31-08-2017

Z- TRANSFORMS

MODULE -5

Introduction:→

The Z- transform is the discrete time counterpart to the laplace transform (i.e Laplace is for continuous time). The Z-transform is the extension of discrete time fourier transform. There are many signals for which DTFT doesnot converge but Z- transform does. The primary objective of Z- transform is the study of system characteristics.

The Z-transform:→ In discrete time LTI system with impulse response $h(n)$, the response $y(n)$ of the system to a complex exponential input z^n (where $z^n = re^{j\Omega}$, r is the magnitude & Ω is the angle) is given by

$$y(n) = H\{x[n]\} = h[n] * x[n] = \sum_{k=-\infty}^{\infty} h[k]x[n-k]$$

we use $x[n] = z^n$ to obtain

$$y[n] = \sum_{k=-\infty}^{\infty} h[k] z^{n-k} = z^n \left\{ \sum_{k=-\infty}^{\infty} h(k) z^{-k} \right\}$$

where the transfer function; $\boxed{H(z) = \sum_{k=-\infty}^{\infty} h(k) z^{-k}}$ → (1)

$$y[n] = H(z) \cdot z^n \longrightarrow (2)$$

Generally Z-Transform of DT signal is given by

$$\boxed{Z\{x(n)\} = X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n}} \longrightarrow (3)$$

$$X(re^{j\Omega}) = \sum_{n=-\infty}^{\infty} x(n) (re^{j\Omega})^{-n}$$

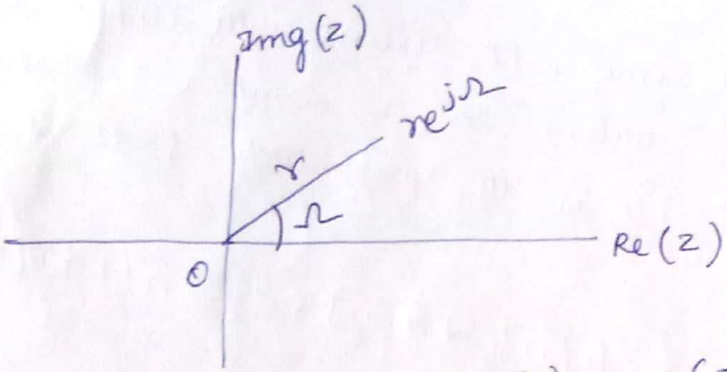
$$= \sum_{n=-\infty}^{\infty} \{x(n) r^{-n}\} e^{-j\Omega n} \longrightarrow (4)$$

for $r=1$ the equation (4) becomes

$$X(e^{j\Omega}) = \sum_{n=-\infty}^{\infty} x(n) \cdot e^{-j\Omega n} = \sum_{n=-\infty}^{\infty} x(n) (e^{j\Omega})^{-n}$$

$$X(e^{j\Omega}) = \sum_{n=-\infty}^{\infty} X(z) \Big|_{z=e^{j\Omega}}$$

A point $z = re^{j\Omega}$ is located at distance r from origin & Ω relative to real axis as shown in figure



The relationship between $x(n)$ & $X(z)$ is expressed as

$$x(n) \xleftrightarrow{Z} X(z)$$

The Z-transform of DT sequence $x(n)$ is given by

$$X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n}$$

$$X(z) = \sum_{n=-\infty}^{\infty} \{x(n) r^{-n}\} e^{-j\Omega n} \longrightarrow (5)$$

For existence of $X(z)$ the summation should converge ie summation should be absolute.

$$\sum_{n=-\infty}^{\infty} |x(n) r^{-n}| < \infty$$

The range of r for which this condition is satisfied is known as "Region of convergence (ROC)"

(2)

Determine the z-transform of the signal $x(n) = \alpha^n u(n)$ {Causal exponential signal}

Soln:- $X(Z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n} = \sum_{n=0}^{\infty} \alpha^n z^{-n}$ {$\because u(n)$ exists only for $0 \to \infty$}

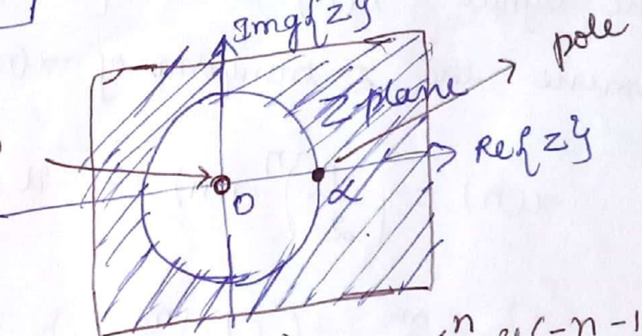
$= \sum_{n=0}^{\infty} (\alpha z^{-1})^n$ $\boxed{\left\{\because \sum_{n=0}^{\infty} \alpha^n = \frac{1}{1-\alpha}\right\}}$

$= \sum_{n=0}^{\infty} \left(\frac{\alpha}{z}\right)^n$

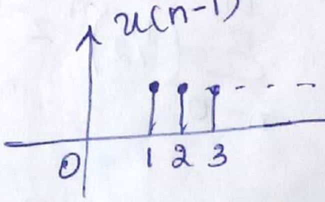
The sum converges, provided $\left|\left(\frac{\alpha}{z}\right)\right| < 1$ or $|z| > |\alpha|$

$\boxed{X(Z) = \frac{1}{1-(\alpha/z)} = \frac{z}{z-\alpha}}$; $|z| > |\alpha|$

$Z = 0$ → Zeros
$Z = \alpha$ → poles

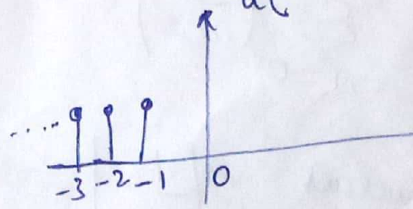


Determine the z-transform of the signal $x(n) = -\alpha^n u(-n-1)$ and also find ROC {anti causal (non causal) exponential signal}



Soln:- $X(Z) = \sum_{n=-\infty}^{\infty} x(n) \cdot z^{-n}$ $-\sum_{n=-\infty}^{-1} (\alpha z^{-1})^n$

$= \sum_{n=-\infty}^{-1} -\alpha^n z^{-n} = \left(-\sum_{n=\infty}^{1} \cancel{(\alpha^{-1} z)^{-n}}\right)$



$= -\sum_{n=1}^{\infty} (\alpha z^{-1})^{-n}$

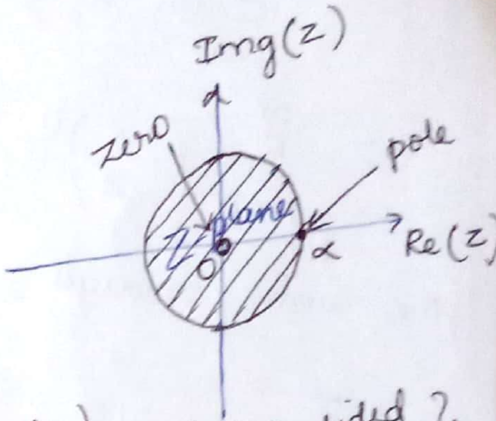
$= -\sum_{n=1}^{\infty} (\alpha^{-1} z)^n$

The sum converges, provided $|\alpha^{-1} z| < 1$ ie $|z/\alpha| < 1$ or $|z| < |\alpha|$

$$X(z) = -\frac{\alpha^{-1}z}{1-\alpha^{-1}z} \qquad \left\{ \because \sum_{n=1}^{\infty} \alpha^n = \frac{\alpha}{1-\alpha} \right.$$

$$= -\frac{z/\alpha}{1 - z/\alpha}$$

$$= -\frac{z}{\alpha - z} = \frac{-z}{-1(z-\alpha)}$$



$$\boxed{X(z) = \frac{z}{z-\alpha}} \quad ; \quad |z| < |\alpha|$$

$$\boxed{\begin{array}{l} z = 0 \rightarrow \text{zero} \\ z = \alpha \rightarrow \text{pole} \end{array}}$$

→ For the signal $x(n) = -u[-n-1] + \left(\frac{1}{2}\right)^n u(n)$ { two sided signal }

Determine the Z-transform of $x(n)$ & ROC

Solu^n:- $x(n) = \left(\frac{1}{2}\right)^n u(n) - u[-n-1]$ $\qquad X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n}$

$$X(z) = \sum_{n=-\infty}^{\infty} \left\{ \left(\frac{1}{2}\right)^n u(n) - u[-n-1] \right\} z^{-n}$$

$$= \sum_{n=0}^{\infty} (1/2)^n \underset{(z^{-1})^n}{z^{-n}} - \sum_{n=-\infty}^{-1} z^{-n}$$

$$= \sum_{n=0}^{\infty} \left(\frac{1}{2z}\right)^n - \sum_{n=1}^{\infty} z^n$$

It converges. $\left|\frac{1}{2z}\right| < 1 \Rightarrow \frac{1}{2} < |z|$ or $\boxed{|z| > 1/2}$

. $|z| < 1$.

$\therefore$ ROC is $\boxed{1/2 < |z| < 1}$

$$X(z) = \frac{1}{1-\frac{1}{2z}} - \frac{z}{1-z} = \cancel{\frac{2z}{2z-1} - \frac{z}{1-z}} = \frac{2z}{2z-1} + \frac{z}{(z-}$$

$$= \frac{1}{1-\frac{1}{2z}} + \left[\frac{-z}{-(z-1)}\right]$$

$$= \frac{1}{1-\frac{1}{2}z} + \frac{z}{(z-1)} = \frac{(z-1)+z(1-\frac{1}{2z})}{(1-\frac{1}{2z})(z-1)}$$

$$X(z) = \cancel{(z-1+z-1/2)} \quad \frac{z(2z-3/2)}{(z-1/2)(z-1)} \quad ; \quad 1/2 < |z| < 1.$$

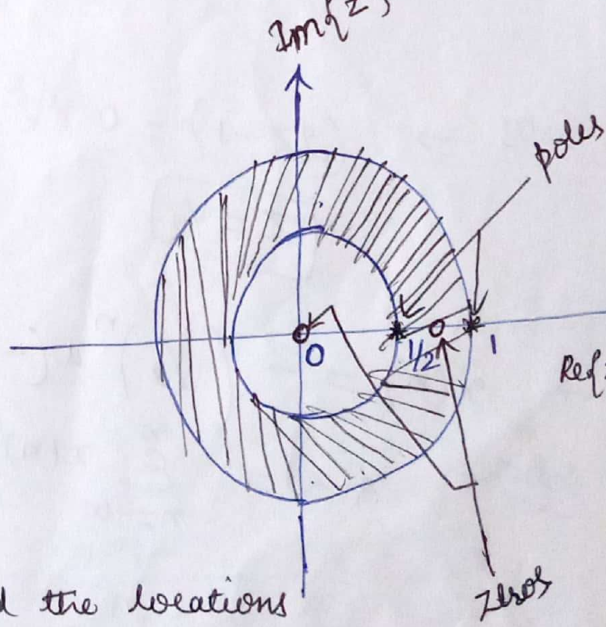
Poles and Zeros :→ The Z-transform of $x(n)$ is given by

$$X(z) = \frac{b_0 + b_1 z^{-1} + \dots + b_M z^{-M}}{a_0 + a_1 z^{-1} + \dots + a_N z^{-N}}$$

Numerator polynomial yields zeros

Denominator polynomial yields poles

$z=0$	$z=3/4$	→ zeros
$z=1/2$	$z=1$	→ poles



→ determine the z-transform, the ROC and the locations of poles and zeros of $X(z)$ for the following signals.

$$x[n] = (1/2)^n u(n) + (-1/3)^n u(n)$$

Soln:- $X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n} = \sum_{n=0}^{\infty} (1/2)^n z^{-n} + \sum_{n=0}^{\infty} (-1/3)^n z^{-n}$

$$\left|\frac{1}{2z}\right| < 1 \quad ; \quad \left(-\frac{1}{3z}\right) < 1$$

$$|z| > \frac{1}{2} \quad ; \quad |z| > \left(-\frac{1}{3}\right)$$

$\therefore$ the ROC is $\boxed{|z| > \frac{1}{2}}$

$$X(z) = \frac{1}{1-\frac{1}{2z}} + \frac{1}{1+\frac{1}{3z}} = \frac{2z}{2z-1} + \frac{3z}{3z+1}$$

$$= \frac{2z(3z+1) + 3z(2z-1)}{(2z-1)(3z+1)} = \frac{6z^2+2z+6z^2-3z}{(2z-1)(3z+1)}$$

$$X(z) = \frac{12z^2 - z}{(2z-1)(3z+1)} = \frac{z(12z-1)}{(2z-1)(3z+1)} \quad ; \quad |z| > 1.$$

Zeroes $\rightarrow$ $\boxed{z=0}$ & $12z-1=0.$

$$12z = 1$$

$$\boxed{z = \frac{1}{12}}$$

Poles $\rightarrow$ $(2z-1) = 0$ $\quad$ $3z+1=0$

$$\boxed{z = \frac{1}{2}} \quad \boxed{z = -\frac{1}{3}}$$

$\rightarrow$ $x(n) = -\left(\frac{1}{2}\right)^n u(-n-1) - \left(-\frac{1}{3}\right)^n u[-n-1]$

Solun:- $X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n} = -\sum_{n=-\infty}^{-1} \left(\frac{1}{2}\right)^n z^{-n} - \sum_{n=-\infty}^{-1} \left(-\frac{1}{3}\right)^n z^{-n}$

$$= -\sum_{n=1}^{\infty} (2z)^n - \sum_{n=1}^{\infty} (-3z)^n$$

The region of convergence ; $|2z| < 1 \rightarrow |z| < \frac{1}{2}$

$$|(-3z)| < 1 \rightarrow |z| < +\frac{1}{3}$$

The ROC is $\boxed{|z| < \frac{1}{3}}$

(4)

$$= -\frac{2z}{1-2z} - \frac{(-3z)}{1+3z} = \frac{-\cancel{1-3z}\ -1+2z}{(1-2z)\ (1+3z)} = \cancel{\frac{-z-2}{(1-2z)(1+3z)}}$$

$$x(z) = \frac{-2z-6z^2+3z-6z^2}{(1-2z)\ (1+3z)} = \frac{z-12z^2}{(1-2z)\ (1+3z)} = \frac{z(1-12z)}{(1-2z)\ (1+3z)}$$

$; \underline{|z| < 1/3}$

Zeros: equating numerator to 0

$$\boxed{z=0 \quad | \quad z=1/12}$$

Poles: equating denominator to 0 $\boxed{z=1/2 \ \ \& \ \ z=-1/3}$

$\rightarrow \quad x(n) = -(3/4)^n u[-n-1] + (-1/3)^n u[n]$

Solun :- $x(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n} = -\sum_{n=-\infty}^{-1} (3/4)^n z^{-n} + \sum_{n=0}^{\infty} (-1/3)^n z^{-n}$

$$= -\sum_{n=1}^{\infty} \left(\frac{4z}{3}\right)^n + \sum_{n=0}^{\infty} (-1/3z)^n$$

The ROC $\left|\frac{4z}{3}\right| < 1 \Rightarrow |z| < 3/4$

$|1/3z| < 1 \Rightarrow |z| > \frac{1}{3}$.

