

MODULE 3

Extended Surface Heat Transfer

3.1 Introduction:

Convection: Heat transfer between a solid surface and a moving fluid is governed by the Newton's cooling law: $q = hA(T_s - T_\infty)$, where T_s is the surface temperature and T_∞ is the fluid temperature. Therefore, to increase the convective heat transfer, one can

- Increase the temperature difference ($T_s - T_\infty$) between the surface and the fluid.
- Increase the convection coefficient h . This can be accomplished by increasing the fluid flow over the surface since h is a function of the flow velocity and the higher the velocity, the higher the h . Example: a cooling fan.
- Increase the contact surface area A . Example: a heat sink with fins.

Many times, when the first option is not in our control and the second option (i.e. increasing h) is already stretched to its limit, we are left with the only alternative of increasing the effective surface area by using fins or extended surfaces. Fins are protrusions from the base surface into the cooling fluid, so that the extra surface of the protrusions is also in contact with the fluid. Most of you have encountered cooling fins on air-cooled engines (motorcycles, portable generators, etc.), electronic equipment (CPUs), automobile radiators, air conditioning equipment (condensers) and elsewhere.

3.2 Extended surface analysis:

In this module, consideration will be limited to steady state analysis of rectangular or pin fins of constant cross sectional area. Annular fins or fins involving a tapered cross section may be analyzed by similar methods, but will involve solution of more complicated equations which result. Numerical methods of integration or computer programs can be used to advantage in such cases.

We start with the General Conduction Equation:

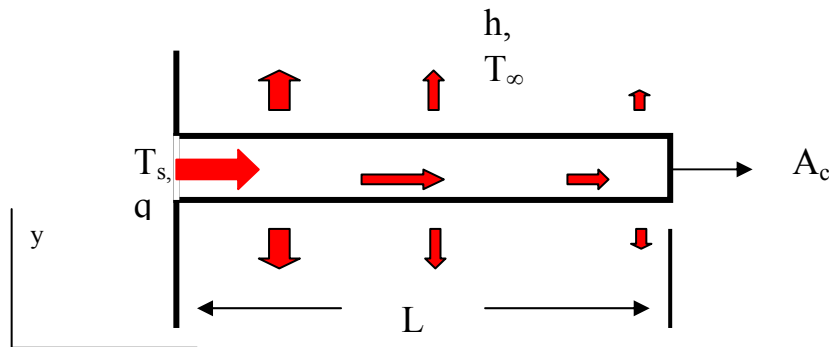
$$\frac{1}{\alpha} \cdot \frac{dT}{d\tau} \Big|_{system} = \nabla^2 T + \frac{\dot{q}}{k} \quad (1)$$

After making the assumptions of Steady State, One-Dimensional Conduction, this equation reduces to the form:

$$\frac{d^2 T}{dx^2} + \frac{\dot{q}}{k} = 0 \quad (2)$$

This is a second order, ordinary differential equation and will require 2 boundary conditions to evaluate the two constants of integration that will arise.

Consider the cooling fin shown below:

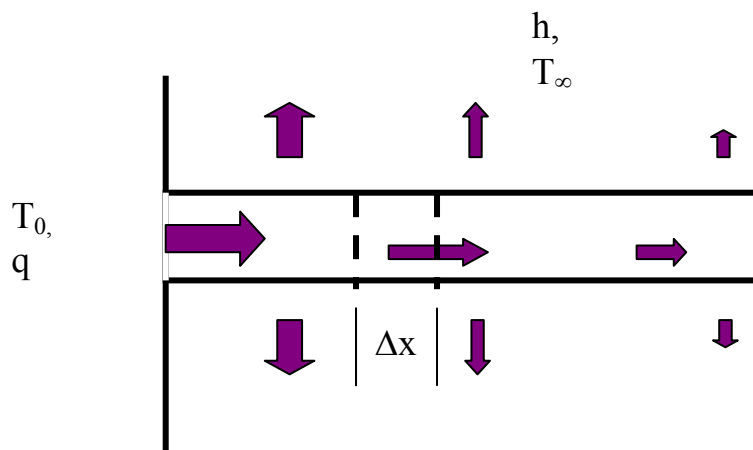


The fin is situated on the surface of a hot surface at T_s and surrounded by a coolant at temperature T_∞ , which cools with convective coefficient, h . The fin has a cross sectional area, A_c , (This is the area through which heat is conducted.) and an overall length, L .

Note that as energy is conducted down the length of the fin, some portion is lost, by convection, from the sides. Thus the heat flow varies along the length of the fin.

We further note that the arrows indicating the direction of heat flow point in both the x and y directions. This is an indication that this is truly a two- or three-dimensional heat flow, depending on the geometry of the fin. However, quite often, it is convenient to analyse a fin by examining an equivalent one-dimensional system. The equivalent system will involve the introduction of heat sinks (negative heat sources), which remove an amount of energy equivalent to what would be lost through the sides by convection.

Consider a differential length of the fin.



Across this segment the heat loss will be $h \cdot (P \cdot \Delta x) \cdot (T - T_\infty)$, where P is the perimeter around the fin. The equivalent heat sink would be $-h \cdot (P \cdot \Delta x) \cdot (T - T_\infty)$.

Equating the heat source to the convective loss:

$$\cancel{\frac{d^2T}{dx^2}} = \frac{-h \cdot P \cdot (T - T_\infty)}{A_c} \quad (3)$$

Substitute this value into the General Conduction Equation as simplified for One-Dimension, Steady State Conduction with Sources:

$$\frac{d^2T}{dx^2} - \frac{h \cdot P}{k \cdot A_c} \cdot (T - T_\infty) = 0 \quad (4)$$

which is the equation for a fin with a constant cross sectional area. This is the Second Order Differential Equation that we will solve for each fin analysis. Prior to solving, a couple of simplifications should be noted. First, we see that h , P , k and A_c are all independent of x in the defined system (They may not be constant if a more general analysis is desired.). We replace this ratio with a constant. Let

$$m^2 = \frac{h \cdot P}{k \cdot A_c} \quad (5)$$

then:

$$\frac{d^2T}{dx^2} - m^2 \cdot (T - T_\infty) = 0 \quad (6)$$

Next we notice that the equation is non-homogeneous (due to the T_∞ term). Recall that non-homogeneous differential equations require both a general and a particular solution. We can make this equation homogeneous by introducing the temperature relative to the surroundings:

$$\theta \equiv T - T_\infty \quad (7)$$

Differentiating this equation we find:

$$\frac{d\theta}{dx} = \frac{dT}{dx} + 0 \quad (8)$$

Differentiate a second time:

$$\frac{d^2\theta}{dx^2} = \frac{d^2T}{dx^2} \quad (9)$$

Substitute into the Fin Equation:

$$\frac{d^2\theta}{dx^2} - m^2 \cdot \theta = 0 \quad (10)$$

This equation is a Second Order, Homogeneous Differential Equation.

3.3 Solution of the Fin Equation

We apply a standard technique for solving a second order homogeneous linear differential equation.

Try $\theta = e^{\alpha \cdot x}$. Differentiate this expression twice:

$$\frac{d\theta}{dx} = \alpha \cdot e^{\alpha \cdot x} \quad (11)$$

$$\frac{d^2\theta}{dx^2} = \alpha^2 \cdot e^{\alpha \cdot x} \quad (12)$$

Substitute this trial solution into the differential equation:

$$\alpha^2 \cdot e^{\alpha \cdot x} - m^2 \cdot e^{\alpha \cdot x} = 0 \quad (13)$$

Equation (13) provides the following relation:

$$\alpha = \pm m \quad (14)$$

We now have two solutions to the equation. The general solution to the above differential equation will be a linear combination of each of the independent solutions.

Then:

$$\theta = A \cdot e^{m \cdot x} + B \cdot e^{-m \cdot x} \quad (15)$$

where A and B are arbitrary constants which need to be determined from the boundary conditions. Note that it is a 2nd order differential equation, and hence we need two boundary conditions to determine the two constants of integration.

An alternative solution can be obtained as follows: Note that the hyperbolic sin, sinh, the hyperbolic cosine, cosh, are defined as:

$$\sinh(m \cdot x) = \frac{e^{m \cdot x} - e^{-m \cdot x}}{2} \quad \cosh(m \cdot x) = \frac{e^{m \cdot x} + e^{-m \cdot x}}{2} \quad (16)$$

We may write:

$$C \cdot \cosh(m \cdot x) + D \cdot \sinh(m \cdot x) = C \cdot \frac{e^{m \cdot x} + e^{-m \cdot x}}{2} + D \cdot \frac{e^{m \cdot x} - e^{-m \cdot x}}{2} = \frac{C+D}{2} \cdot e^{m \cdot x} + \frac{C-D}{2} \cdot e^{-m \cdot x} \quad (17)$$

We see that if (C+D)/2 replaces A and (C-D)/2 replaces B then the two solutions are equivalent.

$$\theta = C \cdot \cosh(m \cdot x) + D \cdot \sinh(m \cdot x) \quad (18)$$

Generally the exponential solution is used for very long fins, the hyperbolic solutions for other cases.

