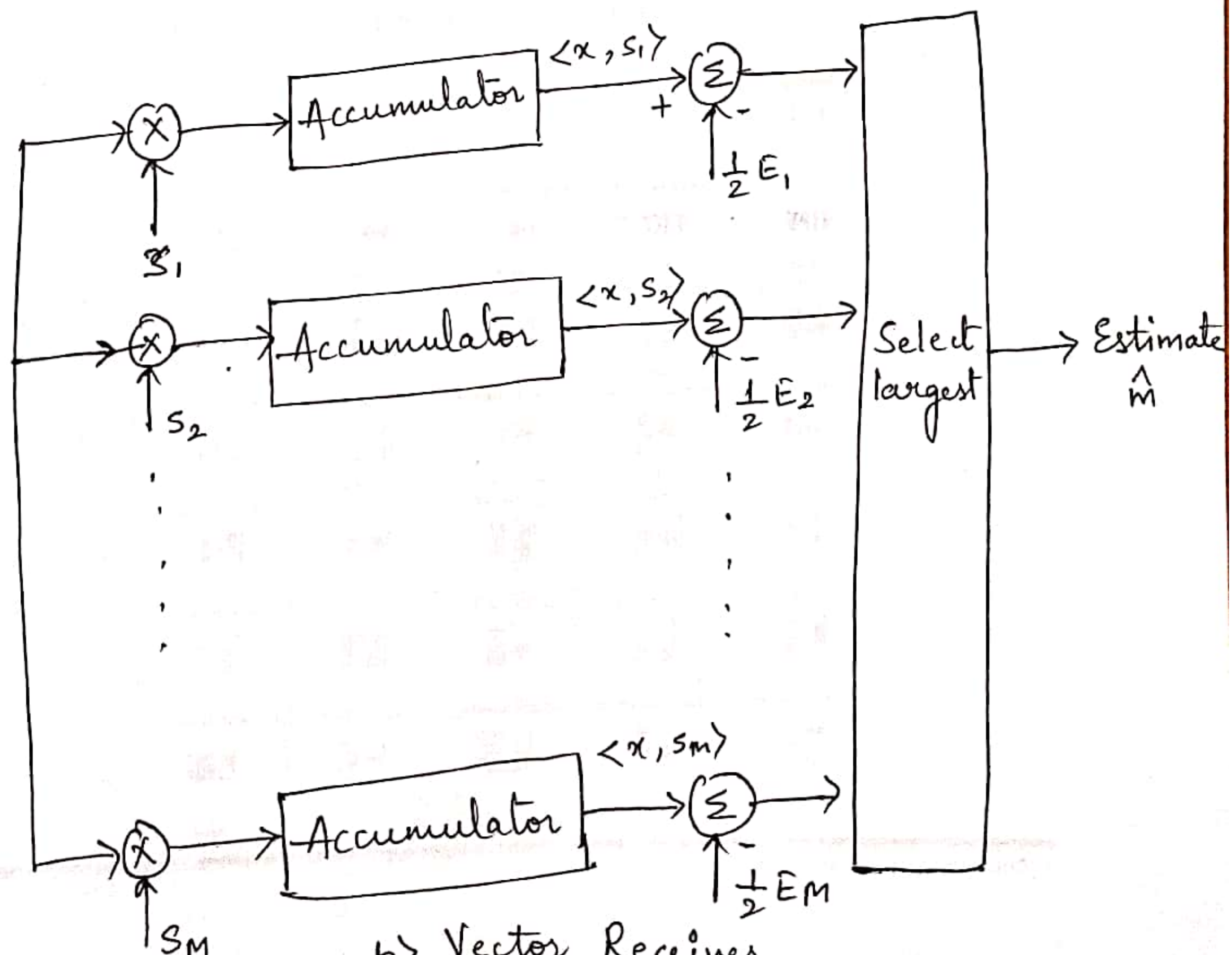
Correlation Receiver

→ The optimum receiver consists of two subsystems as shown in fig below. a) Detector b) Vector Receiver

→ This is called correlation receiver.



a) Detector.



b) Vector Receiver

1) Detector part : It consists of a bank of M product-integrators or correlators supplied with a corresponding set of coherent reference signals or orthonormal basis function $\phi_1(t), \phi_2(t), \ldots \phi_N(t)$ that are generated locally.

This bank of correlators operate on the received signal $x(t)$, to produce the observation vector x,

2) Vector Receiver : It is implemented in the form of a maximum-likelihood detector that operates on the observation vector x to produce an estimate $\hat{m}$ of the transmitted symbol m_i, $i = 1, 2, \ldots M$.

Here N elements of each of the M signal vectors $s_1, s_2, \ldots s_M$ & resulting products are successively summed in accumulators to form the corresponding set of inner products $\{x, s_k\}$ where $k = 1, 2 \ldots M$.

These inner products are corrected for the fact that the transmitted signal energies may be unequal.

Finally, the largest is selected & corresponding decision is ~~take~~ made.