$\therefore$ ROC is $\boxed{1/3 < |z| < 3/4}$

$$x(z) = -\frac{4z/3}{1-4z/3} + \frac{1}{1+\frac{1}{3z}} = \frac{-4z}{3-4z} + \frac{3z}{3z+1}$$

$$x(z) = \frac{-12z^2-4z+9z-12z^2}{(3-4z)\ (3z+1)} = \frac{-24z^2+5z}{(3-4z)(3z+1)} = \frac{z(-24z+5)}{(3-4z)(3z+1)}$$

ROC: $\underline{1/3 < |z| < 3/4}$

Zeros: $z=0$ & $z = 5/24$

Poles: $z = 3/4$ & $z = -1/3$.

$\rightarrow$ $x[n] = e^{j\Omega_0 n} u[n]$

Solun:- $X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n} = \sum_{n=0}^{\infty} \left(\frac{e^{j\Omega_0}}{z}\right)^n$

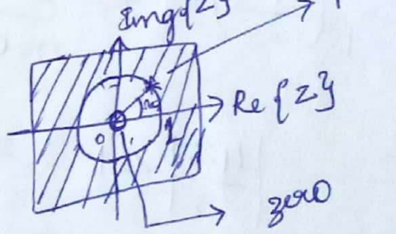
ROC is $\left|\frac{e^{j\Omega_0}}{z}\right| < 1 \Rightarrow$ ~~sin~~ $\underbrace{|\cos\Omega_0 + j\sin\Omega_0|}_{1} < |z|$

$\{ \because e^{i\theta} = \cos\theta + i\sin\theta \;;\; |e^{i\theta}| = |\cos\theta + i\sin\theta|$

$|e^{i\theta}| = \sqrt{\cancel{a^2+b^2}} \sqrt{\cos^2\theta + \sin^2\theta} = 1 \}$

$\therefore$ The ROC is $\underline{|z| > 1}$

$X(z) = \frac{1}{1 - \frac{e^{j\Omega_0}}{z}} = \frac{z}{z - e^{j\Omega_0}} \;;\; |z| > 1$



Zeros: $z=0$

Poles: $z = e^{j\Omega_0}$

$\rightarrow$ $x(n) = 7\left(\frac{1}{3}\right)^n u(n) - 6\left(\frac{1}{2}\right)^n u(n)$

Solun: $X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n} = 7\sum_{n=0}^{\infty} \left(\frac{1}{3z}\right)^n - 6\sum_{n=0}^{\infty}\left(\frac{1}{2z}\right)^n$

ROC:

$\left|\frac{1}{3z}\right| < 1 \Rightarrow |z| > 1/3$ $\therefore$ The ROC is $\underline{|z| > 1/2}$

$\left|\frac{1}{2z}\right| < 1 \Rightarrow |z| > 1/2$

$X(z) = 7\left\{\frac{1}{1 - \frac{1}{3z}}\right\} - 6\left\{\frac{1}{1 - \frac{1}{2z}}\right\}$

$$\frac{1}{2j}\left\{\frac{3z}{3z-e^{j\pi/4}}\right\}-\frac{1}{2j}\left\{\frac{3z}{3z-e^{-j\pi/4}}\right\}$$

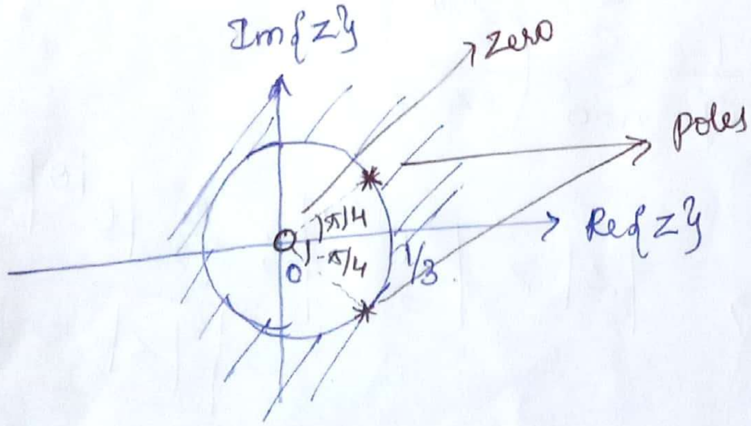
$$=\frac{3z}{2j}\left\{\frac{3z-e^{-j\pi/4}-3z+e^{j\pi/4}}{(3z-e^{j\pi/4})(3z-e^{-j\pi/4})}\right\}$$

$$=\frac{3z}{2j}\left\{\frac{e^{j\pi/4}-e^{-j\pi/4}}{(3z-e^{j\pi/4})(3z-e^{-j\pi/4})}\right\} \rightarrow \sin\pi/4$$

$$X(z)=\frac{3z\sin\pi/4}{(3z-e^{j\pi/4})(3z-e^{-j\pi/4})}=\frac{3z/\sqrt{2}}{(3z-e^{j\pi/4})(3z-e^{-j\pi/4})}$$

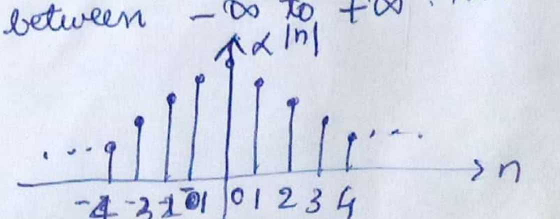
Zeros: $\frac{3z}{\sqrt{2}}=0 \Rightarrow \boxed{z=0}$

Poles: $\boxed{z=\frac{e^{j\pi/4}}{3}}$. $\boxed{z=\frac{e^{-j\pi/4}}{3}.}$



→ Q. Determine z-transform of $x(n)=\alpha^{|n|}$; $|\alpha|<1$. Also find ROC

Soln: $x(n)=\alpha^{|n|}$ where n varies between $-\infty$ to $+\infty$. The waveform of $\alpha^{|n|}$ is shown



$\therefore x(n)=\alpha^{n}u(n)+\alpha^{-n}u(-n-1)$

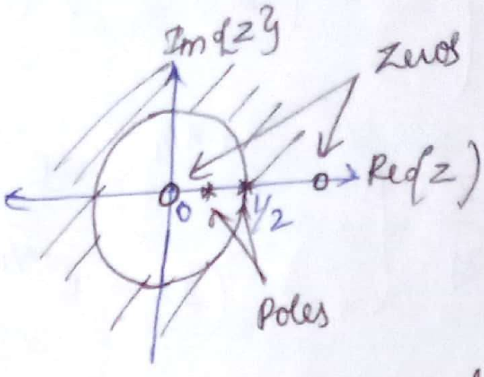
(5)

$$X(z) = \frac{21z}{3z-1} - \frac{12z}{2z-1} = \frac{42z^2 - 21z - 36z^2 + 12z}{(3z-1)(2z-1)}$$

$$X(z) = \frac{6z^2 - 9z}{(3z-1)(2z-1)} = \frac{3z(2z-3)}{(3z-1)(2z-1)} \; ; \quad |z| > 1/2$$

Zeros: $z = 0$ & $z = 3/2$

Poles: $z = 1/3$ & $z = 1/2$



05-04-2017

→ Find Z-transform of the sequence $x(n) = (1/3)^n \sin(\pi/4\, n)\, u(n)$ & also sketch ROC & pole-zero location

Soln:

$$x(n) = (1/3)^n \sin\left(\frac{\pi n}{4}\right) u(n) = \left(\frac{1}{3}\right)^n \left[\frac{e^{j\frac{\pi}{4}n} - e^{-j\frac{\pi}{4}n}}{2j}\right] u(n)$$

$$= \left[\frac{1}{2j}\left(\frac{1}{3}\right)^n e^{j\frac{\pi n}{4}} - \frac{1}{2j} e^{-j\frac{\pi n}{4}} (1/3)^n\right] u(n)$$

$$X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n}$$

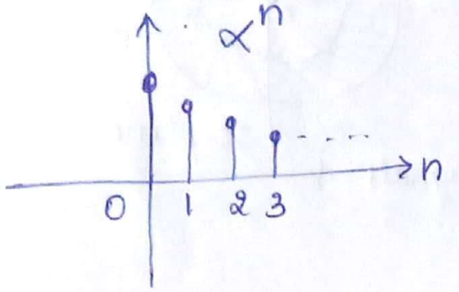
$$= \frac{1}{2j} \sum_{n=0}^{\infty} \left(\frac{e^{j\frac{\pi}{4}}}{3z}\right)^n - \frac{1}{2j} \sum_{n=0}^{\infty} \left(\frac{e^{-j\pi/4}}{3z}\right)^n$$

ROC:

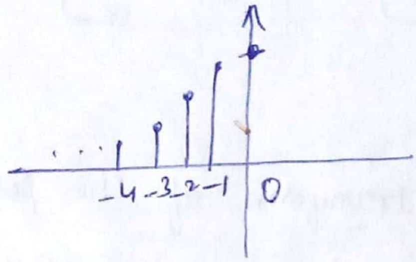
$$\left|\frac{e^{j\frac{\pi}{4}}}{3z}\right| < 1 \Rightarrow |z| > \left|\frac{e^{j\pi/4}}{3}\right| ; \; \{\because |e^{j\theta}| = 1\} \quad \boxed{|z| > 1/3}$$

$$\left|\frac{e^{-j\pi/4}}{3z}\right| < 1 \Rightarrow |z| > \left|\frac{e^{-j\frac{\pi}{4}}}{3}\right| \quad \boxed{|z| > 1/3}$$

...it the signal into 2 signals. One signal interval is from 0 to ∞
other signal is from -1 to $-\infty$



fig(2)



fig(3)

The equation of signal shown in fig(2) is $\alpha^n u(n)$

and for fig (3) is $\alpha^{-n} u(-n-1)$

$$\therefore \quad x(n) = \alpha^{|n|} = \alpha^n u(n) + \alpha^{-n} u(-n-1)$$

$$X(z) = \sum_{n=-\infty}^{\infty} x(n) \cdot z^{-n} = \sum_{n=0}^{\infty} \alpha^n z^{-n} + \sum_{n=-1}^{-\infty} \alpha^{-n} z^{-n}$$

$$= \sum_{n=0}^{\infty} \left(\frac{\alpha}{z}\right)^n + \sum_{n=1}^{\infty} (\alpha z)^n$$

ROC:

$$\left|\frac{\alpha}{z}\right| < 1 \Rightarrow |z| > |\alpha|$$

$$|\alpha z| < 1 \Rightarrow |z| < \left|\frac{1}{\alpha}\right|$$

$\therefore$ The region of convergence is $\boxed{\alpha < |z| < \frac{1}{\alpha}}$

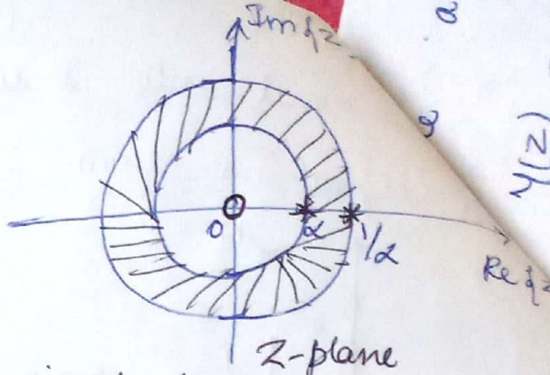
$$X(z) = \frac{1}{1-(\alpha/z)} + \frac{\alpha z}{1-\alpha z} = \frac{z}{(z-\alpha)} + \frac{\alpha z}{(1-\alpha z)}$$

$$= \frac{z - \cancel{\alpha z^2} + \cancel{\alpha z^2} - \alpha^2 z}{(z-\alpha)(1-\alpha z)} = \frac{z - \alpha^2 z}{(z-\alpha)(1-\alpha z)}$$

$$X(z) = \frac{z(1-\alpha^2)}{(z-\alpha)(1-\alpha z)} \; ; \quad \alpha < |z| < \frac{1}{\alpha}$$

Zeros : $\boxed{z=0}$

Poles : $\boxed{z=\alpha}$ $\boxed{z=\frac{1}{\alpha}}$



→ Find the Z-transform of the following signals & determine ROC

(i) $x(n) = (1+n)u(n)$

(ii) $y(n) = (a^n + a^{-n})u(n)$; where a is real

(iii) $w(n) = (-1)^n \, 2^{-n} u(n)$

(i) $x(n) = (1+n)u(n) = u(n) + n u(n)$

$$X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n} = \sum_{n=0}^{\infty} z^{-n} + \sum_{n=0}^{\infty} n z^{-n}$$

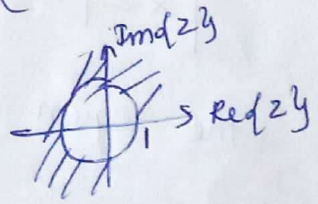
$$= \sum_{n=0}^{\infty} (z^{-1})^n + \sum_{n=0}^{\infty} n (z^{-1})^n \qquad \left\{ \sum_{n=0}^{\infty} n\alpha^n = \begin{cases} \dfrac{\alpha}{(1-\alpha)^2} ; & |\alpha| < 1 \\ \infty ; & |\alpha| > 1 \end{cases} \right.$$

ROC: $|z^{-1}| < 1$; $\boxed{|z| > 1}$

$\cancel{|z| < 1}$

$$X(z) = \frac{1}{1-z^{-1}} + \frac{z^{-1}}{(1-z^{-1})^2} = \frac{1-z^{-1}+z^{-1}}{(1-z^{-1})^2}$$

$$\boxed{X(z) = \frac{1}{(1-z^{-1})^2}} \; ; \; |z| > 1$$



(ii) $y(n) = (a^n + a^{-n})u(n)$

$$Y(z) = \sum_{n=-\infty}^{\infty} y(n) z^{-n} = \sum_{n=0}^{\infty} a^n z^{-n} + \sum_{n=0}^{\infty} (a^{-1} z^{-1})^n$$

ROC : $\left|\frac{a}{z}\right| < 1 \Rightarrow |z| > |a|$; $\left|\frac{1}{az}\right| < 1 \Rightarrow |z| > \left|\frac{1}{a}\right|$

ince 'a' is real ROC is max (a, 1/a)

$$Y(z) = \frac{1}{1-az^{-1}} + \frac{1}{1-\frac{1}{az}} = \frac{z}{(z-a)} + \frac{az}{az-1}$$

$$Y(z) = \frac{az^2 - z + az^2 - a^2 z}{(z-a)(az-1)} = \frac{2az^2 - z - a^2 z}{(z-a)(az-1)} = \frac{z(2az-1-a^2)}{(z-a)(az-1)}$$

(iii) $w(n) = (-1)^n 2^{-n} u(n) = (-2^{-1})^n u(n)$

$$W(z) = \sum_{n=-\infty}^{\infty} w(n) z^{-n} = \sum_{n=0}^{\infty} (-2^{-1})^n z^{-n}$$

$$W(z) = \sum_{n=0}^{\infty} (-2^{-1} z^{-1})^n$$

ROC is $|-2^{-1} z^{-1}| < 1$

$$\boxed{|z| > 1/2}$$

$$W(z) = \frac{1}{1+\frac{1}{2z}} = \frac{2z}{2z+1}$$

$|z| > 1/2$.