Boundary Conditions:

Since the solution results in 2 constants of integration we require 2 boundary conditions. The first one is obvious, as one end of the fin will be attached to a hot surface and will come into thermal equilibrium with that surface. Hence, at the fin base,

$$\theta(0) = T_0 - T_\infty \equiv \theta_0 \quad (19)$$

The second boundary condition depends on the condition imposed at the other end of the fin. There are various possibilities, as described below.

Very long fins:

For very long fins, the end located a long distance from the heat source will approach the temperature of the surroundings. Hence,

$$\theta(\infty) = 0 \quad (20)$$

Substitute the second condition into the exponential solution of the fin equation:

$$\theta(\infty) = 0 = A \cdot e^{m \cdot \infty} + B \cdot e^{-m \cdot \infty} \quad (21)$$

The first exponential term is infinite and the second is equal to zero. The only way that this equation can be valid is if $A = 0$. Now apply the second boundary condition.

$$\theta(0) = \theta_0 = B \cdot e^{-m \cdot 0} \Rightarrow B = \theta_0 \quad (22)$$

The general temperature profile for a very long fin is then:

$$\theta(x) = \theta_0 \cdot e^{-m \cdot x} \quad (23)$$

If we wish to find the heat flow through the fin, we may apply Fourier Law:

$$q = -k \cdot A_c \cdot \frac{dT}{dx} = -k \cdot A_c \cdot \frac{d\theta}{dx} \quad (24)$$

Differentiate the temperature profile:

$$\frac{d\theta}{dx} = -\theta_0 \cdot m \cdot e^{-m \cdot x} \quad (25)$$

So that:

$$q = k \cdot A_c \cdot \theta_0 \cdot \left[\frac{h \cdot P}{k \cdot A_c} \right]^{1/2} \cdot e^{-m \cdot x} = \sqrt{h \cdot P \cdot k \cdot A_c} \cdot e^{-m \cdot x} \cdot \theta_0 = M \theta_0 e^{-mx} \quad (26)$$

where $M = \sqrt{hPkA_c}$.

Often we wish to know the total heat flow through the fin, i.e. the heat flow entering at the base ($x=0$).

$$q = \sqrt{h \cdot P \cdot k \cdot A_c} \cdot \theta_0 = M\theta_0 \quad (27)$$

The insulated tip fin

Assume that the tip is insulated and hence there is no heat transfer:

$$\left. \frac{d\theta}{dx} \right|_{x=L} = 0 \quad (28)$$

The solution to the fin equation is known to be:

$$\theta = C \cdot \cosh(m \cdot x) + D \cdot \sinh(m \cdot x) \quad (29)$$

Differentiate this expression.

$$\frac{d\theta}{dx} = C \cdot m \cdot \sinh(m \cdot x) + D \cdot m \cdot \cosh(m \cdot x) \quad (30)$$

Apply the first boundary condition at the base:

$$\theta(0) = \theta_0 = C \sinh(m \cdot 0) + D \cosh(m \cdot 0) \quad (31)$$

So that $D = \theta_0$. Now apply the second boundary condition at the tip to find the value of C:

$$\left. \frac{d\theta}{dx} \right|_{x=L} = 0 = Cm \sinh(m \cdot L) + \theta_0 m \cosh(m \cdot L) \quad (32)$$

which requires that

$$C = -\theta_0 \frac{\cosh(mL)}{\sinh(mL)} \quad (33)$$

This leads to the general temperature profile:

$$\theta(x) = \theta_0 \frac{\cosh m(L-x)}{\cosh(mL)} \quad (34)$$

We may find the heat flow at any value of x by differentiating the temperature profile and substituting it into the Fourier Law:

$$q = -k \cdot A_c \cdot \frac{dT}{dx} = -k \cdot A_c \cdot \frac{d\theta}{dx} \quad (35)$$

So that the energy flowing through the base of the fin is:

$$q = \sqrt{hPkA_c} \theta_0 \tanh(mL) = M\theta_0 \tanh(mL) \quad (36)$$

If we compare this result with that for the very long fin, we see that the primary difference in form is in the hyperbolic tangent term. That term, which always results in a number equal to or less than one, represents the reduced heat loss due to the shortening of the fin.

Other tip conditions:

We have already seen two tip conditions, one being the long fin and the other being the insulated tip. Two other possibilities are usually considered for fin analysis: (i) a tip subjected to convective heat transfer, and (ii) a tip with a prescribed temperature. The expressions for temperature distribution and fin heat transfer for all the four cases are summarized in the table below.

Table 3.1

Case	Tip Condition	Temp. Distribution	Fin heat transfer
A	Convection heat transfer: $h\theta(L) = -k(d\theta/dx)_{x=L}$	$\frac{\cosh m(L-x) + (h/mk) \sinh m(L-x)}{\cosh mL + (h/mk) \sinh mL}$	$M\theta_0 \frac{\sinh mL + (h/mk) \cosh mL}{\cosh mL + (h/mk) \sinh mL}$
B	Adiabatic $(d\theta/dx)_{x=L} = 0$	$\frac{\cosh m(L-x)}{\cosh mL}$	$M\theta_0 \tanh mL$
C	Given temperature: $\theta(L) = \theta_L$	$\frac{(\theta_L/\theta_b) \sinh m(L-x) + \sinh m(L-x)}{\sinh mL}$	$M\theta_0 \frac{(\cosh mL - \theta_L/\theta_b)}{\sinh mL}$
D	Infinitely long fin $\theta(L) = 0$	e^{-mx}	$M\theta_0$

3.4 Fin Effectiveness

How effective a fin can enhance heat transfer is characterized by the fin effectiveness, ε_f , which is as the ratio of fin heat transfer and the heat transfer without the fin. For an adiabatic fin:

$$\varepsilon_f = \frac{q_f}{q} = \frac{q_f}{hA_c(T_b - T_\infty)} = \frac{\sqrt{hPkA_c} \tanh(mL)}{hA_c} = \sqrt{\frac{kP}{hA_c}} \tanh(mL) \quad (37)$$

If the fin is long enough, $mL > 2$, $\tanh(mL) \rightarrow 1$, and hence it can be considered as infinite fin (case D in Table 3.1). Hence, for long fins,

$$\varepsilon_f \rightarrow \sqrt{\frac{kP}{hA_c}} = \sqrt{\left(\frac{k}{h}\right) \frac{P}{A_c}} \quad (38)$$

In order to enhance heat transfer, ε_f should be greater than 1 (In case $\varepsilon_f < 1$, the fin would have no purpose as it would serve as an insulator instead). However $\varepsilon_f \geq 2$ is considered unjustifiable because of diminishing returns as fin length increases.

To increase ε_f , the fin's material should have higher thermal conductivity, k . It seems to be counterintuitive that the lower convection coefficient, h , the higher ε_f . Well, if h is very high, it is not necessary to enhance heat transfer by adding heat fins. Therefore, heat fins are more effective if h is low.

Observations:

- If fins are to be used on surfaces separating gas and liquid, fins are usually placed on the gas side. (Why?)
- P/A_c should be as high as possible. Use a square fin with a dimension of W by W as an example: $P=4W$, $A_c=W^2$, $P/A_c=(4/W)$. The smaller the W , the higher is the P/A_c , and the higher the ε_f . Conclusion: It is preferred to use thin and closely spaced (to increase the total number) fins.

The effectiveness of a fin can also be characterized by

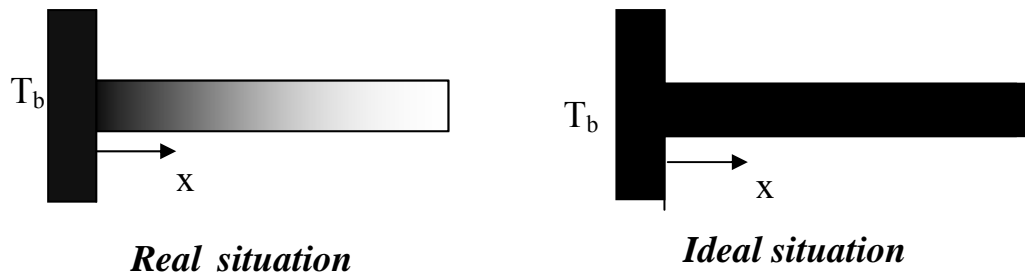
$$\varepsilon_f = \frac{q_f}{q} = \frac{q_f}{hA_c(T_b - T_\infty)} = \frac{(T_b - T_\infty)/R_{t,f}}{(T_b - T_\infty)/R_{t,h}} = \frac{R_{t,h}}{R_{t,f}} \quad (39)$$

It is a ratio of the thermal resistance due to convection to the thermal resistance of a fin. In order to enhance heat transfer, the fin's resistance should be lower than the resistance due only to convection.