Matched filter Receiver

→ Consider a linear filter with impulse response $h_j(t)$. If $x(t)$ is the input to filter then output of the filter $y_j(t)$ is defined by the convolution integral.

i.e $$y_j(t) = \int_{-\infty}^{\infty} x(\tau)\, h_j(t-\tau)\, d\tau$$

where impulse response $h_j(t) = \phi_j(T-t)$

Then resulting filter o/p is

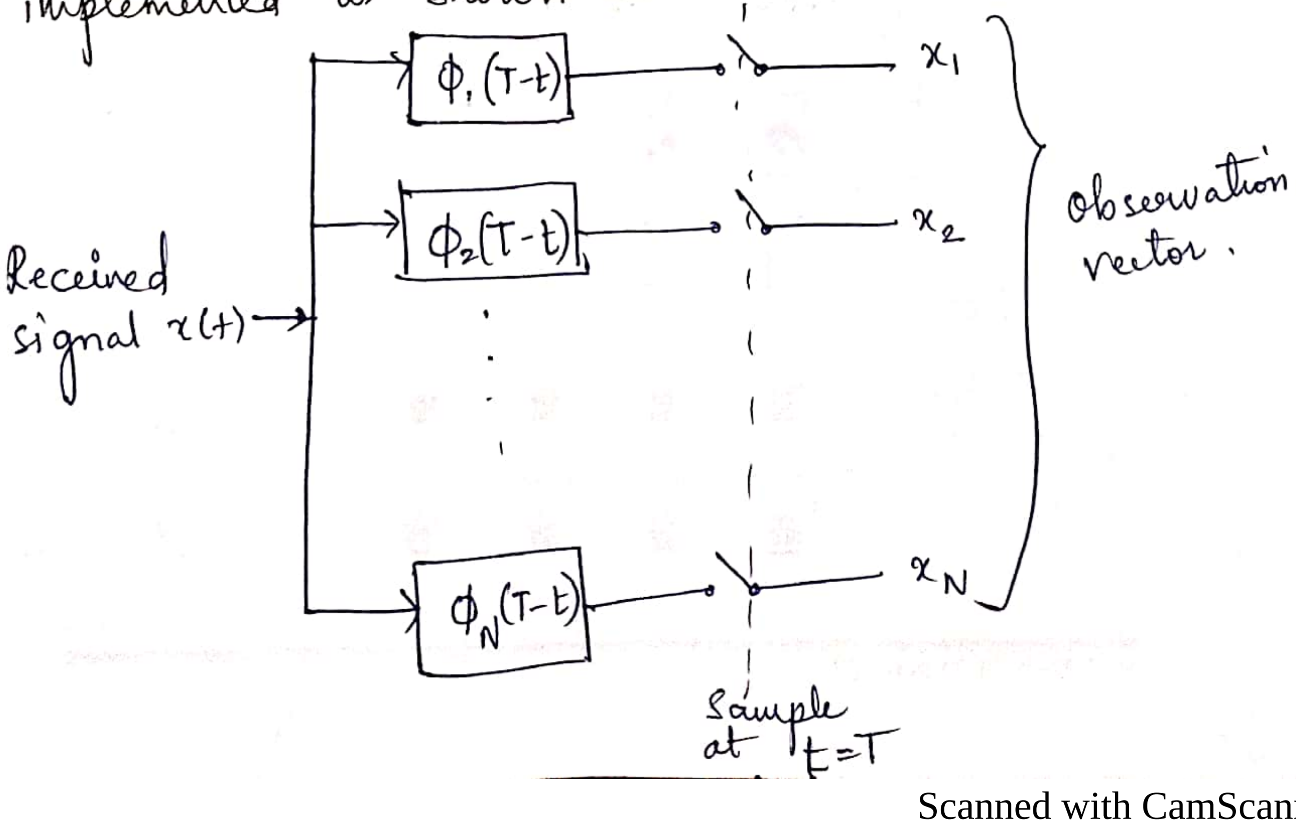
$$y_j(t) = \int_{-\infty}^{\infty} x(\tau)\, \phi_j(T-t+\tau)\, d\tau$$

Sampling this o/p at $t=T$, we get

$$y_j(t) = \int_{-\infty}^{\infty} x(\tau)\, \phi_j(\tau)\, d\tau$$

$= x_j$ → j^{th} correlator o/p produced by the received signal $x(t)$.

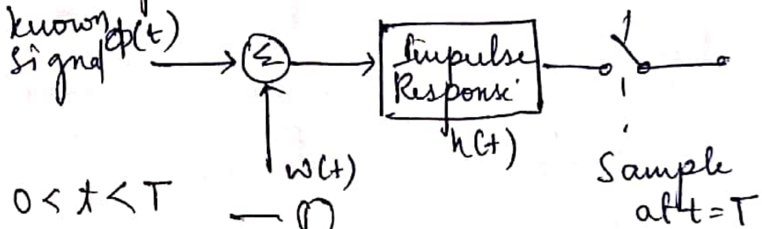
Thus detector part of optimum receiver may also be implemented as shown below.