Find the z-transform of $x(n) = 7\left(\frac{1}{3}\right)^n \cos\left[\frac{2\pi n}{6} + \frac{\pi}{4}\right] u(n)$

$$x(n) = 7\left(\frac{1}{3}\right)^n \left\{ \frac{e^{j\left(\frac{2\pi n}{6}+\frac{\pi}{4}\right)} + e^{-j\left(\frac{2\pi n}{6}+\frac{\pi}{4}\right)}}{2} \right\} u(n)$$

$$X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n}$$

$$= \left(\frac{7}{2}\right)\left[\left\{\sum_{n=0}^{\infty}\left(\frac{1}{3}\right)^n e^{\frac{j2\pi n}{6}} \cdot e^{j\pi/4} z^{-n}\right\} + \left\{\sum_{n=0}^{\infty}\left(\frac{1}{3}\right)^n e^{-j\frac{2\pi n}{6}} \cdot e^{-j\frac{\pi}{4}} z^{-n}\right\}\right]$$

$$= \left(\frac{7}{2}\right)\left[\left\{e^{j\pi/4} \sum_{n=0}^{\infty} \left(\frac{1}{3}\right)^n e^{j\frac{2\pi n}{6}} z^{-n}\right\} + \left\{e^{-j\pi/4} \sum_{n=0}^{\infty}\left(\frac{1}{3}\right)^n e^{-j\frac{2\pi n}{6}} z^{-n}\right\}\right]$$

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$\underline{\text{ROC}}$: $\left|\dfrac{e^{j\frac{2\pi}{6}}}{3z}\right| < 1$; $|z| > \dfrac{1}{3}$

$\left|\dfrac{e^{-j\frac{2\pi}{6}}}{3z}\right| < 1$; $\boxed{|z| > 1/3}$

$$= \frac{7}{2}\left[\frac{e^{j\pi/4}}{1 - \dfrac{e^{j2\pi/6}}{3z}} + \frac{e^{-j\pi/4}}{1 - \dfrac{e^{-j2\pi/6}}{3z}}\right]$$

$$= \left(\frac{7}{2}\right)\left[\frac{3z\, e^{j\pi/4}}{3z - e^{j2\pi/6}} + \frac{3z\, e^{-j\pi/4}}{3z - e^{-j2\pi/6}}\right] = \left(\frac{7}{2}\right)\left[3z\left\{\frac{e^{j\pi/4}}{3z - e^{j2\pi/6}} + \frac{e^{-j\pi/}}{3z - e}\right.\right.$$

$$= \frac{7(3z)}{2}\left[\frac{3z\, e^{j\pi/4} - \left(e^{j\pi/4}\cdot e^{-j2\pi/6}\right) + 3z e^{-j\pi/4} - \left(e^{j2\pi/6}\cdot e^{-j\pi/4}\right)}{\left(3z - e^{j2\pi/6}\right)\left(3z - e^{-j2\pi/6}\right)}\right.$$

$$= \frac{7(3z)}{2}\left[\frac{3z\left(e^{j\pi/4} + e^{-j\pi/4}\right) - \left(e^{-j(2\pi/6 - 2\pi/4)}\right) - e^{j\left(\frac{2\pi}{6} - \frac{\pi}{4}\right.}}{\left(3z - e^{j2\pi/6}\right)\left(3z - e^{-j2\pi/6}\right)}\right.$$

$$X(z) = \frac{7(3z)}{}\left[\frac{3z\cos\pi/4 - \overbrace{\cos\left(\frac{2\pi}{6} - \pi/4\right)}^{\to\ \cos(\pi/12)}}{\left(3z - e^{j2\pi/6}\right)\left(3z - e^{-j2\pi/6}\right)}\right]$$

roperties of Z-Transform :→

1. Linearity
2. Time shifting
3. Scaling in the z-domain
4. Time reversal
5. Time expansion
6. Conjugation
7. Convolution
8. Differentiation in the z-domain
9. Initial value theorem
10. Final value theorem

Linearity :→ If $x(n) \xleftrightarrow{Z} X(z)$ with ROC $= R_1$

and $y(n) \xleftrightarrow{Z} Y(z)$ with ROC $= R_2$

then $ax(n) + b\,y(n) \xleftrightarrow{Z} a\,X(z) + b\,Y(z)$ with ROC atleast $R_1 \cap R_2$

Proof: WKT $Z\{x(n)\} = X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n}$

$$Z\{a\,x(n) + b\,y(n)\} = \sum_{n=-\infty}^{\infty} \{a x(n) + b\,y(n)\} z^{-n}$$

$$= \sum_{n=-\infty}^{\infty} a x(n) z^{-n} + \sum_{n=-\infty}^{\infty} b\,y(n) z^{-n} = a \sum_{n=-\infty}^{\infty} x(n) z^{-n} + b \sum_{n=-\infty}^{\infty} y(n) z^{-n}$$

$$\boxed{Z\{a\,x(n) + b\,y(n)\} = a\,X(z) + b\,Y(z)}$$

Time shifting :→ If $x(n) \xleftrightarrow{Z} X(z)$ with ROC $= R$

then $x(n-n_0) \xleftrightarrow{Z} z^{-n_0} X(z)$ with ROC $= R$ except for the possible addition or deletion of the origin or infinity

Proof: $Z\{x(n)\} = X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n}$

$$Z\{x(n-n_0)\} = \sum_{n=-\infty}^{\infty} x(n-n_0) z^{-n}$$

Put $n - n_0 = l$

$$= \sum_{n=-\infty}^{\infty} x(l) \cdot z^{-(l+n_0)}$$

$$= \sum_{l=-\infty}^{\infty} x(l) \cdot z^{-l} z^{-n_0} = z^{-n_0} \sum_{n=-\infty}^{\infty} x(l) \cdot z^{-l}$$

$$\boxed{Z\{x(n-n_0)\} = z^{-n_0} X(z)}$$

Scaling in the Z-domain :→ If $x(n) \xleftrightarrow{Z} X(z)$; ROR = R

then $\alpha^n x(n) \xleftrightarrow{Z} X\left(\frac{z}{\alpha}\right)$; ROC = $|\alpha| R$

where 'α' is a complex number

Proof: $Z\{x(n)\} = X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n}$

$$Z\{\alpha^n x(n)\} = \sum_{n=-\infty}^{\infty} \alpha^n x(n) z^{-n}$$

$$= \sum_{n=-\infty}^{\infty} x(n) \cdot (\alpha^{-1} z)^{-n} = X(\alpha^{-1} z)$$

$$\boxed{Z\{\alpha^n x(n)\} = X\left(\frac{z}{\alpha}\right)}$$

Note: $e^{j\Omega_0 n} x(n) \xleftrightarrow{Z} X(e^{-j\Omega_0} z)$

Time Reversal :→ If $x(n) \xleftrightarrow{Z} X(z)$ with ROC = R

then $x(-n) \xleftrightarrow{Z} X\left(\frac{1}{z}\right)$ with ROC = $\frac{1}{R}$

Proof: $Z\{x(n)\} = X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n}$

$$Z\{x(-n)\} = \sum_{n=-\infty}^{\infty} x(-n) z^{-n}$$

Put $l = -n$

$$= \sum_{l=-\infty}^{\infty} x(l) z^{l} = \sum_{l=-\infty}^{\infty} x(l) \cdot (z^{-1})^{-l}$$

$$= X(z^{-1})$$

$$\boxed{Z\{x(-n)\} = X\left(\frac{1}{z}\right)}$$

me expansion :→ If $x(n) \xleftrightarrow{Z} X(z)$ with ROC = R

then $x_{(k)}(n) \xleftrightarrow{Z} X(z^k)$ with ROC = $R^{1/k}$

where $x_{(k)}(n) = x\left(\frac{n}{k}\right)$; if n is an integer multiple of k

$= 0$; otherwise

ie, $x_{(k)}(n)$ has $(k-1)$ zeroes inserted between successive values of the original signal.

Proof: $X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n}$

The coefficient of the term z^{-n} equals the value of the signal at time n.

Similarly ; $X(z^k) = \sum_{n=-\infty}^{\infty} x(n) z^{-kn}$

ie the coefficients of the term z^{-m} equals zero if 'm' is not a multiple of 'k' & equal to $x\left(\frac{m}{k}\right)$ if 'm' is a multiple of k. Thus the inverse transform is $x_k(n)$.

Conjugation :→ If $x(n) \xleftrightarrow{Z} X(z)$ with ROC = R

then $x^*(n) \xleftrightarrow{Z} X^*(z^*)$ with ROC = R

If $x(n)$ is real then $X(z) = X^*(z^*)$

Thus, if $X(z)$ has a pole (or zero) at $z = z_0$, it must have a pole (or zero) at the complex conjugate point $z = z_0^*$

Convolution :→ If $x(n) \xleftrightarrow{Z} X(z)$; with ROC = R_1

and $y(n) \xleftrightarrow{Z} Y(z)$; with ROC = R_2

then $x(n) * y(n) \xleftrightarrow{Z} X(z) \cdot Y(z)$; with ROC atleast $R_1 \cap R_2$

Proof : WKT $x(n) * y(n) = \sum_{n=-\infty}^{\infty} x(k) y(n-k)$

$$X(z) = Z\left[x(n) * y(n)\right]$$

$$= \sum_{n=-\infty}^{\infty} \left[\sum_{k=-\infty}^{\infty} x(k)\, y(n-k)\right] z^{-n}$$

Interchanging the order of the summation,

$$= \sum_{k=-\infty}^{\infty} x(k) \left[\sum_{k=-\infty}^{\infty} y(n-k)\, z^{-n}\right]$$

Put $n-k = l$

$$= \sum_{k=-\infty}^{\infty} x(k) \left[\sum_{l=-\infty}^{\infty} y(l)\, z^{-l} \cdot z^{-k}\right]$$

$$= \left[\sum_{k=-\infty}^{\infty} x(k)\, z^{-k}\right] \left[\sum_{l=-\infty}^{\infty} y(l)\, z^{-l}\right]$$

$$\boxed{Z\{x(n) * y(n)\} = X(z)\, Y(z)}$$

Differentiation in the Z- domain :→

If $x(n) \xleftrightarrow{Z} X(z)$ with ROC = R

then $n\, x(n) \xleftrightarrow{Z} -z \dfrac{dX(z)}{dz}$ with ROC = R

Proof: $Z\{x(n)\} = X(z) = \sum_{n=-\infty}^{\infty} x(n)\, z^{-n}$

differentiating both the sides with respect to 'z' we get

$$\frac{dX(z)}{dz} = \sum_{n=-\infty}^{\infty} x(n)\, (-n)\, z^{-n-1}$$

$$= -z^{-1} \sum_{n=-\infty}^{\infty} \left[n\, x(n)\right] z^{-n}$$

$$= -z^{-1}\, Z\{n\, x(n)\}$$

$$\therefore \boxed{Z\{n\, x(n)\} = -z \frac{dX(z)}{dz}}$$

Initial value Theorem :→

If $x(n) = 0$ for $n < 0$ [ie $x(n)$ is causal]

then $\lim_{n\to 0} x(n) = x(0) = \lim_{z\to\infty} X(z)$

Proof: $Z\{x(n)\} = X(z) = \sum_{n=0}^{\infty} x(n) z^{-n}$

$= \sum_{n=0}^{\infty} x(n) z^{-n}$

$= x(0) + x(1) z^{-1} + x(2) z^{-2} + \cdots$

Take limit $z \to \infty$ on both the sides,

$\lim_{z\to\infty} X(z) = x(0) + 0 + 0 + \cdots$

$$\boxed{\lim_{n\to 0} x(n) = x(0) = \lim_{z\to\infty} X(z)}$$

Final Value Theorem :→

If $x(n) \xleftrightarrow{Z} X(z)$ and if $X(z)$ exists and no double or higher order poles outside the unit circle and it has no double or higher order poles on the unit circle centered at the origin of the z-plane,

then $\lim_{n\to\infty} x(n) = x(\infty) = \lim_{z\to 1} (z-1) X(z)$

Proof: $Z\{x(n)\} = X(z)$ → (1)

$Z\{x(n+1)\} = zX(z) . z x(0)$ → (2)

equation (1) − (2)

$Z\{x(n+1)\} - Z\{x(n)\} = zX(z) - z x(0) - X(z)$

$\sum_{n=0}^{\infty} x(n+1) z^{-n} - \sum_{n=0}^{\infty} x(n) z^{-n} = (z-1) X(z) - z x(0)$

Taking limit $z \to 1$ on both the sides we get,

$\lim_{z\to 1} [(z-1)X(z) - z x(0)] = \lim_{z\to 1} \sum_{n=0}^{\infty} \{x(n+1) - x(n)\} z^{-n}$

$$\boxed{\lim_{n\to\infty} x(n) = x(\infty) = \lim_{z\to 1} (z-1) X(z)}$$

Some common Z-transform pairs :→

Signal	Transform	ROC
$\delta(n)$	1	All Z
$\delta(n-k)$	z^{-k}	All Z except 0 if $(k>0)$ or ∞ (if $k<0$)
$u(n)$	$\frac{1}{1-z^{-1}} = \frac{z}{z-1}$	$\lvert z\rvert > 1$
$-u(-n-1)$	$\frac{1}{1-z^{-1}} = \frac{z}{z-1}$	$\lvert z\rvert < 1$
$\alpha^n u(n)$	$\frac{1}{1-\alpha z^{-1}} = \frac{z}{z-\alpha}$	$\lvert z\rvert > \lvert\alpha\rvert$
$-\alpha^n u(-n-1)$	$\frac{1}{1-\alpha z^{-1}} = \frac{z}{z-\alpha}$	$\lvert z\rvert < \lvert\alpha\rvert$
$n\alpha^n u(n)$	$\frac{\alpha z^{-1}}{(1-\alpha z^{-1})^2}$	$\lvert z\rvert > \lvert\alpha\rvert$
$-n\alpha^n u(-n-1)$	$\frac{\alpha z^{-1}}{(1-\alpha z^{-1})^2}$	$\lvert z\rvert < \lvert\alpha\rvert$
$\cos \Omega_0 n \cdot u(n)$	$\frac{1-(\cos\Omega_0)z^{-1}}{1-(2\cos\Omega_0)z^{-1}+z^{-2}}$	$\lvert z\rvert > 1$
$\sin \Omega_0 n \cdot u(n)$	$\frac{(\sin\Omega_0)z^{-1}}{1-(2\cos\Omega_0)z^{-1}+z^{-2}}$	$\lvert z\rvert > 1$

(8C)

$\alpha^n \cos \Omega_0 n \cdot u(n)$	$\dfrac{1-(\alpha\cos\Omega_0)z^{-1}}{1-(2\alpha\cos\Omega_0)z^{-1}+z^{-2}}$	$\lvert z\rvert > \lvert\alpha\rvert$
$\alpha^n \sin \Omega_0 n \; u(n)$	$\dfrac{(\alpha\sin\Omega_0)z^{-1}}{1-(2\alpha\cos\Omega_0)z^{-1}+z^{-2}}$	$\lvert z\rvert > \lvert\alpha\rvert$

(9)

…nd the Z- transform of the signal $x(n) = u(-n)$

Solun: $x(n) = u(-n)$

WKT $u(n) \xleftrightarrow{Z} \dfrac{1}{1-z^{-1}}$; $|z|>1$

using time reversal property ; $u(-n) \xleftrightarrow{Z} \dfrac{1}{1-\left(\frac{1}{z}\right)^{-1}}$; $\left|\frac{1}{z}\right|>1$

$u(-n) \xleftrightarrow{Z} \dfrac{1}{1-z}$; $|z|<1$

Find the Z- transform of the signal $x(n) = 3 \cdot 2^n u(-n)$ using appropriate properties.