3.5 Fin Efficiency

The fin efficiency is defined as the ratio of the energy transferred through a real fin to that transferred through an ideal fin. An ideal fin is thought to be one made of a perfect or infinite conductor material. A perfect conductor has an infinite thermal conductivity so that the entire fin is at the base material temperature.

$$\eta = \frac{q_{real}}{q_{ideal}} = \frac{\sqrt{h \cdot P \cdot k \cdot A_c} \cdot \theta_L \cdot \tanh(m \cdot L)}{h \cdot (P \cdot L) \cdot \theta_L} \quad (40)$$



Simplifying equation (40):

$$\eta = \sqrt{\frac{k \cdot A_c}{h \cdot P}} \frac{\theta_L \cdot \tanh(m \cdot L)}{L \cdot \theta_L} = \frac{\tanh(m \cdot L)}{m \cdot L} \quad (41)$$

The heat transfer through any fin can now be written as:

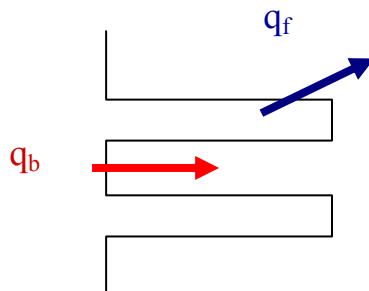
$$q \left[\frac{1}{\eta \cdot h \cdot A_f} \right] = (T - T_\infty) \quad (42)$$

The above equation provides us with the concept of fin thermal resistance (using electrical analogy) as

$$R_{t,f} = \frac{1}{\eta \cdot h \cdot A_f} \quad (43)$$

Overall Fin Efficiency:

Overall fin efficiency for an array of fins



Define terms: A_b : base area exposed to coolant

A_f : surface area of a single fin

A_t : total area including base area and total finned surface, $A_t = A_b + N A_f$

N : total number of fins

Heat Transfer from a Fin Array:

$$\begin{aligned}
 q_t &= q_b + Nq_f = hA_b(T_b - T_\infty) + N\eta_f hA_f(T_b - T_\infty) \\
 &= h[(A_b - NA_f) + N\eta_f A_f](T_b - T_\infty) = h[A_b - NA_f(1 - \eta_f)](T_b - T_\infty) \\
 &= hA_t[1 - \frac{NA_f}{A_t}(1 - \eta_f)](T_b - T_\infty) = \eta_o hA_t(T_b - T_\infty)
 \end{aligned}$$

Define overall fin efficiency: $\eta_o = 1 - \frac{NA_f}{A_t}(1 - \eta_f)$

$$q_t = hA_t\eta_o(T_b - T_\infty) = \frac{T_b - T_\infty}{R_{t,o}} \text{ where } R_{t,o} = \frac{1}{hA_t\eta_o}$$

Compare to heat transfer without fins

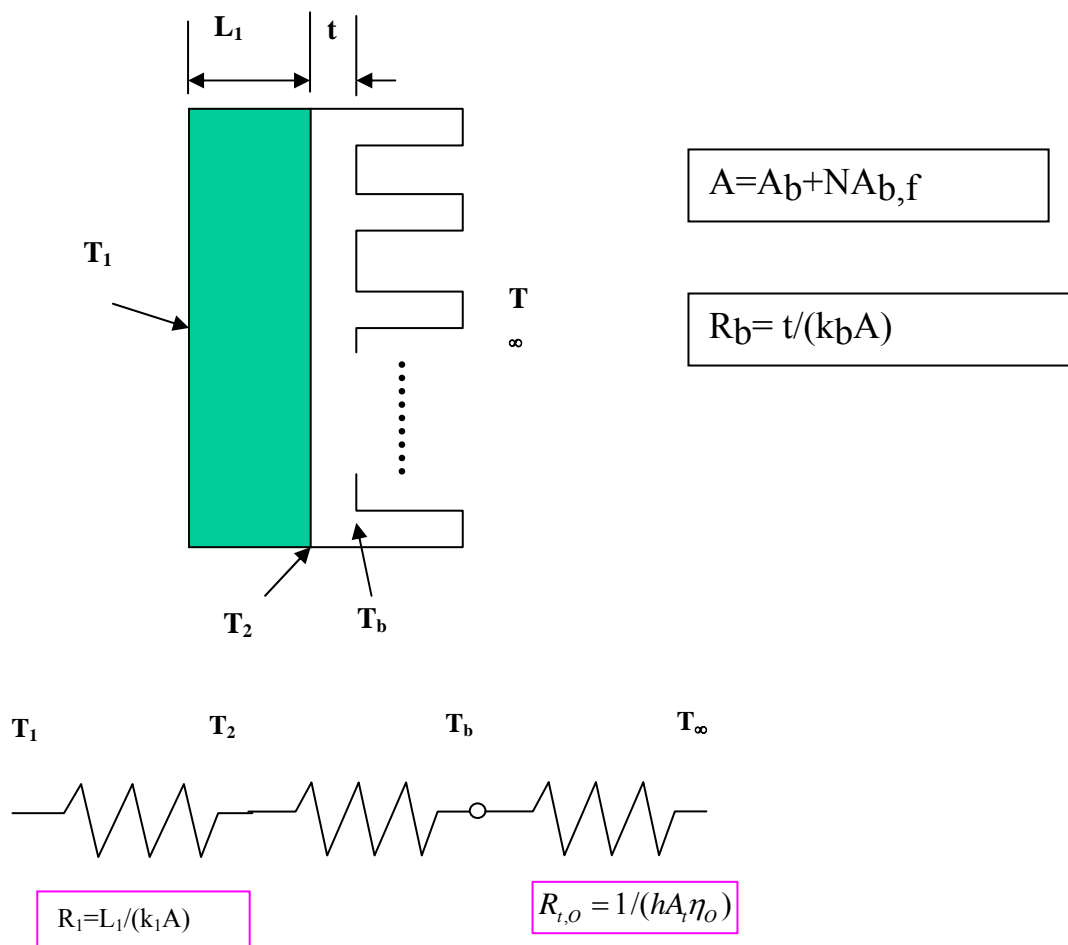
$$q = hA(T_b - T_\infty) = h(A_b + NA_{b,f})(T_b - T_\infty) = \frac{1}{hA}$$

where $A_{b,f}$ is the base area (unexposed) for the fin

To enhance heat transfer $A_t\eta_o \gg A$

That is, to increase the effective area $\eta_o A_t$.

Thermal Resistance Concept:



$$q = \frac{T_1 - T_\infty}{\sum R} = \frac{T_1 - T_\infty}{R_1 + R_b + R_{t,o}}$$

MODULE 7

HEAT EXCHANGERS

7.1 What are heat exchangers?

Heat exchangers are devices used to transfer heat energy from one fluid to another. Typical heat exchangers experienced by us in our daily lives include condensers and evaporators used in air conditioning units and refrigerators. Boilers and condensers in thermal power plants are examples of large industrial heat exchangers. There are heat exchangers in our automobiles in the form of radiators and oil coolers. Heat exchangers are also abundant in chemical and process industries.

There is a wide variety of heat exchangers for diverse kinds of uses, hence the construction also would differ widely. However, in spite of the variety, most heat exchangers can be classified into some common types based on some fundamental design concepts. We will consider only the more common types here for discussing some analysis and design methodologies.

7.2 Heat Transfer Considerations

The energy flow between hot and cold streams, with hot stream in the bigger diameter tube, is as shown in Figure 7.1. Heat transfer mode is by convection on the inside as well as outside of the inner tube and by conduction across the tube. Since the heat transfer occurs across the smaller tube, it is this internal surface which controls the heat transfer process. By convention, it is the outer surface, termed A_o , of this central tube which is referred to in describing heat exchanger area. Applying the principles of thermal resistance,

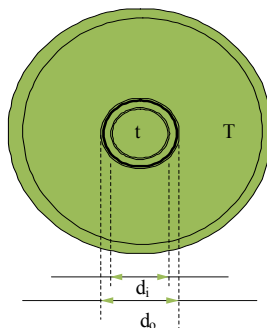


Figure 7.1: End view of a tubular heat exchanger

$$R = \frac{1}{h_o A_o} + \frac{\ln\left(\frac{r_o}{r_i}\right)}{2\pi \cdot kl} + \frac{1}{h_i A_i}$$

If we define overall the heat transfer coefficient, U_c , as:

$$U_c \equiv \frac{1}{RA_o}$$

Substituting the value of the thermal resistance R yields:

$$\frac{1}{U_c} = \frac{1}{h_o} + \frac{r_o \ln\left(\frac{r_o}{r_i}\right)}{k} + \frac{A_o}{h_i A_i}$$

Standard convective correlations are available in text books and handbooks for the convective coefficients, h_o and h_i . The thermal conductivity, k , corresponds to that for the material of the internal tube. To evaluate the thermal resistances, geometrical quantities (areas and radii) are determined from the internal tube dimensions available.