⇒ Here, the filter whose impulse response is a time-reversed & delayed version of some signal $\phi_j(t)$ is matched to $\phi_j(t)$ [basis orthonormal impulse function]. Correspondingly, the optimum receiver based on the detector shown in above filter figure, is referred to as the matched filter receiver.

1) Maximization of Output Signal-to-noise ratio.

Consider the linear filter of impulse response $h(t)$ with an input that consists of known signal $\phi(t)$ & an additive noise component, $w(t)$ as shown in fig below.



Hence, we can write

$x(t) = \phi(t) + w(t) \quad 0 < t < T$ — ①

where $T \rightarrow$ observation instant.

let choose $\phi(t) \rightarrow$ one of basis orthonormal f^n.
$w(t) \rightarrow$ Gaussian noise process of $\mu = 0$ & $\sigma_w = 1$.
& power spectral density $S(f) = \frac{N_0}{2}$.

Since filter is linear, the resulting o/p $y(t)$ can be expressed as

$$y(t) = \phi_0(t) + n(t) \quad \text{— ②}$$

where $\phi_0(t)$ & $n(t)$ are produced by signal & noise components of the signal $x(t)$.

Here $|\phi_0(t)| > |n(t)|$

Thus maximum, Signal-to-noise ratio $(SNR)_0 = \frac{|\phi_0(T)|^2}{E[n^2(t)]}$ — ③ when matched to known signal $\phi(t)$.

Now if $\phi_o(t) = \int_{-\infty}^{\infty} H(f)\,\phi(f)\exp(j2\pi f t)\,df$

then

$$|\phi_o(T)|^2 = \left|\int_{-\infty}^{\infty} H(f)\,\phi(f)\exp(j2\pi f T)\,df\right|^2 \quad \text{--- (4)}$$

& power spectral density of noise for $n(t)$ is given as

$$S_N(f) = \frac{N_o}{2}|H(f)|^2$$

Then average power of the noise $n(t)$ is

$$E[n^2(t)] = \int_{-\infty}^{\infty} S_N(f)\,df = \frac{N_o}{2}\int_{-\infty}^{\infty}|H(f)|^2\,df \quad \text{--- (5)}$$

Thus substituting eqn. (4) & (5) in eqn. (3), we get

$$(SNR)_0 = \frac{\left|\int_{-\infty}^{\infty} H(f)\,\phi(f)\exp(j2\pi f T)\,df\right|^2}{\frac{N_o}{2}\int_{-\infty}^{\infty}|H(f)|^2\,df}$$

From Schwarz's inequality, we known that

$$\left|\int_{-\infty}^{\infty} H(f)\,\phi(f)\exp(j2\pi f T)\,df\right|^2 \leq \int_{-\infty}^{\infty}|H(f)|^2\,df \int_{-\infty}^{\infty}|\phi(f)|^2\,df$$

$$\therefore\ (SNR)_0 = \frac{\cancel{\int_{-\infty}^{\infty}|H(f)|^2\,df}\int_{-\infty}^{\infty}|\phi(f)|^2\,df}{\frac{N_o}{2}\cancel{\int_{-\infty}^{\infty}|H(f)|^2\,df}}$$

$$\therefore\ \boxed{(SNR)_0 = \frac{2}{N_o}\int_{-\infty}^{\infty}|\phi(f)|^2\,df}$$

2) Properties of Matched filter

WKT, A filter matched to a known signal $\phi(t)$ of duration T is characterised by an impulse response that is time reversed & delayed version of the input $\phi(t)$ as shown by

$$h_{opt}(t) = \phi(T-t)$$

Applying FT, we get

$$H_{opt}(f) = \phi^*(f)\exp(-j2\pi fT).$$

Based on these fundamental pair of relations, we derive some important properties of matched filters.

Property 1:

The spectrum of the output signal of a matched filter with the matched signal as input is proportional to the energy spectral density of the input signal.

i.e $$\phi_0(f) = |\phi(f)|^2 \exp(-j2\pi fT).$$

Property 2:

The output signal of a matched filter is proportional to a shifted version of the autocorrelation function of the input signal to the which the filter is matched.

i.e $$\phi_0(t) = R_\phi(t-T).$$

sampled at $t=T$, we have

$$\phi_0(t) = R_\phi(0) = E$$ where E is the signal energy

Property 3:

The output signal-to-noise ratio of a matched filter depends on the ratio of the signal energy to the power spectral density of the white noise at the filter o/p.

i.e $(SNR)_{0,max} = \frac{2E}{N_0}$ or $\frac{E}{(N_0/2)}$.

Property 4 :

The matched-filtering operation may be separated into 2 matching condⁿ:

1) Spectral phase matching, that produces the desired output peak at time T &
2) Spectral ~~de~~ amplitude matching that gives this peak value its optimum signal to noise density ratio.

i.e Spectral phase matched to the signal $\phi(t)$ if transfer fⁿ of filter is defined as

$$H(f) = |H(f)| \exp[-j\theta(f) - j2\pi fT]$$

& Spectral amplitude matching, $|H(f)| = |\phi(f)|$