Solun: Given $x(n) = 3 \cdot 2^n u(-n) = 3 \cdot \left(\frac{1}{2}\right)^{-n} u(-n)$

WKT $\left(\frac{1}{2}\right)^n u(n) \xleftrightarrow{Z} \dfrac{1}{1-\frac{1}{2}z^{-1}}$, $|z|>\frac{1}{2}$

using time reversal property ; $\left\{\because \left|\frac{1}{z}\right| > \frac{1}{2}\right\}$

$\left(\frac{1}{2}\right)^{-n} u(-n) \xleftrightarrow{Z} \dfrac{1}{1-\frac{1}{2}z}$; $|z| < 2$

using linearity property

$3\left(\frac{1}{2}\right)^{-n} u(-n) \xleftrightarrow{Z} \dfrac{3}{1-\frac{1}{2}z}$; $|z|<2$

using appropriate properties find the Z-transform of

$x(n) = n^2\left(\frac{1}{2}\right)^n u(n-3)$

Solun:- $x(n) = \frac{1}{8} n^2 \left(\frac{1}{2}\right)^{n-3} u(n-3)$

WKT $\left(\frac{1}{2}\right)^n u(n) \xleftrightarrow{Z} \dfrac{1}{1-\frac{1}{2}z^{-1}}$; $|z|>1/2$

using time shifting property; $\left(\frac{1}{2}\right)^{n-3} u(n-3) \xleftrightarrow{Z} \dfrac{z^{-3}}{1-\frac{1}{2}z^{-1}}$; $|z|>\frac{1}{2}$; except $z=0$

$\xleftrightarrow{Z} \dfrac{1}{z^3 - \frac{1}{2}z^2}$

using differentiation in z domain property

$$n\left(\frac{1}{2}\right)^{n-3} u(n-3) \xleftrightarrow{Z} -z\,\frac{d}{dz}\left[\frac{1}{z^3-\frac{1}{2}z^2}\right];$$

$$\xleftrightarrow{Z} -z\left\{\frac{-3z^2+z}{\left(z^3-\frac{1}{2}z^2\right)^2}\right\} \Rightarrow \frac{3z^3-z^2}{\left(z^3-\frac{1}{2}z^2\right)^2}$$

$$\left\{\frac{d}{dx}(u/v) = \frac{v\,du/dx - u\,dv/dx}{v^2}\right\}$$

Again using differentiation in z-domain:

$$n^2\left(\frac{1}{2}\right)^{n-3} u(n-3) \xleftrightarrow{Z} -z\,\frac{d}{dz}\left\{\frac{3z^3-z^2}{\left(z^3-\frac{1}{2}z^2\right)^2}\right\}$$

$$= -z\left[\frac{\left(z^3-\frac{1}{2}z^2\right)^2\left(9z^2-2z\right) - 2\left(3z^3-z^2\right)\left(z^3-\frac{1}{2}z^2\right)\left(3z^2-z\right)}{\left(z^3-\frac{1}{2}z^2\right)^3}\right]$$

$$= -z\left[\frac{\left(z^6+\frac{1}{4}z^4-z^5\right)\left(9z^2-2z\right) - \left(6z^3-2z^2\right)\left(3z^5-z^4-\frac{3}{2}z^4-\frac{1}{2}z\right.}{\left(z^3-\frac{1}{2}z^2\right)^3}\right]$$

$$= \cancel{-z\left[9z^3\right.} \xleftrightarrow{Z} \frac{z^4\left(9z^2-5.5z+1\right)}{\left(z^3-\frac{1}{2}z^2\right)^3};$$

using linearity property

$$x(n) = \frac{1}{8}\,n^2\left(\frac{1}{2}\right)^{n-3} u(n-3) \xleftrightarrow{Z} \frac{1}{8}\cdot\frac{z^4\left(9z^2-5.5z+1\right)}{\left(z^3-\frac{1}{2}z^2\right)^3}$$

...d the z transform of the signal ; $x(n) = n \sin(\pi/_2 n) u(-n)$

Soln:- ~~WKT~~ $x(n)$ can be written as

$$x(n) = -n \sin(-\pi/_2 n)\, u(-n)$$

WKT $\sin(\pi/_2 \cdot n)\, u(n) \xleftrightarrow{z} \dfrac{(\sin \pi/2)\cdot z^{-1}}{1-(2\cos \pi/2)z^{-1} + z^{-2}}$; $|z|>1$

$$\xleftrightarrow{z} \frac{z^{-1}}{1+z^{-2}} = \frac{z}{z^2+1}$$

$\frac{d}{dz}\left(\frac{z}{z^2+1}\right)$

using differentiation in z domain property

$$n \sin(\pi/_2 n)\, u(n) \xleftrightarrow{z} -z\left[\frac{(z^2+1)-2z^2}{(z^2+1)^2}\right]$$

$$\xleftrightarrow{z} \frac{z^3-z}{(z^2+1)^2}$$

using time reversal property ;

$$-n \sin(-\pi/_2 n)\, u(-n) \xleftrightarrow{z} \frac{(1/z)^3-(1/z)}{\left((1/z)^2+1\right)^2} \quad ; \quad |z|<1$$

$$= \frac{\frac{(1-z^2)}{z^3}}{\frac{(1+z^2)^2}{z^4}} = \frac{z(1-z^2)}{(1+z^2)^2} \quad ; \quad |z|<1$$

→ Find the z-transform of the signal ; $x(n) = \left(\frac{1}{2}\right)^n u(n) * \left(\frac{1}{3}\right)^n u(n)$

Let $x_1(n) = \left(\frac{1}{2}\right)^n u(n)$; $X_1(z) = \dfrac{1}{1-\frac{1}{2}z^{-1}}$; $|z| > 1/2$

$x_2(n) = \left(\frac{1}{3}\right)^n u(n)$; $X_2(z) = \dfrac{1}{1-\frac{1}{3}z^{-1}}$; $|z| > 1/3$

$$x(n) = x_1(n) * x_2(n)$$

using convolution property

$$Z\{x_1(n) * x_2(n)\} = X_1(z).X_2(z) = \frac{1}{(1-\frac{1}{2}z^{-1})(1-\frac{1}{3}z^{-1})}$$

$$\Rightarrow \quad x(n) = n\left[\left(\frac{1}{2}\right)^n u(n) * \left(\frac{1}{2}\right)^n u(n)\right]$$

$$x_1(n) = \left(\frac{1}{2}\right)^n u(n) \quad ; \quad X_1(z) = \frac{1}{1-\frac{1}{2}z^{-1}} \; ; \; |z| > \frac{1}{2}$$

$$x_2(n) = \left(\frac{1}{2}\right)^n u(n) \quad ; \quad X_2(z) = \frac{1}{1-\frac{1}{2}z^{-1}} \; ; \; |z| > \frac{1}{2}$$

$$Z\{x_1(n) * x_2(n)\} = X_1(z).X_2(z) = \frac{1}{(1-\frac{1}{2}z^{-1})(1-\frac{1}{2}z^{-1})}$$

$$= \frac{1}{(1-\frac{1}{2}z^{-1})^2} \quad ; \quad |z| > \frac{1}{2}$$

differentiation property

$$Z\left\{n\left(x_1(n) * x_2(n)\right)\right\} = -z\left[\frac{\cancel{-2}(1-\frac{1}{2}z^{-1})(-\cancel{\frac{1}{2}}z^{-2})}{(1-\frac{1}{2}z^{-1})^3}\right]$$

$$= -z\left[-\frac{z^{-2} - \frac{1}{2}z^{-3}}{(1-\frac{1}{2}z^{-1})^3}\right] = \frac{-z^{-1} + \frac{1}{2}z^{-2}}{(1-\frac{1}{2}z^{-1})^3}$$

$$= \frac{z^{-1}}{(1-\frac{1}{2}z^{-1})^3} \quad ; \quad |z| > \frac{1}{2}$$

→ Given $x(n) \overset{Z}{\longleftrightarrow} X(z) = \frac{z}{z^2+4}$ with ROC: $|z|<2$ using the Z-transform properties determine the Z-transform of the following signals

(i) $y_1(n) = 2^n x(n)$ (ii) $y_2(n) = n x(n)$

$y_1(n) = 2^n x(n) \longrightarrow$ using scaling property

$$Y_1(z) = \frac{z/2}{z^2/4 + 4} = \frac{2z}{z^2+16} \; ; \; |z| < 4$$

(ii)

$y_2(n) = n\,x(n)$;

Differentiation property.

$$Y_2(z) = -z\frac{d}{dz}\left[\frac{z}{z^2+4}\right] = -z\left[\frac{(z^2+4) - z(2z)}{(z^2+4)^2}\right]$$

$$Y_2(z) = \frac{z^3 - 4z}{(z^2+4)^2} \;;\; |z| < 2$$

→ Determine the z-transform of the following signals & sketch the ROC

(i) $x_1(n) = (1/3)^n \;;\; n \geq 0$

$\qquad = (1/2)^{-n} \;;\; n < 0$

(ii) $x_2(n) = x_1(n+4)$

Soln:- (i) $X_1(z) = \sum_{n=-\infty}^{\infty} x_1(n) z^{-n} = \sum_{n=0}^{\infty} (1/3)^n z^{-n} + \sum_{n=-\infty}^{-1} (1/2)^{-n} z^{-n}$

$$= \sum_{n=0}^{\infty} (1/3)^n z^{-n} + \sum_{n=1}^{\infty} (1/2)^n z^n$$

ROC is: $\left|\frac{1}{3z}\right| < 1 \rightarrow |z| > 1/3$ The ROC is $\boxed{1/3 < |z| < 2}$

$|1/2\, z| < 1 \rightarrow |z| < 2$

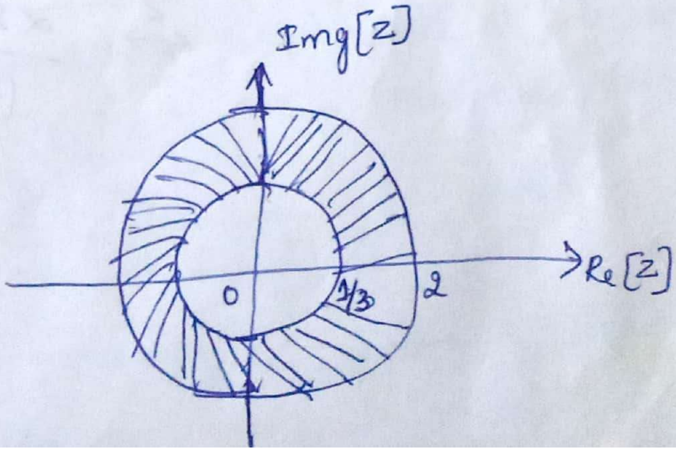
$$X_1(z) = \frac{1}{1 - 1/3\, z^{-1}} + \frac{1/2\, z}{1 - 1/2\, z} = \frac{5/6}{(1 - 1/2\, z)(1 - 1/3\, z^{-1})}$$

(ii) $x_2(n) = x_1(n+4)$

$X_2(z) = z^4 X_1(z) \longrightarrow$ using time shifting property

$$X_2(z) = \frac{5/6\, z^4}{(1 - 1/2\, z)(1 - 1/3\, z^{-1})}$$

ROC: $1/3 < |z| < 2$



→ Determine the Z-transform of the sequence ; $x(n) = n^2 u(n)$

Soln. – $x(n) = n^2 u(n)$

WKT $$u(n) \overset{Z}{\longleftrightarrow} \frac{z}{z-1} \quad ; \quad |z| > 1$$

using differentiation in Z-domain property

$$n\,u(n) \overset{Z}{\longleftrightarrow} -z \frac{d}{dz}\left[\frac{z}{z-1}\right] = \frac{z}{(z-1)^2} \quad ; \quad |z| > 1$$

Again using differentiation in Z-domain property.

$$n^2 u(n) \overset{Z}{\longleftrightarrow} -z \frac{d}{dz}\left[\frac{z}{(z-1)^2}\right] = \frac{z(z+1)}{(z-1)^3} \quad ; \quad |z| > 1$$

→ Find the Z-transform of the signal using appropriate property
$x(n) = -n \alpha^n u(-n-1)$

Soln: $x(n) = -n \alpha^n u(-n-1)$

WKT $$-u(-n-1) \overset{Z}{\longleftrightarrow} \frac{z}{z-1} \quad ; \quad |z| < 1$$

using differentiation in Z-domain property

$$\cancel{-n(\alpha} \quad -n\,u(-n-1) \overset{Z}{\longleftrightarrow} -z \frac{d}{dz}\left[\frac{z}{z-1}\right] = \frac{z}{(z-1)^2} \quad ; |z| < 1$$

using scaling in the Z-domain property

$$-n \alpha^n u(-n-1) \overset{Z}{\longleftrightarrow} \left. \frac{z}{(z-1)^2} \right|_{z = z/\alpha}$$

$$\overset{Z}{\longleftrightarrow} \frac{z/\alpha}{(z/\alpha - 1)^2}$$

$$\overset{Z}{\longleftrightarrow} \frac{\alpha z}{(z-\alpha)^2} \quad ; \quad |z| \alpha |\alpha|$$

(1) The ROC for a finite duration sequence includes entire Z plane except $z=0$ and / or $|z|=\infty$

(2) ROC doesnot contain any poles

(3) ROC is the ring in the z plane centered about origin

(4) ROC of causal sequence is of the form $|z|>r$

(5) ROC of left sided sequence is of the form $|z|<r$

(6) ROC of two sided sequence is the concentric ring in z-plane

(7) If $x(n)$ is finite causal sequence, then its ROC is entire z-plane except $z=0$

Inverse Z-transform: $\rightarrow$ The time domain signal $x(n)$ can be obtained from its Z-transform $X(z)$ by using equation

$$x(n) = \frac{1}{2\pi j} \oint X(z) z^{n-1} dz \rightarrow (1)$$

But it is very tedious to carry out the integration. The two alternative methods to get the inverse z-transforms are:

(i) Partial fraction Expansion Method

(ii) Power series Expansion Method.