7.3 Fouling

Material deposits on the surfaces of the heat exchanger tubes may add more thermal resistances to heat transfer. Such deposits, which are detrimental to the heat exchange process, are known as fouling. Fouling can be caused by a variety of reasons and may significantly affect heat exchanger performance. With the addition of fouling resistance, the overall heat transfer coefficient, U_c , may be modified as:

$$\frac{1}{U_d} = \frac{1}{U_c} + R''$$

where R'' is the fouling resistance.

Fouling can be caused by the following sources:

- 1) *Scaling* is the most common form of fouling and is associated with inverse solubility salts. Examples of such salts are CaCO_3 , CaSO_4 , $\text{Ca}_3(\text{PO}_4)_2$, CaSiO_3 , $\text{Ca}(\text{OH})_2$, $\text{Mg}(\text{OH})_2$, MgSiO_3 , Na_2SO_4 , LiSO_4 , and Li_2CO_3 .
- 2) *Corrosion fouling* is caused by chemical reaction of some fluid constituents with the heat exchanger tube material.
- 3) *Chemical reaction fouling* involves chemical reactions in the process stream which results in deposition of material on the heat exchanger tubes. This commonly occurs in food processing industries.

- 4) *Freezing fouling* is occurs when a portion of the hot stream is cooled to near the freezing point for one of its components. This commonly occurs in refineries where paraffin frequently solidifies from petroleum products at various stages in the refining process. , obstructing both flow and heat transfer.
- 5) *Biological fouling* is common where untreated water from natural resources such as rivers and lakes is used as a coolant. Biological micro-organisms such as algae or other microbes can grow inside the heat exchanger and hinder heat transfer.
- 6) *Particulate fouling* results from the presence of microscale sized particles in solution. When such particles accumulate on a heat exchanger surface they sometimes fuse and harden. Like scale these deposits are difficult to remove.

With fouling, the expression for overall heat transfer coefficient becomes:

$$\frac{1}{U_d} = \frac{1}{h_i \cdot \left(\frac{r_i}{r_o} \right)} + \frac{\ln\left(\frac{r_o}{r_i}\right)}{k} + \frac{1}{h_o} + R''$$

7.4 Basic Heat Exchanger Flow Arrangements

Two basic flow arrangements are as shown in Figure 7.2. Parallel and counter flow provide alternative arrangements for certain specialized applications. In parallel flow both the hot and cold streams enter the heat exchanger at the same end and travel to the opposite end in parallel streams. Energy is transferred along the length from the hot to the cold fluid so the outlet temperatures asymptotically approach each other. In a counter flow arrangement, the two streams enter at opposite ends of the heat exchanger and flow in parallel but opposite directions. Temperatures within the two streams tend to approach one another in a nearly linearly fashion resulting in a much more uniform heating pattern. Shown below the heat exchangers are representations of the axial temperature profiles for each. Parallel flow results in rapid initial rates of heat exchange near the entrance, but heat transfer rates rapidly decrease as the temperatures of the two streams approach one another. This leads to higher exergy loss during heat exchange. Counter flow provides for relatively uniform temperature differences and, consequently, lead toward relatively uniform heat rates throughout the length of the unit.

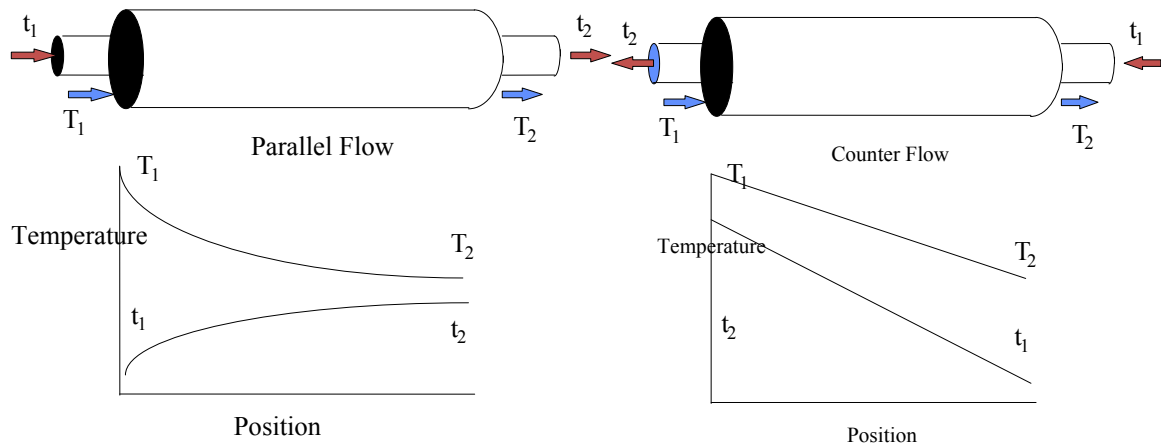


Fig. 7.2 Basic Flow Arrangements for Tubular Heat Exchangers.

7.5 Log Mean Temperature Differences

Heat flows between the hot and cold streams due to the temperature difference across the tube acting as a driving force. As seen in the Figure 7.3, the temperature difference will vary along the length of the HX, and this must be taken into account in the analysis.

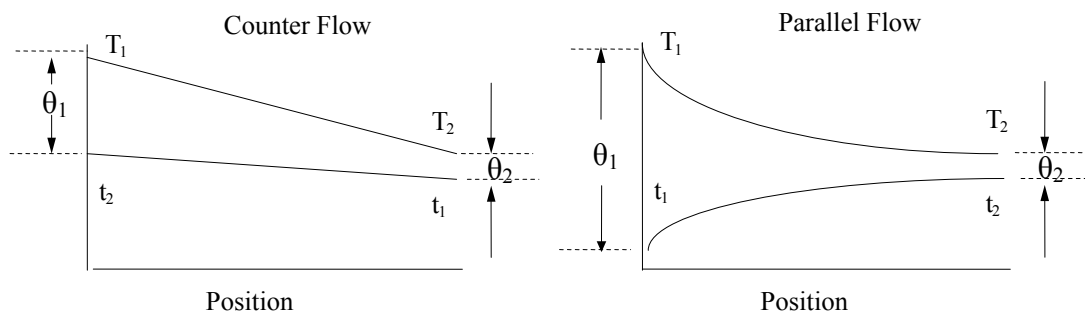


Fig. 7.3 Temperature Differences Between Hot and Cold Process Streams

From the heat exchanger equations shown earlier, it can be shown that the integrated average temperature difference for either parallel or counter flow may be written as:

$$\Delta\theta = LMTD = \frac{\theta_1 - \theta_2}{\ln\left(\frac{\theta_1}{\theta_2}\right)}$$

The effective temperature difference calculated from this equation is known as the log mean temperature difference, frequently abbreviated as LMTD, based on the type of mathematical average that it describes. While the equation applies

to either parallel or counter flow, it can be shown that $\Delta\theta_{\text{eff}}$ will always be greater in the counter flow arrangement.

Another interesting observation from the above Figure is that counter flow is more appropriate for maximum energy recovery. In a number of industrial applications there will be considerable energy available within a hot waste stream which may be recovered before the stream is discharged. This is done by recovering energy into a fresh cold stream. Note in the Figures shown above that the hot stream may be cooled to t_1 for counter flow, but may only be cooled to t_2 for parallel flow. Counter flow allows for a greater degree of energy recovery. Similar arguments may be made to show the advantage of counter flow for energy recovery from refrigerated cold streams.

7.6 Applications for Counter and Parallel Flows

We have seen two advantages for counter flow, (a) larger effective LMTD and (b) greater potential energy recovery. The advantage of the larger LMTD, as seen from the heat exchanger equation, is that a larger LMTD permits a smaller heat exchanger area, A_o , for a given heat transfer, Q . This would normally be expected to result in smaller, less expensive equipment for a given application.

Sometimes, however, parallel flows are desirable (a) where the high initial heating rate may be used to advantage and (b) where it is required the temperatures developed at the tube walls are moderate. In heating very viscous fluids, parallel flow provides for rapid initial heating and consequent decrease in fluid viscosity and reduction in pumping requirement. In applications where moderation of tube wall temperatures is required, parallel flow results in cooler walls. This is especially beneficial in cases where the tubes are sensitive to fouling effects which are aggravated by high temperature.