Partial fraction expansion method: $\rightarrow$ consider $X(z)$ given by

$$X(z) = \frac{B(z)}{A(z)} = \frac{\sum_{k=0}^{M} b_k z^{-k}}{\sum_{k=0}^{N} a_k z^{-k}} = \frac{b_0 + b_1 z^{-1} + \dots + b_M z^{-M}}{1 + a_1 z^{-1} + \dots + a_N z^{-N}} \rightarrow (2)$$

where $a_0 = 1$

If $M < N$, we can use partial fraction expansion directly by factorizing the denominator polynomial

If $M \geq N$ then by long division bring $X(z)$ to the form

$$X(z) = \sum_{k=0}^{M-N} C_k z^{-k} + \frac{\tilde{B}(z)}{A(z)} \rightarrow (3)$$

where the numerator polynomial $\tilde{B}(z)$ has order one less than that of denominator polynomial $A(z)$. Then apply partial fraction expansion for the second term in equation (3)

Partial fraction Expansion of $X(z)$ with simple poles: $\rightarrow$

If $X(z)$ has (no of zeros are lesser than poles) simple N poles at $z = \varphi_l$ which are distinct, where $0 \leq l \leq N$. Then the partial fraction expansion is given by $X(z) = \sum_{l=1}^{N} \frac{K_l}{(1 - \varphi_l z^{-1})}$

where K_l are called the residues given by $K_l = (1 - \varphi_l z^{-1}) X(z) \Big|_{z=\varphi_l}$

Find the discrete time sequence $x(n)$ which has Z-transform

$$X(z) = \frac{-1+5z^{-1}}{\left(1-3/2\,z^{-1}+1/2\,z^{-2}\right)} \quad \text{with ROC : } |z|>1$$

Soln :-

$$X(z) = \frac{-1+5z^{-1}}{1-3/2\,z^{-1}+1/2\,z^{-2}} \; ; \quad |z|>1$$

~~Put x = z^{-1}~~

$$X(z) = \frac{-1+5z^{-1}}{1-1/2\,z^{-1}-z^{-1}+1/2\,z^{-2}}$$

$$X(z) = \frac{-1+5z^{-1}}{1\left(1-1/2\,z^{-1}\right)-z^{-1}\left(1-1/2\,z^{-1}\right)} = \frac{-1+5z^{-1}}{\left(1-1/2\,z^{-1}\right)\left(1-z^{-1}\right)}$$

$M = 1 \quad N = 2 \quad \therefore M < N$

using partial fraction expansion

$$X(z) = \frac{K_1}{\left(1-1/2\,z^{-1}\right)} + \frac{K_2}{\left(1-z^{-1}\right)}$$

$$K_1 = \left(1-1/2\,z^{-1}\right)X(z)\Big|_{z=1/2} = \cancel{\left(1-1/2\,z^{-1}\right)}\left[\frac{-1+5z^{-1}}{\cancel{\left(1-1/2\,z^{-1}\right)}\left(1-z^{-1}\right)}\right]_{z=1/2}$$

$$K_1 = \frac{-1+5z^{-1}}{1-z^{-1}}\Bigg|_{z=1/2} = \frac{-1+10}{1-2} = \underline{-9}$$

$$K_2 = \left(1-z^{-1}\right)X(z)\Big|_{z=1} = \frac{-1+5z^{-1}}{\left(1-1/2\,z^{-1}\right)}\Bigg|_{z=1} = \frac{4}{1/2} = \underline{8}$$

$$\therefore X(z) = \frac{-9}{\left(1-1/2\,z^{-1}\right)} + \frac{8}{\left(1-z^{-1}\right)} \; ; \quad |z|>1$$

$\therefore$ The ROC is $|z|>1$ ie exits outside the circle of radius 1

$\therefore$ we can conclude that the discrete time sequence corresponding to both the term must be right ~~beg~~ sided : $x(n) = -9(1/2)^n u(n) + 8u(n)$

$$\boxed{x(n) = \left[-9\,(1/2)^n + 8\right]u(n)}$$

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Find the inverse Z-transform of the following using partial fraction expansion :

$$x(z) = \frac{1-\frac{1}{2}z^{-1}}{1+\frac{3}{4}z^{-1}+\frac{1}{8}z^{-2}} \quad ; \ |z| > \frac{1}{2}$$

Solu^n

Factorising the denominator

$$x(z) = \frac{1-\frac{1}{2}z^{-1}}{(1+\frac{1}{2}z^{-1})(1+\frac{1}{4}z^{-2})} \quad ; \ |z| > \frac{1}{2}$$

$M=1 \quad N=2 \qquad M<N$

using partial fraction expansion directly :

$$X(z) = \frac{K_1}{(1+\frac{1}{2}z^{-1})} + \frac{K_2}{(1+\frac{1}{4}z^{-2})} \quad ; \ |z| > \frac{1}{2}$$

$$K_1 = (1+\tfrac{1}{2}z^{-1})\, x(z) \Big|_{z=-0.5 \text{ or } -\frac{1}{2}}$$

$$K_1 = \frac{1-\frac{1}{2}z^{-1}}{(1+\frac{1}{4}z^{-2})}\Bigg|_{z=-\frac{1}{2}} = \frac{2}{\frac{1}{2}} = 4$$

$$K_2 = (1+\tfrac{1}{4}z^{-1})\, x(z)\Big|_{z=-\frac{1}{4}} = \frac{1-\frac{1}{2}z^{-1}}{(1+\frac{1}{2}z^{-1})}\Bigg|_{z=-\frac{1}{4}} = \frac{3}{-1} = -3$$

$$x(z) = \frac{4}{(1+\frac{1}{2}z^{-1})} + \frac{-3}{(1+\frac{1}{4}z^{-1})} \quad ; \ |z| > \frac{1}{2}$$

Right sided sequence

$$\therefore\ x(n) = 4\left(-\tfrac{1}{2}\right)^n u(n) - 3\left(-\tfrac{1}{4}\right)^n u(n)$$

$$\boxed{x(n) = \cancel{4\left(-\tfrac{1}{2}\right)^n u(n)} - \left[4\left(-\tfrac{1}{2}\right)^n - 3\left(-\tfrac{1}{4}\right)^n\right]u(n)}$$

$\cancel{(1-z^{-1})\left(1-\frac{1}{3}z\right)}$

→ using partial fraction expansion method obtain the time domain signal corresponding to the Z-transform given below

(i) $X(z) = \dfrac{\frac{1}{4}z^{-1}}{(1-\frac{1}{2}z^{-1})(1-\frac{1}{4}z^{-1})}$; $|z| > \frac{1}{2}$

(ii) $X(z) = \dfrac{\frac{1}{4}z^{-1}}{(1-\frac{1}{2}z^{-1})(1-\frac{1}{4}z^{-1})}$; $|z| < \frac{1}{4}$

(iii) $X(z) = \dfrac{\frac{1}{4}z^{-1}}{(1-\frac{1}{2}z^{-1})(1-\frac{1}{4}z^{-1})}$; $\frac{1}{4} < |z| < \frac{1}{2}$

(i) $X(z) = \dfrac{K_1}{(1-\frac{1}{2}z^{-1})} + \dfrac{K_2}{(1-\frac{1}{4}z^{-1})}$; $|z| > \frac{1}{2}$

$K_1 = (1-\frac{1}{2}z^{-1})\,X(z)\Big|_{z=\frac{1}{2}} = \dfrac{1/2}{1/2} = 1$

$K_2 = (1-\frac{1}{4}z^{-1})\,X(z)\Big|_{z=\frac{1}{4}} = -1$

$X(z) = \dfrac{1}{(1-\frac{1}{2}z^{-1})} + \dfrac{-1}{(1-\frac{1}{4}z^{-1})}$; $|z| > \frac{1}{2}$

Right sided sequence

$$\boxed{x(n) = \left(\tfrac{1}{2}\right)^n u(n) - \left(\tfrac{1}{4}\right)^n u(n)}$$

(ii) ROC $|z| < \frac{1}{4}$; ∵ ROC is inside the circle it must be left sided sequence

$$\boxed{x(n) = -\left(\tfrac{1}{2}\right)^n u(-n-1) + \left(\tfrac{1}{4}\right)^n u(-n-1)}$$

ROC : $\frac{1}{4} < |z| < \frac{1}{2}$

$\because$ ROC of the given $X(z)$ exists in between $\frac{1}{4}$ & $\frac{1}{2}$ radius in Z-plane it is two sided sequence

$$\therefore \boxed{x(n) = -\left(\tfrac{1}{2}\right)^n u(-n-1) - \left(\tfrac{1}{4}\right)^n u(n)}$$

4. Find the inverse Z-transform of $X(z) = \dfrac{z^2 - 3z}{z^2 + \frac{3}{2}z - 1}$

ROC : $\frac{1}{2} < |z| < 2$

Soln:-

$$X(z) = \frac{z^2 - 3z}{z^2 + \frac{3}{2}z - 1}$$

Dividing both numerator & denominator by z^2

$$X(z) = \frac{1 - 3z^{-1}}{1 + \frac{3}{2}z^{-1} - z^{-2}} = \frac{1 - 3z^{-1}}{1 + \frac{1}{2}z^{-1} + z^{-1} - z^{-2}}$$

$$X(z) = \cancel{\frac{1 - 3z^{-1}}{1(1 + \frac{1}{2}z^{-1}) + 2z^{-1}(}} \quad \frac{1 - 3z^{-1}}{(1 + 2z^{-1})(1 - \frac{1}{2}z^{-1})} ; \quad \frac{1}{2} < |z| < 2$$

using partial fraction method;

$$X(z) = \frac{K_1}{(1 + 2z^{-1})} + \frac{K_2}{(1 - \frac{1}{2}z^{-1})}$$

$$K_1 = \left.(1 + 2z^{-1})\,X(z)\right|_{z=-2} = 2$$

$$K_2 = \left.(1 - \tfrac{1}{2}z^{-1})\,X(z)\right|_{z=\frac{1}{2}} = -1$$

$$X(z) = \frac{2}{(1 + 2z^{-1})} - \frac{1}{(1 - \frac{1}{2}z^{-1})} ; \quad \frac{1}{2} < |z| < 2$$

$$\boxed{x(n) = -2(-2)^n u(-n-1) - \left(\tfrac{1}{2}\right)^n u(n)}$$

→ $x(z) = \dfrac{1-2z^{-1}}{1-5/2\,z^{-1}+z^{-2}}$ Find the sequence associated with given z-transform using partial expansion & $x(n)$ is absolutely summ

$$x(z) = \frac{K_1}{(1-2z^{-1})} + \frac{K_2}{(1-\frac{1}{2}z^{-1})}$$

$$K_1 = (1-2z^{-1})\,x(z)\Big|_{z=2} =$$

$$x(z) = \frac{\cancel{(1-2z^{-1})}}{\cancel{(1-2z^{-1})}(1-\frac{1}{2}z^{-1})}$$

$$x(z) = \frac{1}{(1-\frac{1}{2}z^{-1})}$$

∴ Only one pole exits. Therefore no need to expand

Taking inverse z-transform we get;

$$\therefore \boxed{x(n) = \left(\tfrac{1}{2}\right)^n u(n)}$$

→ Find the inverse z-transform of the following using partial fraction expansion method. $X(z) = \dfrac{1+2z^{-1}+z^{-2}}{1-3/2\,z^{-1}+\frac{1}{2}z^{-2}}$; with ROC $|z|>1$

Solun:- $M=2$ $N=2$ ∴ $M \nless N$.

By long division method;

$$\begin{array}{r|l} & \quad 2 \\ \frac{1}{2}z^{-2} - \frac{3}{2}z^{-1} + 1 & z^{-2} + 2z^{-1} + 1 \\ & z^{-2} - 3z^{-1} + 2 \\ & (-) \quad (+) \quad (-) \\ \hline & \quad 5z^{-1} - 1 \end{array}$$

$$\therefore x(z) = 2 + \frac{5z^{-1}-1}{1-\frac{3}{2}z^{-1}+\frac{1}{2}z^{-2}} = 2 + \underbrace{\frac{5z^{-1}-1}{(1-z^{-1})(1-\frac{1}{2}z^{-1})}}_{X_1(z)}$$

apply partial fraction expansion for the second term

$$X_1(z) = \frac{K_1}{(1-z^{-1})} + \frac{K_2}{(1-\tfrac{1}{2}z^{-1})}$$

$$\therefore K_1 = (1-z^{-1})X_1(z)\Big|_{z=1} = 8$$

$$K_2 = (1-\tfrac{1}{2}z^{-1})\,X(z)\Big|_{z=1/2} = -9$$

$$\therefore X(z) = 2 + \frac{8}{(1-z^{-1})} - \frac{9}{(1-\tfrac{1}{2}z^{-1})}$$

$\because |z|>1 \rightarrow$ outside the circle $\rightarrow$ Hence right sided sequence

$$\boxed{x(n) = 2\delta(n) + 8u(n) - 9\,(\tfrac{1}{2})^n u(n)}$$

$\rightarrow$ $X(z) = \dfrac{z^3 + z^2 + \frac{3}{2}z + \frac{1}{2}}{z^3 + \frac{3}{2}z^2 + \frac{1}{2}z}$; ROC $|z| < 1/2$

Soln:
Dividing both numerator & denominator by z^3

$$X(z) = \frac{1 + z^{-1} + \frac{3}{2}z^{-2} + \frac{1}{2}z^{-3}}{1 + \frac{3}{2}z^{-1} + \frac{1}{2}z^{-2}}$$

$M=3 \quad N=2 \qquad M>N$

by long division method;

$$\begin{array}{r|l} & \quad z^{-1} \\ \frac{1}{2}z^{-2} + \frac{3}{2}z^{-1} + 1 & \frac{1}{2}z^{-3} + \frac{3}{2}z^{-2} + z^{-1} + 1 \\ & \frac{1}{2}z^{-3} + \frac{3}{2}z^{-2} + z^{-1} \\ & (-) \qquad (-) \qquad (-) \\ \hline & \qquad\qquad 1 \end{array}$$

$$\therefore X(z) = z^{-1} + \frac{1}{1 + \frac{3}{2}z^{-1} + \frac{1}{2}z^{-1}} = z^{-1} + \underbrace{\frac{1}{(1+\frac{1}{2}z^{-1})(1+z^{-1})}}_{X_1(z)}$$

$$x_1(z) = \frac{K_1}{(1+z^{-1})} + \frac{K_2}{(1+\frac{1}{2}z^{-1})}$$

$$K_1 = (1+z^{-1})\,x_1(z)\Big|_{z=-1} = 2$$

$$K_2 = (1+\tfrac{1}{2}z^{-1})\,x_1(z)\Big|_{z=-\frac{1}{2}} = -1$$

$$\therefore\ x(z) = z^{-1} + \frac{2}{(1+z^{-1})} - \frac{1}{(1+\frac{1}{2}z^{-1})}$$