7.7 Multipass Flow Arrangements

In order to increase the surface area for convection relative to the fluid volume, it is common to design for multiple tubes within a single heat exchanger. With multiple tubes it is possible to arrange to flow so that one region will be in parallel and another portion in counter flow. An arrangement where the tube side fluid passes through once in parallel and once in counter flow is shown in the Figure 7.4. Normal terminology would refer to this arrangement as a 1-2 pass heat exchanger, indicating that the shell side fluid passes through the unit once, the tube side twice. By convention the number of shell side passes is always listed first.

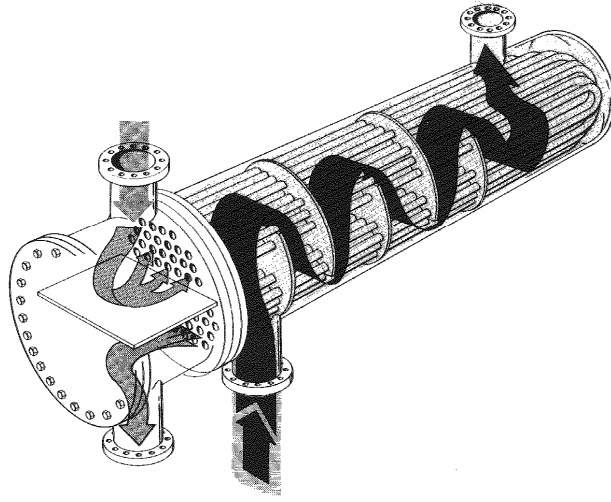


Fig. 7.4 Multipass flow arrangement

The primary reason for using multipass designs is to increase the average tube side fluid velocity in a given arrangement. In a two pass arrangement the fluid flows through only half the tubes and any one point, so that the Reynold's number is effectively doubled. Increasing the Reynolds's number results in increased turbulence, increased Nusselt numbers and, finally, in increased convection coefficients. Even though the parallel portion of the flow results in a lower effective ΔT , the increase in overall heat transfer coefficient will frequently compensate so that the overall heat exchanger size will be smaller for a specific service. The improvement achievable with multipass heat exchangers is substantially large. Accordingly, it is a more accepted practice in modern industries compared to conventional true parallel or counter flow designs.

The LMTD formulas developed earlier are no longer adequate for multipass heat exchangers. Normal practice is to calculate the LMTD for counter flow, $LMTD_{cf}$, and to apply a correction factor, F_T , such that

$$\Delta\theta_{eff} = F_T \cdot LMTD_{CF}$$

The correction factors, F_T , can be found theoretically and presented in analytical form. The equation given below has been shown to be accurate for any arrangement having 2, 4, 6,, $2n$ tube passes per shell pass to within 2%.

$$F_T = \frac{\sqrt{R^2 + 1} \ln \left[\frac{1 - P}{1 - R \cdot P} \right]}{(R - 1) \ln \left[\frac{2 - P(R + 1 - \sqrt{R^2 + 1})}{2 - P(R + 1 + \sqrt{R^2 + 1})} \right]}$$

where the capacity ratio, R , is defined as:

$$R = \frac{T_1 - T_2}{t_2 - t_1}$$

The effectiveness may be given by the equation:

$$P = \frac{1 - X^{1/N_{shell}}}{R - X^{1/N_{shell}}}$$

provided that $R \neq 1$. In the case that $R=1$, the effectiveness is given by:

$$P = \frac{P_o}{N_{shell} - P_o \cdot (N_{shell} - 1)}$$

where

$$P_o = \frac{t_2 - t_1}{T_1 - t_1}$$

and

$$X = \frac{P_o \cdot R - 1}{P_o - 1}$$

7.8 Effectiveness-NTU Method:

Quite often, heat exchanger analysts are faced with the situation that only the inlet temperatures are known and the heat transfer characteristics (UA value) are known, but the outlet temperatures have to be calculated. Clearly, LMTH method will not be applicable here. In this regard, an alternative method known as the ε -NTU method is used.

Before we introduce this method, let us ask ourselves following question: How will existing Heat Exchange perform for given inlet conditions? Define effectiveness: The effectiveness, ε , is the ratio of the energy recovered in a HX to that recoverable in an ideal HX.

$$\varepsilon = \frac{\dot{Q}_{actual}}{\dot{Q}_{max}}, \text{ where } \dot{Q}_{max} \text{ is for an infinitely long H.Ex.}$$

$$\text{One fluid } \Delta T \rightarrow \Delta T_{max} = T_{h,in} - T_{c,in}$$

$$\text{and since } \dot{Q} = (\dot{m}c_A)\Delta T_A = (\dot{m}c_B)\Delta T_B = C_A\Delta T_A = C_B\Delta T_B$$

then only the fluid with lesser of C_A, C_B heat capacity rate can have ΔT_{max}

$$\text{i.e. } \dot{Q}_{max} = C_{min}\Delta T_{max} \text{ and } \varepsilon = \frac{\dot{Q}}{C_{min}(T_{h,in} - T_{c,in})}$$

$$\text{or, } \dot{Q} = \varepsilon C_{min}(T_{h,in} - T_{c,in})$$

We want expression for ε which does not contain outlet T's.

Substitute back into $\dot{Q} = UA(LMTD)$

$$\varepsilon = \frac{1 - \exp\left[\frac{-UA}{C_{min}}\left(1 - \frac{C_{min}}{C_{max}}\right)\right]}{1 - \frac{C_{min}}{C_{max}} \exp\left[\frac{-UA}{C_{min}}\left(1 - \frac{C_{min}}{C_{max}}\right)\right]}$$

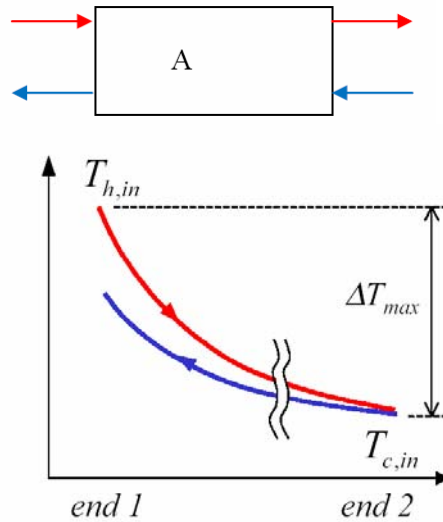


Fig. 7.5 Calculation of effectiveness-NTU

$$\therefore \varepsilon = \varepsilon \left(NTU, \frac{C_{min}}{C_{max}} \right)$$

and No. of transfer units (size of HEx.) $NTU = \frac{UA}{C_{min}}$

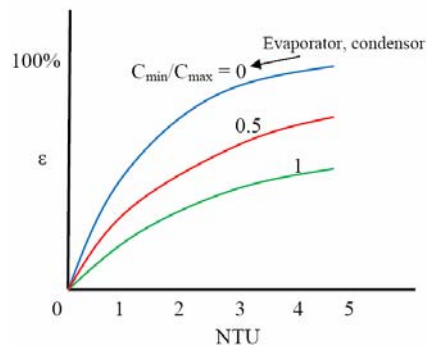
Charts for each Configuration

Procedure:

Determine C_{max} , C_{min}/C_{max}

Get $UA/C_{min} \rightarrow \varepsilon$ from chart

$$\dot{Q} = \varepsilon C_{min} (T_{h,in} - T_{c,in})$$



$$NTU_{max} = \frac{UA}{C_{min}} \Rightarrow A = \frac{NTU_{max} C_{min}}{U}$$

- NTU_{max} can be obtained from figures in textbooks/handbooks
- First, however, we must determine which fluid has C_{min} .

MODULE I

BASICS OF HEAT TRANSFER

While teaching heat transfer, one of the first questions students commonly ask is the difference between *heat* and *temperature*. Another common question concerns the difference between the subjects of *heat transfer* and *thermodynamics*. Let me begin this chapter by trying to address these two questions.

1.1 Difference between heat and temperature

In heat transfer problems, we often interchangeably use the terms *heat* and *temperature*. Actually, there is a distinct difference between the two. *Temperature* is a measure of the amount of energy possessed by the molecules of a substance. It manifests itself as a degree of hotness, and can be used to predict the direction of heat transfer. The usual symbol for temperature is T . The scales for measuring temperature in SI units are the Celsius and Kelvin temperature scales. *Heat*, on the other hand, is energy in transit. Spontaneously, heat flows from a hotter body to a colder one. The usual symbol for heat is Q . In the SI system, common units for measuring heat are the Joule and calorie.