$\because$ ROC $|z| < \frac{1}{2}$; $x(n)$ must be left sided sequence. By using linearity property

$$\boxed{x(n) = \delta(n-1) - 2\,(-1)^n u(-n-1) + (-\tfrac{1}{2})^n\,u(-n-1)}$$

Partial fraction Expansion of $x(z)$ with multiple poles:→

If $x(z)$ (no of zeros are lesser than poles) has multiple poles, the partial fraction is slightly different

Eg Consider that $x(z)$ has 'N' poles. If the pole $z=\beta$ repeats L times & the remaining $N-L$ poles are simple at $z=\varphi_l$ $1 \le l \le N-L$, then the partial fraction expansion is given by

$$x(z) = \sum_{l=1}^{N-L} \frac{\quad}{1-\varphi_l z^{-1}} + \sum_{i=1} \frac{K_l}{(1-\beta z^{-1})^i} \ ; \ \gamma_i \longrightarrow (1)$$

where the constants are computed using the formula

$$\gamma_i = \frac{1}{(L-i)!\,(-\beta)^{L-i}} \frac{d^{L-i}}{d(z^{-1})^{L-i}} \left[(1-\beta z^{-1})^L x(z)\right]\Big|_{z=\beta} \ ; 1 \le i \le L \ \hookrightarrow (2)$$

$$\& \quad K_L = (1-\varphi_l z^{-1})\,x(z)\Big|_{z=\varphi_l} \longrightarrow (3)$$

Find the inverse z-transform of $X(z) = \dfrac{4-3z^{-1}+3z^{-2}}{(1+2z^{-1})(1-3z^{-1})^2}$:

$|z|>3$

Soln: Here there are 2 poles (ie $l=2$) at $z=3$ & $M<N$

The partial fraction expansion of $X(z)$ is

$$X(z) = \frac{K_1}{(1+2z^{-1})} + \frac{\gamma_1}{(1-3z^{-1})} + \frac{\gamma_2}{(1-3z^{-1})^2}$$

$$\frac{4-3z^{-1}+3z^{-2}}{(1+2z^{-1})\cancel{(1-3z^{-1})^2}} = \frac{K_1(\cancel{1-3z^{-1}})(1-3z^{-1})^2 + \gamma_1(1+2z^{-1})(1-3z^{-1}) + \gamma_2(1+2z^{-1})}{(1+2z^{-1})\cancel{(1-3z^{-1})^2}}$$

$$4-3z^{-1}+3z^{-2} = K_1(\cancel{1-3z^{-1}})(1-3z^{-1})^2 + \gamma_1(1+2z^{-1})(1-3z^{-1}) + \gamma_2(1+2z^{-1}) \rightarrow (1)$$

Put $z^{-1} = -1/2$ in equation (1)

$$4-3(-1/2)+3(-1/2)^2 = K_1(1-3(-1/2))^2$$

$$K_1(1+3/2)^2 = 6.25$$

$$\boxed{K_1 = 1}$$

Put $z^{-1} = 1/3$ in equation (1)

$$4-3(1/3)+3(1/3)^2 = \gamma_2(1+2(1/3))$$

$$\boxed{\gamma_2 = 2}$$

Put $z^{-1}=1$ & use $K_1=1$ and $\gamma_2=2$ in equation (1)

$$\cancel{4}-\cancel{3}+\cancel{3} = \cancel{4} + \gamma_1(-6) + 4$$

$$-6\gamma_1 = -6$$

$$\therefore \boxed{\gamma_1 = 1}$$

$$X(z) = \frac{1}{(1+2z^{-1})} + \frac{2}{(1-3z^{-1})} + \frac{2}{(1-3z^{-1})^2} \; ; \; \underline{|z| > 3}$$

Right sided sequence

$$\frac{1}{1+2z^{-1}} \xleftrightarrow{IZT} (-2)^n u(n)$$

$$\frac{2}{(1-3z^{-1})} \xleftrightarrow{IZT} 2(3)^n u(n)$$

consider the 3rd term

$$\frac{1}{(1-3z^{-1})^2} \xleftrightarrow{IZT} (3)^n u(n)$$

$$X_1(z) \longleftrightarrow x_1(n)$$

$$-z\frac{d}{dz}X_1(z) \longleftrightarrow n x_1(n)$$

$$-z\frac{d}{dz}\left(\frac{1}{1-3z^{-1}}\right) \longleftrightarrow n(3)^n u(n)$$

$$-z\left[\frac{0-3z^{-2}}{(1-3z^{-1})^2}\right] \longleftrightarrow n(3)^n u(n)$$

$$\frac{3z^{-1}}{(1-3z^{-1})^2} \longleftrightarrow n(3)^n u(n)$$

$$\frac{\cancel{3z^{-1}}}{}$$

multiply & divide the 3rd term by $(3z^{-1})$ → $n(3)^n u(n)$

$$\frac{2 \cdot 3z^{-1}}{3z^{-1}(1-3z^{-1})^2} = \left(\frac{2}{3}\right)\frac{3z^{-1}\,z^{1}}{(1-3z^{-1})^2}$$

z^1 → time shift

$$\therefore \left(\frac{2}{3}\right)(n+1)(3)^{n+1}u(n+1)$$

$$\therefore \boxed{x(n) = (-2)^n u(n) + (3)^n u(n) + \frac{2}{3}\cdot(n+1)(3)^{n+1}u(n+1)}$$

or

$$X(z) = \frac{K_1}{(1+2z^{-1})} + \frac{r_1}{(1-3z^{-1})} + \frac{r_2}{(1-3z^{-1})^2}$$

$$\frac{d}{dz^{-1}}\left(\frac{z^{-1}}{z}\right)$$

$z^{(-1)}$

$$K_1 = (1+2z^{-1})\,X(z)\Big|_{z=-2} = 1$$

$$r_1 = \frac{1}{(2-1)!\,(-3)^{2-1}}\,\frac{d^{2-1}}{d(z^{-1})^{2-1}}\left[(1-3z^{-1})^2 X(z)\right]\Big|_{z=3}$$

$$= \frac{1}{(-3)}\,\frac{d}{dz^{-1}}\left[\frac{4-3z^{-1}+3z^{-2}}{(1+2z^{-1})}\right]\Big|_{z=3}$$

$$\boxed{r_1 = 1}$$

$$r_2 = \frac{1}{(2-2)!\,(-3)^{2-2}}\,\frac{d^{2-2}}{d(z^{-1})^{2-2}}\left[(1-3z^{-1})^2\; X(z)\right]\Big|_{z=3}$$

$$\boxed{r_2 = 2}$$

→ Find the IZT of $X(z) = \dfrac{3-2z^{-1}+z^{-2}}{(1-z^{-1})\,(1-\frac{1}{3}z^{-1})^2}$ using partial fraction expansion method

Soln:-

$$X(z) = \frac{K_1}{(1-z^{-1})} + \frac{r_1}{(1-\frac{1}{3}z^{-1})} + \frac{r_2}{(1-\frac{1}{3}z^{-1})^2}$$

$$K_1 = (1-z^{-1})\,X(z)\Big|_{z=1} = \frac{3-2z^{-1}+z^{-2}}{(1-\frac{1}{3}z^{-1})^2}\Big|_{z=1} = \underline{\underline{4.5}}$$

$$r_1 = \frac{1}{(2-1)!\,(-\frac{1}{3})^{2-1}}\,\frac{d^{2-1}}{d(z^{-1})^{2-1}}\left[(1-\tfrac{1}{3}z^{-1})^2\,X(z)\right]\Big|_{z=1/3}$$

$$= (-3)\,\frac{d}{dz^{-1}}\left[\frac{3-2z^{-1}+z^{-2}}{(1-z^{-1})}\right]\Big|_{z=1/3} = (-3)\left\{\frac{(1-z^{-1})(-2+2z^{-1}) - (3-2z^{-1}+z^{-2})(-1)}{(1-z^{-1})^2}\right\}_{z=1/3}$$

$$\boxed{r_1 = 1.5}$$

$$r_2 = \frac{1}{(2-2)!\,(-1/3)^{2-2}} \frac{d^{2-2}}{d(z^{-1})^{2-2}} \left[(1 - 1/3\, z^{-1})^2 X(z) \right]_{z = 1/3}$$

$$r_2 = \left. \frac{3 - 2z^{-1} + z^{-2}}{(1 - z^{-1})} \right|_{z = 1/3} = \underline{-3}$$

$$\therefore X(z) = \frac{4.5}{(1 - z^{-1})} + \frac{1.5}{(1 - 1/3\, z^{-1})} - \frac{3}{(1 - 1/3\, z^{-1})^2}$$

<u>IZT</u>

$$\frac{1}{1 - z^{-1}} \longleftrightarrow u(n) \quad ; \quad \frac{1}{(1 - 1/3\, z^{-1})} \longleftrightarrow (1/3)^n u(n)$$

<u>3rd term</u>

Multiply and divide by $1/3\, z^{-1}$

$$\frac{3 \cdot 1/3\, z^{-1}}{1/3\, z^{-1} (1 - 1/3\, z^{-1})^2} \rightarrow n (1/3)^n u(n)$$

$$= \frac{9 (1/3\, z^{-1})\, z^{1}}{(1 - 1/3\, z^{-1})^2} \longleftrightarrow 9(n+1) (1/3)^{n+1} u(n+1)$$

$$\therefore \boxed{x(n) = 4.5\, u(n) + 1.5 (1/3)^n u(n) - 9 (n+1) (1/3)^{n+1} u(n+1)}$$

→ Determine the IZT of the following sequences using partial fraction expansion

(i) $X_1(z) = \dfrac{4 - 3z^{-1} + 3z^{-2}}{(z+2)(z-3)^2}$; $|z| > 3$

(ii) $X_2(z) = \dfrac{4 - 3z^{-1} + 3z^{-2}}{(z+2)(z-3)^2}$; $|z| < 2$

(iii) $X_3(z) = \dfrac{4 - 3z^{-1} + 3z^{-2}}{(z+2)(z-3)^2}$; $2 < |z| < 3$

c

Soln :- Given $X_1(z) = \dfrac{4-3z^{-1}+3z^{-2}}{(z+2)(z-3)^2}$

$$= \frac{4-3z^{-1}+3z^{-2}}{z^3(1+2z^{-1})(1-3z^{-1})^2} = z^{-3}\underbrace{\left[\frac{4-3z^{-1}+3z^{-2}}{(1+2z^{-1})(1-3z^{-1})^2}\right]}_{Xa(z)}$$

Partial fraction of $Xa(z)$

$$Xa(z) = \frac{K_1}{(1+2z^{-1})} + \frac{r_1}{(1-3z^{-1})} + \frac{r_2}{(1-3z^{-1})^2}$$

$$K_1 = (1+2z^{-1})\,X(z)\Big|_{z=-2} = \underline{\underline{1}}$$

$$r_1 = \frac{1}{(2-1)!\,(-3)^{2-1}}\;\frac{d^{2-1}}{d(z^{-1})^{2-1}}\left[(1-3z^{-1})^2\,Xa(z)\right]_{z=3} = \underline{\underline{1}}$$

$$r_2 = \frac{1}{(2-2)!\,(-3)^0}\;\frac{d^0}{d(z^{-1})^0}\left[(1-3z^{-1})^2\,Xa(z)\right]_{z=3} = \underline{\underline{2}}$$

$$\therefore\; X(z) = z^{-3}\left[\frac{1}{1+2z^{-1}} + \frac{1}{1-3z^{-1}} + \frac{2}{(1-3z^{-1})^2}\right]$$

$$X(z) = \frac{z^{-3}}{1+2z^{-1}} + \frac{z^{-3}}{1-3z^{-1}} + \frac{(2/3)\,(3z^{-1})\,z^1 . z^{-3}}{(1-3z^{-1})^2}$$

(a) Given ROC: $|z|>3$ ∴ $x(n)$ must be right sided sequ...

$$x(n) = (-2)^{n-3}u(n-3) + (3)^{n-3}u(n-3) + \left(\frac{2}{3}\right)(n-2)(3)^{n-2}u(n-$$

(b) Given: ROC $|z|<2$; $x(n)$ is left sided sequence & applying time shifting property

$$x(n) = -(-2)^{n-3}u(-n+2) - (3)^{n-3}u(-n+2) - (2/3)(n-2)(3)^{n-2}u(-n+1$$

$$-\alpha^n u(-n-1) \xrightarrow{\text{Replace } n \text{ by } (n-3)} -\alpha^n u(-(n-3)-1)$$

(c) Given: ROC : $2<|z|<3$ $x(n)$ must be two sided sequence

$$x(n) = (-2)^{n-3}u(n-3) - (3)^{n-3}u(-n+2) - \left(\frac{2}{3}\right)(n-2)(3)^{n-2}u(-n+1)$$

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→ $$X(z) = \frac{1}{(1+z^{-1})(1-z^{-1})^2} ; |z|>1$$

Soln

$$X(z) = \frac{K_1}{(1+z^{-1})} + \frac{r_1}{(1-z^{-1})} + \frac{r_2}{(1-z^{-1})^2} ; |z|>1$$

$$K_1 = (1+z^{-1})X(z)\Big|_{z=-1} = \frac{\cancel{(1+z^{-1})}}{\cancel{(1+z^{-1})}(1-z^{-1})^2}\Bigg|_{z=-1} ; |z|>1 = 1/2$$

$$r_1 = \frac{1}{(-1)}\frac{d}{dz^{-1}}\left[\frac{\cancel{(1-z^{-1})^2}\cdot 1}{(1+z^{-1})\cancel{(1-z^{-1})^2}}\right]_{z^{-1}=1} = -1\left[-1(1+z^{-1})(1)\right]_{z^{-1}=1}$$

$$\boxed{r_1 = (1+z^{-1}) = 2}$$

$$\boxed{r_2 = \frac{1}{(1+z^{-1})}\Bigg|_{z^{-1}=1} = 1/2}$$

$$X(z) = +\frac{2}{(1-z^{-1})} + \frac{1/2}{(1-z^{-1})^2} = 2u(n) + 1/2(n+1)\alpha^{(n+1)}u(n+1)$$

$$\frac{1/2\, z^{-1}}{z^{-1}(1-z^{-1})^2}$$

$(n+1)\alpha^{(n+1)}u(n+1)$

...er series Expansion Method :→

WKT $X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n} = x(-\infty) z^{\infty} + \dots + x(-3)z^{3} + x(-2) z^{2} + x(-1) z + x(0) + x(1) z^{-1} + x(2) z^{-2} + x(3) z^{-3} + \dots + x(\infty) z^{-\infty} \rightarrow (1)$