1.2 Difference between thermodynamics and heat transfer

Thermodynamics tells us:

- how much heat is transferred (δQ)
- how much work is done (δW)
- final state of the system

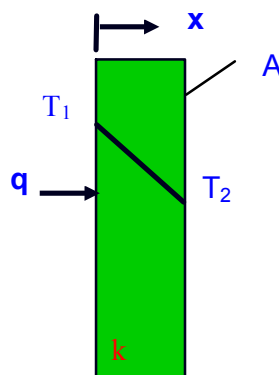
Heat transfer tells us:

- how (with what **modes**) δQ is transferred
- at what **rate** δQ is transferred
- temperature distribution inside the body



1.3 Modes of Heat Transfer

- **Conduction:** An energy transfer across a system boundary due to a temperature difference by the mechanism of inter-molecular interactions. Conduction needs matter and does not require any bulk motion of matter.



Conduction rate equation is described by the Fourier Law:

$$\vec{q} = -kA\nabla T$$

where: q = heat flow vector, (W)
 k = thermal conductivity, a thermodynamic property of the material.
 (W/m K)
 A = Cross sectional area in direction of heat flow. (m²)
 ∇T = Gradient of temperature (K/m)
 $= \partial T/\partial x \mathbf{i} + \partial T/\partial y \mathbf{j} + \partial T/\partial z \mathbf{k}$

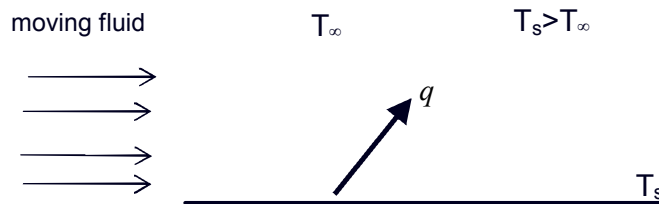
Note: Since this is a vector equation, it is often convenient to work with one component of the vector. For example, in the x direction:

$$q_x = -k A_x dT/dx$$

In circular coordinates it may convenient to work in the radial direction:

$$q_r = -k A_r dT/dr$$

- **Convection:** An energy transfer across a system boundary due to a temperature difference by the combined mechanisms of intermolecular interactions and bulk transport. Convection needs fluid matter.



Newton's Law of Cooling:

$$q = h A_s \Delta T$$

where: q = heat flow from surface, a scalar, (W)
 h = heat transfer coefficient (which is not a thermodynamic property of the material, but may depend on geometry of surface, flow characteristics, thermodynamic properties of the fluid, etc. (W/m² K)
 A_s = Surface area from which convection is occurring. (m²)
 $\Delta T = T_s - T_\infty$ = Temperature Difference between surface and coolant. (K)

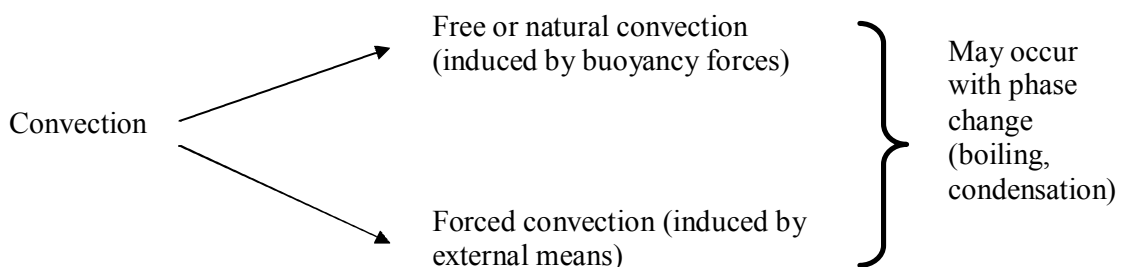


Table 1. Typical values of h (W/m²K)

Free convection	gases: 2 - 25 liquid: 50 – 100
Forced convection	gases: 25 - 250 liquid: 50 - 20,000
Boiling/Condensation	2500 -100,000

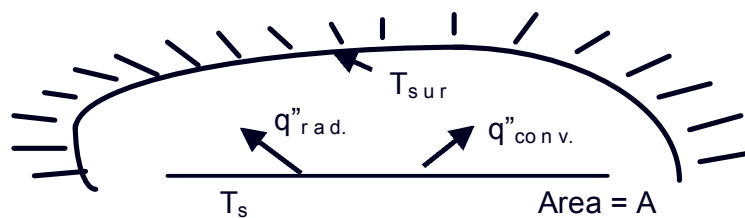
- **Radiation:** Radiation heat transfer involves the transfer of heat by electromagnetic radiation that arises due to the temperature of the body. Radiation does not need matter.

Emissive power of a surface:

$$E = \sigma \epsilon T_s^4 \text{ (W/ m}^2\text{)}$$

where: ϵ = emissivity, which is a surface property ($\epsilon = 1$ is black body)
 σ = Steffan Boltzman constant = $5.67 \times 10^{-8} \text{ W/m}^2 \text{ K}^4$.
 T_s = Absolute temperature of the surface (K)

The above equation is derived from Stefan Boltzman law, which describes a gross heat emission rather than heat transfer. The expression for the actual radiation heat transfer rate between surfaces having arbitrary orientations can be quite complex, and will be dealt with in Module 9. However, the rate of radiation heat exchange between a small surface and a large surrounding is given by the following expression:



$$q = \epsilon \cdot \sigma \cdot A \cdot (T_s^4 - T_{sur}^4)$$

where: ϵ = Surface Emissivity
 A = Surface Area
 T_s = Absolute temperature of surface. (K)
 T_{sur} = Absolute temperature of surroundings.(K)

1.4 Thermal Conductivity, k

As noted previously, thermal conductivity is a thermodynamic property of a material. From the State Postulate given in thermodynamics, it may be recalled that thermodynamic properties of pure substances are functions of two independent thermodynamic intensive properties, say temperature and pressure. Thermal conductivity of real gases is largely independent of pressure and may be considered a function of temperature alone. For solids and liquids, properties are largely independent of pressure and depend on temperature alone.

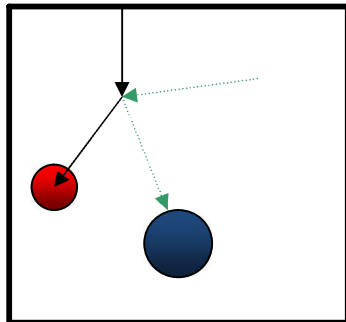
$$k = k(T)$$

Table 2 gives the values of thermal conductivity for a variety of materials.

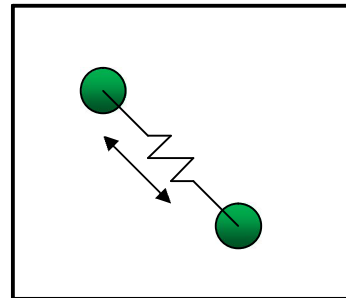
Table 2. Thermal Conductivities of Selected Materials at Room Temperature.

Material	Thermal Conductivity, W/m K
Copper	401
Silver	429
Gold	317
Aluminum	237
Steel	60.5
Limestone	2.15
Bakelite	1.4
Water	0.613
Air	0.0263

Let us try to gain an insight into the basic concept of thermal conductivity for various materials. The fundamental concept comes from the molecular or atomic scale activities. Molecules/atoms of various materials gain energy through different mechanisms. Gases, in which molecules are free to move with a mean free path sufficiently large compared to their diameters, possess energy in the form of kinetic energy of the molecules. Energy is gained or lost through collisions/interactions of gas molecules.



*Kinetic energy transfer
between gas molecules.*



Lattice vibration may be transferred
between molecules as nuclei
attract/repel each other.

Solids, on the other hand, have atoms/molecules which are more closely packed which cannot move as freely as in gases. Hence, they cannot effectively transfer energy through these same mechanisms. Instead, solids may exhibit energy through vibration or rotation of the nucleus. Hence, the energy transfer is typically through lattice vibrations.

Another important mechanism in which materials maintain energy is by shifting electrons into higher orbital rings. In the case of electrical conductors the electrons are weakly bonded to the molecule and can drift from one molecule to another, transporting their energy in the process. Hence, flow of electrons, which is commonly observed in metals, is an effective transport mechanism, resulting in a correlation that materials which are excellent electrical conductors are usually excellent thermal conductors.

MODULE 2

ONE DIMENSIONAL STEADY STATE HEAT CONDUCTION

2.1 Objectives of conduction analysis:

The primary objective is to determine the temperature field, $T(x,y,z,t)$, in a body (i.e. how temperature varies with position within the body)

$T(x,y,z,t)$ depends on:

- Boundary conditions
- Initial condition
- Material properties (k , c_p , ρ)
- Geometry of the body (shape, size)

Why we need $T(x, y, z, t)$?

- To compute heat flux at any location (using Fourier's eqn.)
- Compute thermal stresses, expansion, deflection due to temp. Etc.
- Design insulation thickness
- Chip temperature calculation
- Heat treatment of metals

2.2 General Conduction Equation

Recognize that heat transfer involves an energy transfer across a system boundary. The analysis for such process begins from the 1st Law of Thermodynamics for a closed system:

$$\left. \frac{dE}{dt} \right|_{\text{system}} = \dot{Q}_{\text{in}} - \dot{W}_{\text{out}}$$

The above equation essentially represents Conservation of Energy. The sign convention on work is such that negative work out is positive work in.

$$\left. \frac{dE}{dt} \right|_{\text{system}} = \dot{Q}_{\text{in}} + \dot{W}_{\text{in}}$$

The work in term could describe an electric current flow across the system boundary and through a resistance inside the system. Alternatively it could describe a shaft turning across the system boundary and overcoming friction within the system. The net effect in either case would cause the internal energy of the system to rise. In heat transfer we generalize all such terms as "heat sources".

$$\left. \frac{dE}{dt} \right|_{\text{system}} = \dot{Q}_{\text{in}} + \dot{Q}_{\text{gen}}$$

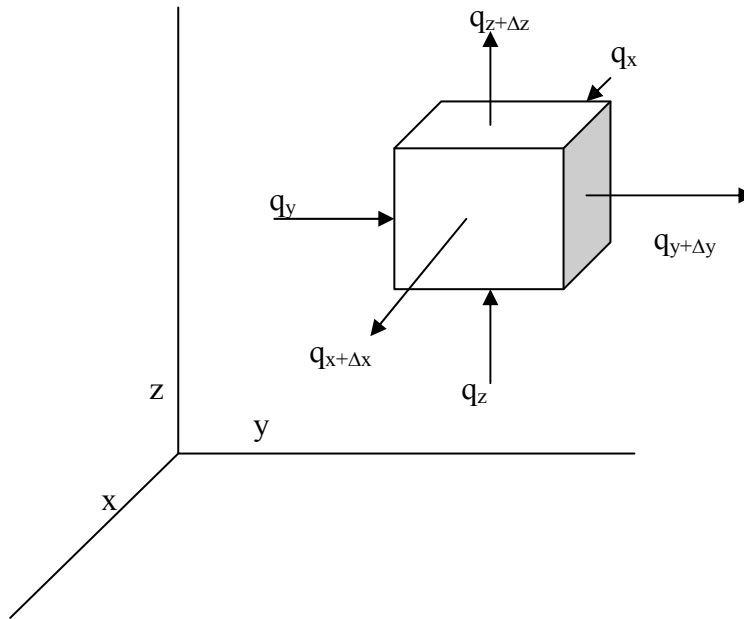
The energy of the system will in general include internal energy, U , potential energy, $\frac{1}{2} mgz$, or kinetic energy, $\frac{1}{2} mv^2$. In case of heat transfer problems, the latter two terms could often be neglected. In this case,

$$E = U = m \cdot u = m \cdot c_p \cdot (T - T_{\text{ref}}) = \rho \cdot V \cdot c_p \cdot (T - T_{\text{ref}})$$

where T_{ref} is the reference temperature at which the energy of the system is defined as zero. When we differentiate the above expression with respect to time, the reference temperature, being constant, disappears:

$$\rho \cdot c_p \cdot V \cdot \frac{dT}{dt} \Big|_{\text{system}} = \dot{\mathcal{Q}}_{\text{in}} + \dot{\mathcal{Q}}_{\text{gen}}$$

Consider the differential control element shown below. Heat is assumed to flow through the element in the positive directions as shown by the 6 heat vectors.



In the equation above we substitute the 6 heat inflows/outflows using the appropriate sign:

$$\rho \cdot c_p \cdot (\Delta x \cdot \Delta y \cdot \Delta z) \cdot \frac{dT}{dt} \Big|_{\text{system}} = q_x - q_{x+\Delta x} + q_y - q_{y+\Delta y} + q_z - q_{z+\Delta z} + \dot{\mathcal{Q}}_{\text{gen}}$$

Substitute for each of the conduction terms using the Fourier Law:

$$\begin{aligned} \rho \cdot c_p \cdot (\Delta x \cdot \Delta y \cdot \Delta z) \cdot \frac{\partial T}{\partial t} \Big|_{\text{system}} = & \left\{ -k \cdot (\Delta y \cdot \Delta z) \cdot \frac{\partial T}{\partial x} - \left[-k \cdot (\Delta y \cdot \Delta z) \cdot \frac{\partial T}{\partial x} + \frac{\partial}{\partial x} \left(-k \cdot (\Delta y \cdot \Delta z) \cdot \frac{\partial T}{\partial x} \right) \cdot \Delta x \right] \right\} \\ & + \left\{ -k \cdot (\Delta x \cdot \Delta z) \cdot \frac{\partial T}{\partial y} - \left[-k \cdot (\Delta x \cdot \Delta z) \cdot \frac{\partial T}{\partial y} + \frac{\partial}{\partial y} \left(-k \cdot (\Delta x \cdot \Delta z) \cdot \frac{\partial T}{\partial y} \right) \cdot \Delta y \right] \right\} \\ & + \left\{ -k \cdot (\Delta x \cdot \Delta y) \cdot \frac{\partial T}{\partial z} + \left[-k \cdot (\Delta x \cdot \Delta y) \cdot \frac{\partial T}{\partial z} + \frac{\partial}{\partial z} \left(-k \cdot (\Delta x \cdot \Delta y) \cdot \frac{\partial T}{\partial z} \right) \cdot \Delta z \right] \right\} \\ & + \dot{\mathcal{Q}}_{\text{gen}} (\Delta x \cdot \Delta y \cdot \Delta z) \end{aligned}$$

where $\dot{\mathcal{Q}}_{\text{gen}}$ is defined as the internal heat generation per unit volume.

The above equation reduces to:

$$\rho \cdot c_p \cdot (\Delta x \cdot \Delta y \cdot \Delta z) \cdot \frac{dT}{dt} \Big|_{\text{system}} = \left\{ - \left[\frac{\partial}{\partial x} \left(-k \cdot (\Delta y \cdot \Delta z) \cdot \frac{\partial T}{\partial x} \right) \right] \cdot \Delta x \right\}$$

$$+ \left\{ - \left[\frac{\partial}{\partial y} \left(-k \cdot (\Delta x \cdot \Delta z) \cdot \frac{\partial T}{\partial y} \right) \cdot \Delta y \right] \right\} \\ + \left\{ \left[\frac{\partial}{\partial z} \left(-k \cdot (\Delta x \cdot \Delta y) \cdot \frac{\partial T}{\partial z} \right) \cdot \Delta z \right] \right\} + \cancel{\rho \cdot c_p \cdot \Delta x \cdot \Delta y \cdot \Delta z}$$

Dividing by the volume ($\Delta x \cdot \Delta y \cdot \Delta z$),

$$\rho \cdot c_p \cdot \frac{dT}{dt} \Big|_{\text{system}} = - \frac{\partial}{\partial x} \left(-k \cdot \frac{\partial T}{\partial x} \right) - \frac{\partial}{\partial y} \left(-k \cdot \frac{\partial T}{\partial y} \right) - \frac{\partial}{\partial z} \left(-k \cdot \frac{\partial T}{\partial z} \right) + \cancel{\rho \cdot c_p}$$

which is the **general conduction equation** in three dimensions.

In the case where k is independent of x , y and z then

$$\frac{\rho \cdot c_p}{k} \cdot \frac{dT}{dt} \Big|_{\text{system}} = \frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2} + \frac{\cancel{\rho \cdot c_p}}{k}$$

Define the thermodynamic property, α , the thermal diffusivity:

$$\alpha \equiv \frac{k}{\rho \cdot c_p}$$

Then

$$\frac{1}{\alpha} \cdot \frac{dT}{dt} \Big|_{\text{system}} = \frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2} + \frac{\cancel{\rho \cdot c_p}}{k}$$

or, :

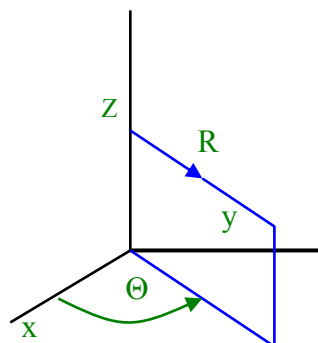
$$\frac{1}{\alpha} \cdot \frac{dT}{dt} \Big|_{\text{system}} = \nabla^2 T + \frac{\cancel{\rho \cdot c_p}}{k}$$

The vector form of this equation is quite compact and is the most general form. However, we often find it convenient to expand the spatial derivative in specific coordinate systems:

Cartesian Coordinates

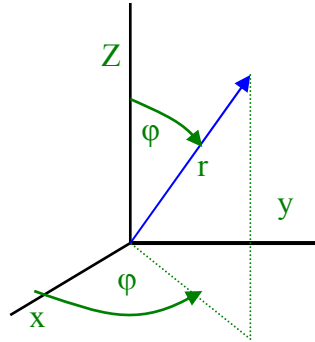
$$\frac{1}{\alpha} \cdot \frac{\partial T}{\partial \tau} = \frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2} + \frac{\cancel{\rho \cdot c_p}}{k}$$