By comparing the given transform with equation (1) we can obtain corresponding discrete time sequence $x(n)$ because the coefficients of z^{-n} is the sequence values $x(n)$

Advantage : Ability to find the inverse z-transform of the signals that are not a ratio of polynomials in 'z'

Find the z-transform of $X(z) = \frac{1}{1 - 1.5z^{-1} + 0.5z^{-2}}$, ROC $|z| > 1$

Soln : It is right sided sequence. Therefore convert $X(z)$ into power series having only negative powers of 'z'

$$
\begin{array}{r|l}
 & 1 + 1.5z^{-1} + 1.75z^{-2} + 1.875z^{-3} + 1.9375z^{-4} + \dots \\
\hline
1 - 1.5z^{-1} + 0.5z^{-2} & 1 \\
 & 1 - 1.5z^{-1} + 0.5z^{-2} \\
\hline
 & 1.5z^{-1} - 0.5z^{-2} \\
 & 1.5z^{-1} - 2.25z^{-2} + 0.75z^{-3} \\
\hline
 & 1.75z^{-2} - 0.75z^{-3} \\
 & 1.75z^{-2} - 2.625z^{-3} + 0.875z^{-4} \\
\hline
 & 1.875z^{-3} - 0.875z^{-4} \\
 & 1.875z^{-3} - 2.8125z^{-4} + 0.9375z^{-5} \\
\hline
 & 1.9375z^{-4} - 0.9375z^{-5} \\
 & 1.9375z^{-4} - 2.906z^{-5} + 0.96875z^{-6} \\
\hline
\end{array}
$$

$\therefore X(z) = 1 + 1.5z^{-1} + 1.75z^{-2} + 1.875z^{-3} + 1.9375z^{-4} + \dots$

Comparing with equan (1)

$x(n) = 0$; for $n < 0$

$x(0) = 1$; $x(1) = 1.5$; $x(2) = 1.75$; $x(3) = 1.875$

$x(4) = 1.9375$ . . . . . .

$$\therefore \boxed{x(n) = \{\underset{\uparrow}{1}, 1.5, 1.75, 1.875, 1.9375 \ldots\}}$$

→ find the IZT of $X(z) = \dfrac{1}{1 - 1.5z^{-1} + 0.5z^{-2}}$; $|z| < 0.5$

Soln: It is left handed sequence. Convert $X(z)$ into power series having only positive powers of 'z'

$$\cancel{1 - 1.5z^{-1} + 0.5z^{-2} \,)\, 1}$$

$$\cancel{1 - 1.5z^{-1} + 0.5z^{-2}}$$

$$\cancel{1.5z^{-1} - 0.5z^{-2}}$$

$$0.5z^{-2} - 1.5z^{-1} + 1 \,\overline{)\, 1} \quad 2z^2 + 6z^3 + 14z^4 + 30z^5 + 62z^6 + \ldots$$

$1 - 3z + 2z^2$

$3z - 2z^2$

$3z - 9z^2 + 6z^3$

$7z^2 - 6z^3$

$7z^2 - 21z^3 + 14z^4$

$15z^3 - 14z^4$

$15z^3 - 45z^4 + 30z^5$

$31z^4 - 30z^5$

$$\therefore X(z) = 2z^2 + 6z^3 + 14z^4 + 30z^5 + 62z^6 + \ldots\ldots$$

Comparing with expansion equation

$x(n) = 0$; for $n > 0$

$x(0) = 0$

$x(-1) = 0$

$x(-2) = 2$

$x(-3) = 6$

$x(-4) = 14$

$x(-5) = 30$

$x(-6) = 62$

$$\therefore \boxed{x(n) = \{\ldots\ldots 62, 30, 14, 6, 2, 0, \underset{\uparrow}{0}\}}$$

Find the IZT of $X(Z) = z^2(1-\frac{1}{2}z^{-1})(1+z^{-1})(1-z^{-1})$

$$X(z) = (z^2 - \frac{1}{2}z)(1-z^{-1}) = z^2 - \frac{1}{2}z - 1 + \frac{1}{2}z^{-1}$$

By comparing with expansion equation

$x(-2) = 1$; $x(1) = \frac{1}{2}$

$x(-1) = -\frac{1}{2}$; $x(n) = 0$; otherwise

$x(0) = -1$;

Alternatively : $x(n) = \delta(n+2) - \frac{1}{2}\delta(n+1) - \delta(n) + \frac{1}{2}\delta(n-1)$

→ Find the IZT of $X(Z) = \sum_{K=5}^{10} (\frac{1}{K}) z^{-K}$; $|z| > 0$

$$X(z) = \frac{1}{5}Z^{-5} + \frac{1}{6}z^{-6} + \frac{1}{7}z^{-7} + \frac{1}{8}z^{-8} + \frac{1}{9}z^{-9} + \frac{1}{10}z^{-10}$$

Taking IZT;

$x(5) = \frac{1}{5}$ $x(7) = \frac{1}{7}$ $x(9) = \frac{1}{9}$

$x(6) = \frac{1}{6}$ $x(8) = \frac{1}{8}$ $x(10) = \frac{1}{10}$

(or)

$$x(n) = \frac{1}{5}\delta(n-5) + \frac{1}{6}\delta(n-6) + \frac{1}{7}\delta(n-7) + \frac{1}{8}\delta(n-8) + \frac{1}{9}\delta(n-9) + \frac{1}{10}\delta(n-10)$$

(or)

$$x(n) = \sum_{K=5}^{10} \frac{1}{K}\delta(n-K)$$

02-05-2017

→ Using power series expansion method determine the IZT of

$X(Z) = \cos(2Z)$; $|z| < \infty$

Soln: Given $X(z)$ is left handed sequence because ROC is $|z| < \infty$

WKT $\cos\theta = 1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \frac{\theta^6}{6!} + \ldots$

$$\boxed{\cos\theta = \sum_{k=0}^{\infty} (-1)^k \frac{\theta^{2k}}{(2k)!}}$$

$$X(z) = \cos 2z$$

$$= \sum_{k=0}^{\infty} \left[(-1)^k \frac{(2z)^{2k}}{(2k)!} \right] = \sum_{k=0}^{\infty} \left[(-1)^k \frac{2^{2k} \cdot z^{2k}}{(2k)!} \right]$$

$$= \sum_{k=0}^{\infty} \left((-4)^k \frac{z^{2k}}{(2k)!} \right)$$

Taking IZT : $\boxed{x(n) = \sum_{k=0}^{\infty} \frac{(-4)^k}{(2k)!} \delta(n+2k)}$

→ using power series expansion method, Determine the IZT

$$X(z) = \cos(z^{-2}) \quad ; \ |z|>0$$

Right handed sequence

$$\cos\theta = \sum_{k=0}^{\infty} (-1)^k \frac{\theta^{2k}}{(2k)!}$$

$$X(z) = \cos(z^{-2}) = \sum_{k=0}^{\infty} \frac{(-1)^k \cdot (z^{-2})^{2k}}{(2k)!} = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!} z^{-4k}$$

Taking IZT $\boxed{x(n) = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!} \delta(n-4k)}$

→ using power series expansion method, determine the IZT of $X(z) = \ln(1+z^{-1})$

$|z| > 0$

Solu^n: $\ln(1+\theta) = \theta - \frac{\theta^2}{2} + \frac{\theta^3}{3} - \frac{\theta^4}{4} + \dots$ if $|\theta| < 1$

$$\ln(1+\theta) = \sum_{k=1}^{\infty} \frac{(-1)^{k-1} \theta^k}{k}$$

$$X(z) = \ln(1+z^{-1}) \overset{(21)}{=} \sum_{k=1}^{\infty} \frac{(-1)^{k-1}(z^{-1})^k}{k} = \sum_{k=1}^{\infty} \frac{(-1)^{k-1} z^{-k}}{k}$$

Taking IZT on both the sides we get;

$$x(n) = \sum_{k=1}^{\infty} \frac{(-1)^{k-1} \delta(n-k)}{k}.$$

→ $X(z) = \ln(1-2z)$; $|z| < 1/2$

$$\ln(1-\theta) = -\theta - \frac{\theta^2}{2} - \frac{\theta^3}{3} - \frac{\theta^4}{4} - \dots = -\sum_{k=1}^{\infty} \frac{\theta^k}{k}$$

$$X(z) = \ln(1-2z) = -\sum_{k=1}^{\infty} \frac{(2z)^k}{k} = -\sum_{k=1}^{\infty} \frac{2^k \cdot z^k}{k}$$

Taking IZT on both the sides we get;

$$\boxed{x(n) = -\sum_{k=1}^{\infty} \frac{2^k}{k} \cdot \delta(n+k)}$$

→ Determine the IZT of $X(z) = \ln\left(\frac{\alpha}{\alpha - z^{-1}}\right)$; $|z| > \frac{1}{|\alpha|}$

Soln: $X(z) = \ln\left(\frac{\alpha}{\alpha - z^{-1}}\right) = \ln\left(\frac{1}{1-(\alpha z)^{-1}}\right) = -\ln\left(1-(\alpha z)^{-1}\right)$

ROC $(\alpha z)^{-1} < 1$

$$X(z) = +\sum_{k=1}^{\infty} \frac{(\alpha z)^{-k}}{k} = \sum_{k=1}^{\infty} \frac{\alpha^{-k} z^{-k}}{k} ; \left\{ |z| > \frac{1}{|\alpha|} \right\}$$

IZT

$$\boxed{x(n) = \sum_{k=1}^{\infty} \frac{\alpha^{-k}}{k} \delta(n-k)}$$

→ $X(z) = \frac{1}{1-z^{-3}}$; $|z| > 1$

$$\sum_{k=0}^{\infty} \alpha^k = \frac{1}{1-\alpha} ; \quad |\alpha| < 1$$

ROC

$$\left\{ \begin{array}{l} |z^{-3}| < 1 \\ \text{ie } |z| > 1 \end{array} \right\}$$

Given $X(z) = \frac{1}{1-z^{-3}} = \sum_{k=0}^{\infty} z^{-3k}$

Taking IZT $\boxed{x(n) = \sum \delta(n-3k)}$; $x(n) = 1$; when n is integer multiple of 3 & $n \geq 0$

$= 0$; elsewhere

Transform Analysis of LTI systems :→ The Z-transform plays an important role in the analysis & representation of discrete-time LTI systems

Transfer function & impulse response :

consider a DT LTI system having impulse response $h(n)$

$$x(n) \longrightarrow \boxed{h(n)} \longrightarrow y(n)$$

fig: discrete time LTI system

$$y(n) = h(n) * x(n)$$

Taking Z-transform

$$Y(z) = H(z) \cdot X(z) \quad ;$$

$X(z)$ = ZT of the input $x(n)$

$Y(z)$ = ZT of the output $y(n)$

$H(z)$ = ZT of the input $h(n)$

$$\boxed{H(z) = \frac{Y(z)}{X(z)}}$$

$H(z)$ is refered to as system function or transfer function of the system & this equation is valid for all values of 'z' for which $X(z)$ is non zero.

03-05-2017 → A causal system has input $x(n)$ & output $y(n)$ find the impulse response of the system if $x(n) = \delta(n) + \frac{1}{4}\delta(n-1) - \frac{1}{8}\delta(n-2)$

$$y(n) = \delta(n) - 3/4\ \delta(n-1)$$

Soln :- Taking Z-transform of $x(n)$ & $y(n)$

$$X(z) = 1 + \frac{1}{4}z^{-1} - \frac{1}{8}z^{-2}$$

$$Y(z) = 1 - 3/4\, z^{-1}$$

Transfer function of the system is given by $H(z) = \dfrac{Y(z)}{X(z)}$

(22)

$$H(z) = \frac{1-\frac{3}{4}z^{-1}}{1+\frac{1}{4}z^{-1}-\frac{1}{8}z^{-2}} = \frac{1-3/4\,z^{-1}}{(1-1/4\,z^{-1})(1+1/2\,z^{-1})}$$

By partial fraction method

$$H(z) = \frac{-2/3}{(1-1/4\,z^{-1})} + \frac{5/3}{(1+1/2\,z^{-1})}$$

It is given that the system is causal [ie $h(n)=0$ for $n<0$]

Taking inverse Z-transform

$$\boxed{h(n) = \frac{-2}{3}\left(\frac{1}{4}\right)^n u(n) + \frac{5}{3}\left(-1/2\right)^n u(n)}$$

→ Design a causal system with the property that if the input is

$x(n) = (1/2)^n u(n) - 1/4\,(1/2)^{n-1} u(n-1)$, then the output is

$y(n) = (1/3)^n u(n)$

Determine the impulse response & the system function $H(z)$ of the system that satisfies this condition.

Soln :— Taking ZT of $x(n)$ & $y(n)$

$$X(z) = \frac{1}{(1-1/2\,z^{-1})} - \frac{1/4\,z^{-1}}{(1-1/2\,z^{-1})} = \frac{(1-1/4\,z^{-1})}{(1-1/2\,z^{-1})}$$

$$Y(z) = \frac{1}{1-1/3\,z^{-1}}$$

The system function $H(z) = \frac{Y(z)}{X(z)} = \frac{(1-1/2\,z^{-1})}{(1-1/3\,z^{-1})(1-1/4\,z^{-1})}$

By partial fraction expansion method

$$H(z) = \frac{-2}{(1-1/3\,z^{-1})} + \frac{3}{(1-1/4\,z^{-1})}$$

Taking IZT; $\boxed{h(n) = -2(1/3)^n u(n) + 3(1/4)^n u(n)}$

→ A system has impulse response $h(n) = (1/2)^n u(n)$. determine input to the system if the output is given by

$$y(n) = 1/3\, u(n) + 2/3\,(-1/2)^n u(n)$$

<u>Soln:</u> Taking ZT of $h(n)$ & $y(n)$

$$H(z) = \frac{1}{1 - 1/2\, z^{-1}} \; ; \quad Y(z) = \frac{1}{1 - 1/3\, z^{-1}} + \frac{2/3}{1 + 1/2\, z^{-1}}$$

$$Y(z) = \frac{1 + 1/2\, z^{-1} + 2/3 - 2/9\, z^{-1}}{(1 - 1/3\, z^{-1})(1 + 1/2\, z^{-1})} = \frac{5}{3} - \frac{5 z^{-1}}{18}$$

$$Y(z) = \frac{1/3}{1 - z^{-1}} + \frac{2/3}{(1 + 1/2\, z^{-1})} = \frac{(1 - 1/2\, z^{-1})}{(1 - z^{-1})(1 + 1/2\, z^{-1})}$$

The ZT of $x(n)$ is given by

$$X(z) = \frac{Y(z)}{H(z)} = \frac{(1 - 1/2\, z^{-1})^2}{(1 - 1/2\, z^{-1} - 1/2\, z^{-2})} = \frac{1 - z^{-1} + 1/4\, z^{-2}}{1 - 1/2\, z^{-1} - 1/2\, z^{-2}}$$

Since $M \nless N$ by long division method

$$X(z) = -1/2 + \frac{-5/4\, z^{-1} + 3/2}{(1 - 1/2\, z^{-1} - 1/2\, z^{-2})} = -1/2 + \frac{1/6}{(1 - z^{-1})} + \frac{4/3}{(1 + 1/2\, z^{-1})}$$

Taking IZT; $\boxed{x(n) = -1/2\, \delta(n) + 1/6\, u(n) + 4/3\,(-1/2)^n u(n)}$

<u>Relationship between transfer function & the difference equation</u>: →

Consider a discrete-time LTI system for which the input & output satisfy a linear constant-coefficient difference equation of the form;

$$\sum_{k=0}^{N} a_k\, y(n-k) = \sum_{k=0}^{M} b_k\, x(n-k)$$

Taking ZT on both the sides

$$\sum_{k=0}^{N} a_k\, z^{-k}\, Y(z) = \sum_{k=0}^{M} b_k\, z^{-k}\, X(z)$$

(23)

$$Y(z) \sum_{k=0}^{N} a_k z^{-k} = X(z) \sum_{k=0}^{M} b_k z^{-k}$$

$\therefore$ Transfer function $H(z) = \dfrac{Y(z)}{X(z)} = \dfrac{\sum_{k=0}^{M} b_k z^{-k}}{\sum_{k=0}^{N} a_k z^{-k}}$

The transfer function of a system described by a linear constant coefficient difference equation is a ratio of polynomials in z^{-1} & it is referred to as 'rational transfer function'

→ Find the TF & the impulse response of the system described by the difference equation; $y(n) - \frac{1}{2} y(n-1) = 2x(n-1)$

Solu^n:— $Y(z) - \frac{1}{2} z^{-1} Y(z) = 2z^{-1} X(z)$

$$Y(z)\left[1 - \tfrac{1}{2} z^{-1}\right] = 2z^{-1} X(z)$$

$$H(z) = \frac{Y(z)}{X(z)} = \frac{2z^{-1}}{(1 - \frac{1}{2}z^{-1})}$$

Taking IZT ; $\boxed{h(n) = 2\left(\tfrac{1}{2}\right)^{n-1} u(n-1)}$

→ $y(n) - \frac{1}{4} y(n-1) - \frac{3}{8} y(n-2) = -x(n) + 2x(n-1)$

Solu^n: $Y(z) - \frac{1}{4} z^{-1} Y(z) - \frac{3}{8} z^{-2} Y(z) = -X(z) + 2z^{-1} X(z)$

$$Y(z)\left[1 - \tfrac{1}{4} z^{-1} - \tfrac{3}{8} z^{-2}\right] = X(z)\left[-1 + 2z^{-1}\right]$$

$$H(z) = \frac{Y(z)}{X(z)} = \frac{-1 + 2z^{-1}}{1 - \frac{1}{4}z^{-1} - \frac{3}{8}z^{-2}} = \frac{-1 + 2z^{-1}}{\left(1 - \frac{3}{4}z^{-1}\right)\left(1 + \frac{1}{2}z^{-1}\right)}$$

By partial fraction expansion ; $H(z) = \dfrac{-2}{(1 + \frac{1}{2}z^{-1})} + \dfrac{1}{(1 - \frac{3}{4}z^{-1})}$

Taking IZT ; $\boxed{h(n) = (-2)\left(-\tfrac{1}{2}\right)^n u(n) + \left(\tfrac{3}{4}\right)^n u(n)}$

05-05-2017

A causal LTI system is described by the difference equation $y(n) = y(n-1) + y(n-2) + x(n-1)$. Find the system function $H(z)$ & the poles & zeros & indicate the ROC. Also determine the impulse response of the system.

Soln: $y(n) = y(n-1) + y(n-2) + x(n-1)$

Taking ZT on both the sides

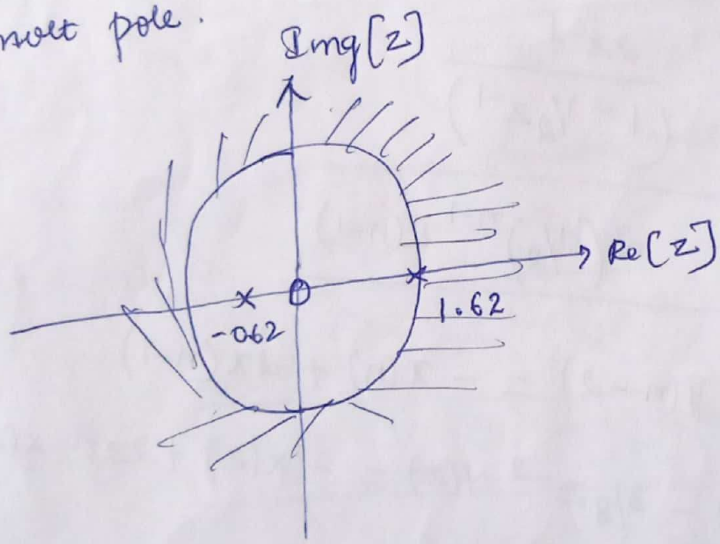
$$Y(z) = z^{-1}Y(z) + z^{-2}Y(z) + z^{-1}X(z)$$

$$H(z) = \frac{Y(z)}{X(z)} = \frac{z^{-1}}{(1-z^{-1}-z^{-2})} = \frac{z}{(z+0.62)(z-1.62)}$$

Zeros at $\boxed{z=0}$

Poles at $\boxed{z=-0.62}$ and $\boxed{z=1.62}$

Since the system is causal, the ROC must be outside the outermost pole.



$$H(z) = \frac{K_1}{1+0.62z^{-1}} + \frac{K_2}{1-1.62z^{-1}}$$

By Partial fraction method;

$$H(z) = \frac{-0.45}{(1+0.62z^{-1})} + \frac{0.45}{(1-1.62z^{-1})}$$

Taking IZT we get

$$\boxed{h(n) = -0.45(-0.62)^n u(n) + 0.45(1.62)^n u(n)}$$

H(z).
e
Stability and Causality :→ characteristics of a system such as causality & stability can be determined from the pole-zero pattern and the ROC of the TF H(z).

→ For the system to be causal, the impulse response must be equal to 0 for $n<0$ [ie. $h(n)=0$ for $n<0$]. Alternatively if the system is causal, then the ROC for H(z) will be outside the outermost pole.

→ If the system is stable, then its impulse response h(n) must be absolutely summable. Alternatively a causal system is stable if the poles of H(z) lies inside the unit circle in the Z-plane.

→ For a system ie both stable and causal, the ROC must include the unit circle & it must be outside the outermost pole. Alternatively for a system to be stable & causal, all the poles should lie inside the unit circle in the Z-plane.

→ Determine, whether the system described below is causal & stable

$$H(z) = \frac{2z+1}{z^2+z-5/16}$$

Soln: $$H(z) = \frac{2(z+1/2)}{(z+5/4)(z-1/4)}$$

We identify that, the location of poles are $z=-5/4=-1.25$ and $z=1/4$

Since one of the pole is lying outside the unit circle (ie at $z=-1.25$) the system is not both causal & stable

→ $$H(z) = \frac{1+2z^{-1}}{1+14/8\,z^{-1}+49/64\,z^{-2}}$$

Multiplying numerator & denominator by z^2

$$H(z) = \frac{z^2+2z}{z^2+14/8\, z+49/64} = \frac{z(z+2)}{(z+7/8)^2}$$

∴ The location of both the poles are at $\boxed{z = -7/8}$

Since all the poles are lying inside the unit circle, the system is both stable & causal.

→ A LTI discrete time system, is given by the system function

$$H(z) = \frac{3-4z^{-1}}{1-3.5z^{-1}+1.5z^{-2}}$$

Specify the ROC of the H(z) & determine h(n) for the following conditions (i) The system is stable (ii) The system is causal

Soln: By partial fraction expansion method

$$H(z) = \frac{1}{(1-1/2\, z^{-1})} + \frac{2}{(1-3z^{-1})}$$

The location of poles are at $\boxed{z = 1/2}$ and $\boxed{z = 3}$

(i) For the system to be stable its ROC must include the unit circle

∴ ROC is $1/2 < |z| < 3$

Taking IZT: $\boxed{h(n) = (1/2)^n u(n) - 2(3)^n u(-n-1)}$

(ii) For the system to be causal, the ROC should be outside the outermost pole.

∴ ROC is $|z| > 3$

∴ $\boxed{h(n) = (1/2)^n u(n) + 2(3)^n u(n)}$

→ A causal LTI system is described by the difference equation;

$y(n) = y(n-1) + y(n-2) + x(n-1)$

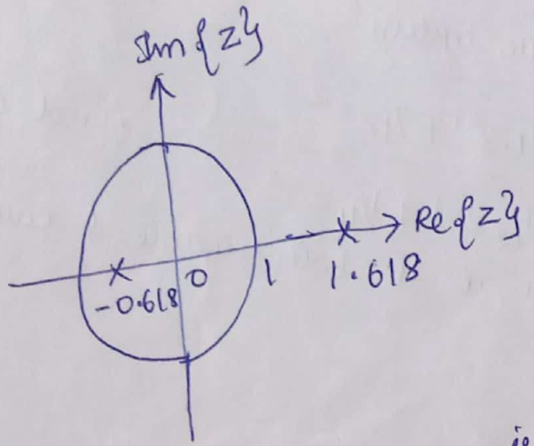
(i) Find the system function (ii) plot the poles & zeros (iii) Indicate the ROC (iv) Find the unit sample response that satisfies the difference equation.

$\underline{sol^n}$: $Y(z)\left[1 - z^{-1} - z^{-2}\right] = z^{-1} X(z)$

(i) System function $H(z) = \frac{Y(z)}{X(z)} = \frac{z^{-1}}{1 - z^{-1} - z^{-2}} = \frac{z}{z^2 - z - 1}$

$$H(z) = \frac{z}{(z+0.618)(z-1.618)}$$

(ii) The pole - zero pattern



(iii) Since the given system is causal, ROC must be outside the outermost pole. $\therefore$ ROC is $|z| > 1.618$.

(iv) By Partial fraction expansion : $H(z) = \frac{(-0.447)}{(1+0.618z^{-1})} + \frac{(0.447)}{(1-1.618z^{-1})}$

IZT ; $\boxed{h(n) = -0.447(-0.618)^n u(n) + 0.447(1.618)^n u(n)}$

(v) For a system to be stable, (non - causal) the ROC should include the unit circle.

$\boxed{h(n) = -0.447(-0.618)^n u(n) - 0.447(1.618)^n u(-n-1)}$

06-05-2017

$\underline{\text{Inverse system}}$: $\rightarrow$ For a system having impulse response $h(n)$, its inverse system impulse response $h^{-1}(n)$ such that

$$h^{-1}(n) * h(n) = \delta(n)$$

Taking ZT on both the sides

$$H^{-1}(z).H(z) = 1$$

$$\therefore \boxed{H^{-1}(z) = \frac{1}{H(z)}}$$

$\therefore$ The TF of an inverse system is the inverse of TF of the system we want to invert ie the zeros of $H(z)$ are the poles of $H^{-1}(z)$ & the poles of $H(z)$ are the zeros of $H^{-1}(z)$.

→ For a system $H^{-1}(z)$ to be both stable & causal its ROC must include unit circle. & it must be outside the outermost pole. ie $H^{-1}(z)$ is both stable & causal if all (~~the~~) of its poles are inside the unit circle.

∴ The poles of $H^{-1}(z)$ are nothing but all the zeros of $H(z)$, for $H^{-1}(z)$ to be causal & stable, all the zeros of $H(z)$ must be inside the unit circle.

→ A system with all its poles & zeros inside the unit circle in the z-plane is referred to as "minimum phase system".

→ For the system having TF; $H(z) = \dfrac{1-4z^{-1}+4z^{-2}}{1-\frac{1}{2}z^{-1}+\frac{1}{4}z^{-2}}$ find the TF of the inverse system & check whether it is both stable & causal.

Soln:
$$H(z) = \frac{1-4z^{-1}+4z^{-2}}{1-\frac{1}{2}z^{-1}+\frac{1}{4}z^{-2}}$$

The TF of the inverse system is obtained by inverting $H(z)$

$$H^{-1}(z) = \frac{1}{H(z)} = \frac{1-\frac{1}{2}z^{-1}+\frac{1}{4}z^{-2}}{1-4z^{-1}+4z^{-2}}$$

Multiplying numerator & denominator by z^2 we get

$$H^{-1}(z) = \frac{z^2-\frac{1}{2}z+\frac{1}{4}}{z^2-4z+4}$$

Factorising
$$H^{-1}(z) = \frac{(z-\frac{1}{4})^2+3/16}{(z-2)^2}$$

So there are 2 poles at $\boxed{z=2}$ ie 2 poles are existing outside the unit circle

∴ The Inverse system cannot be both causal & stable.

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Find the TF of inverse system & check whether it is both stable & causal for the following difference equation : $y(n) - \frac{1}{4}y(n-2) = 6x(n) - 7x(n-1) + 3x(n-2)$

Solun: Taking ZT on both the sides we get;

$$Y(z)\left[1 - \frac{1}{4}z^{-2}\right] = X(z)\left[6 - 7z^{-1} + 3z^{-2}\right]$$

$$H(z) = \frac{Y(z)}{X(z)} = \frac{6 - 7z^{-1} + 3z^{-2}}{1 - \frac{1}{4}z^{-2}}$$

TF of the inverse system is: $H^{-1}(z) = \frac{1}{H(z)} = \frac{1 - \frac{1}{4}z^{-2}}{6 - 7z^{-1} + 3z^{-2}}$

$$\therefore H^{-1}(z) = \frac{z^2 - \frac{1}{4}}{6z^2 - 7z + 3}$$

Location of poles $z = \frac{7 \pm j\sqrt{23}}{12}$

$$\therefore |z| = 0.707$$

Since both the poles exist inside the unit circle, the system can be both causal & stable.

→ A system has impulse response given by
$h(n) = 2\delta(n) + \frac{5}{2}\left(\frac{1}{2}\right)^n u(n) - \frac{7}{2}\left(-\frac{1}{4}\right)^n u(n)$

Find the TF of the inverse system.

Solun:- Taking ZT on both the sides:

$$H(z) = 2 + \frac{5/2}{1 + \frac{1}{2}z^{-1}} - \frac{7/2}{1 + \frac{1}{4}z^{-1}} = \frac{1 - \frac{15}{8}z^{-1} - \frac{1}{4}z^{-2}}{1 - \frac{1}{4}z^{-1} - \frac{1}{8}z^{-2}}$$

The TF of inverse system is obtained by inverting H(z)

$$\boxed{\therefore H^{-1}(z) = \frac{1}{H(z)} = \frac{1 - \frac{1}{4}z^{-1} - \frac{1}{8}z^{-2}}{1 - \frac{15}{8}z^{-1} - \frac{1}{4}z^{-2}}}$$