Circular Coordinates



$$\frac{1}{a} \cdot \frac{\partial T}{\partial \tau} = \frac{1}{r} \frac{\partial}{\partial r} \left(r \cdot \frac{\partial T}{\partial r} \right) + \frac{1}{r^2} \cdot \frac{\partial^2 T}{\partial \theta^2} + \frac{\partial^2 T}{\partial z^2} + \frac{\dot{q}}{k}$$

Spherical Coordinates



$$\frac{1}{a} \cdot \frac{\partial T}{\partial \tau} = \frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 \cdot \frac{\partial T}{\partial r} \right) + \frac{1}{r^2 \cdot \sin^2 \theta} \cdot \frac{\partial^2 T}{\partial \phi^2} + \frac{1}{r^2 \sin \theta} \cdot \frac{\partial}{\partial \theta} \left(\sin \theta \cdot \frac{\partial T}{\partial \theta} \right) + \frac{\dot{q}}{k}$$

In each equation the dependent variable, T, is a function of 4 independent variables, (x,y,z,τ); (r,θ,z,τ); (r,φ,θ,τ) and is a 2nd order, partial differential equation. The solution of such equations will normally require a numerical solution. For the present, we shall simply look at the simplifications that can be made to the equations to describe specific problems.

- **Steady State:** Steady state solutions imply that the system conditions are not changing with time. Thus $\partial T / \partial \tau = 0$.
- **One dimensional:** If heat is flowing in only one coordinate direction, then it follows that there is no temperature gradient in the other two directions. Thus the two partials associated with these directions are equal to zero.
- **Two dimensional:** If heat is flowing in only two coordinate directions, then it follows that there is no temperature gradient in the third direction. Thus, the partial derivative associated with this third direction is equal to zero.
- **No Sources:** If there are no volumetric heat sources within the system then the term, $\dot{q} = 0$.

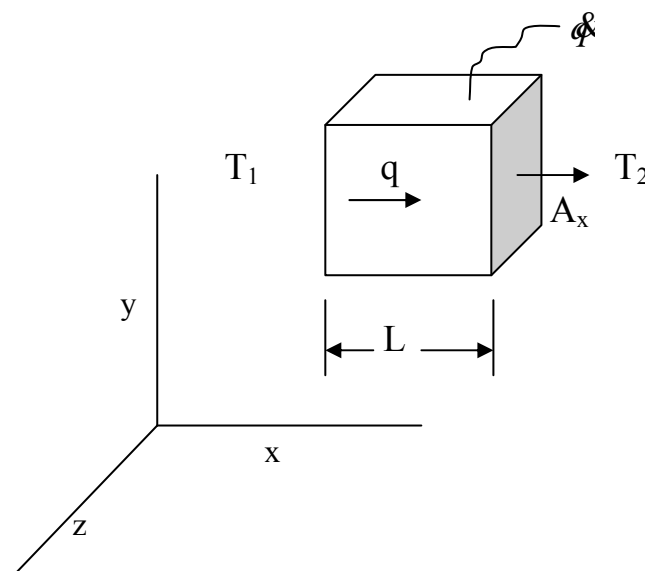
Note that the equation is 2nd order in each coordinate direction so that integration will result in 2 constants of integration. To evaluate these constants two boundary conditions will be required for each coordinate direction.

2.3 Boundary and Initial Conditions

- The objective of deriving the heat diffusion equation is to determine the temperature distribution within the conducting body.

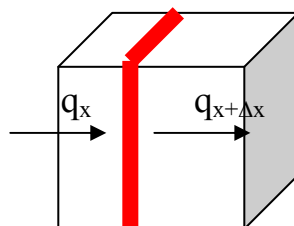
- We have set up a differential equation, with T as the dependent variable. The solution will give us $T(x,y,z)$. Solution depends on boundary conditions (BC) and initial conditions (IC).
- How many BC's and IC's ?
 - Heat equation is second order in spatial coordinate. Hence, 2 BC's needed for each coordinate.
 - * 1D problem: 2 BC in x-direction
 - * 2D problem: 2 BC in x-direction, 2 in y-direction
 - * 3D problem: 2 in x-dir., 2 in y-dir., and 2 in z-dir.
 - Heat equation is first order in time. Hence one IC needed.

2.4 Heat Diffusion Equation for a One Dimensional System



Consider the system shown above. The top, bottom, front and back of the cube are insulated, so that heat can be conducted through the cube only in the x direction. The internal heat generation per unit volume is ϕ (W/m^3).

Consider the heat flow through a differential element of the cube.



From the 1st Law we write for the element:

$$(\dot{E}_{in} - \dot{E}_{out}) + \dot{E}_{gen} = \dot{E}_{st} \quad (2.1)$$

$$q_x - q_{x+\Delta x} + A_x(\Delta x)\dot{\phi} = \frac{\partial E}{\partial t} \quad (2.2)$$

$$q_x = -kA_x \frac{\partial T}{\partial x} \quad (2.3)$$

$$q_{x+\Delta x} = q_x + \frac{\partial q_x}{\partial x} \Delta x \quad (2.4)$$

$$-kA \frac{\partial T}{\partial x} + kA \frac{\partial T}{\partial x} + A \frac{\partial}{\partial x} \left(k \frac{\partial T}{\partial x} \right) \Delta x + A \Delta x \dot{\phi} = \rho A c \Delta x \frac{\partial T}{\partial t} \quad (2.5)$$

$$\frac{\partial}{\partial x} \left(k \frac{\partial T}{\partial x} \right) + \dot{\phi} = \rho c \Delta x \frac{\partial T}{\partial t}$$

Longitudinal conduction
Internal heat generation
Thermal inertia

(2.6)

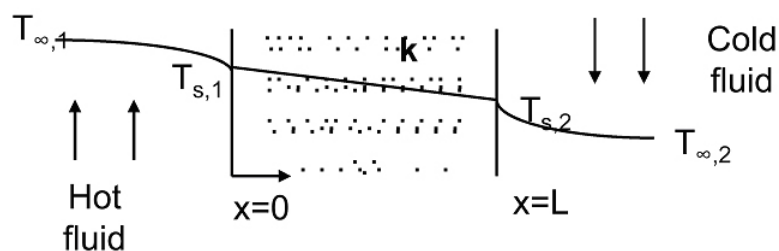
If k is a constant, then

$$\frac{\partial^2 T}{\partial x^2} + \frac{\dot{\phi}}{k} = \frac{\rho c}{k} \frac{\partial T}{\partial t} = \frac{1}{\alpha} \frac{\partial T}{\partial t} \quad (2.7)$$

- For T to rise, LHS must be positive (heat input is positive)
- For a fixed heat input, T rises faster for higher α
- In this special case, heat flow is 1D. If sides were not insulated, heat flow could be 2D, 3D.

2.5 One Dimensional Steady State Heat Conduction

The plane wall:



The differential equation governing heat diffusion is: $\frac{d}{dx} \left(k \frac{dT}{dx} \right) = 0$

With constant k , the above equation may be integrated twice to obtain the general solution:

$$T(x) = C_1 x + C_2$$

where C_1 and C_2 are constants of integration. To obtain the constants of integration, we apply the boundary conditions at $x = 0$ and $x = L$, in which case

$$T(0) = T_{s,1} \quad \text{and} \quad T(L) = T_{s,2}$$

Once the constants of integration are substituted into the general equation, the temperature distribution is obtained:

$$T(x) = (T_{s,2} - T_{s,1}) \frac{x}{L} + T_{s,1}$$

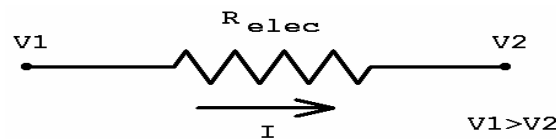
The heat flow rate across the wall is given by:

$$q_x = -kA \frac{dT}{dx} = \frac{kA}{L} (T_{s,1} - T_{s,2}) = \frac{T_{s,1} - T_{s,2}}{L/kA}$$

Thermal resistance (electrical analogy):

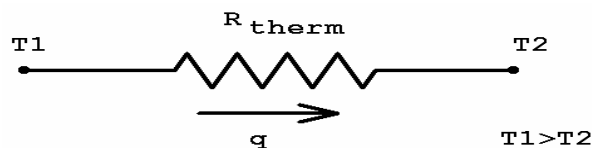
Physical systems are said to be analogous if that obey the same mathematical equation. The above relations can be put into the form of Ohm's law:

$$V = IR_{\text{elec}}$$



Using this terminology it is common to speak of a thermal resistance:

$$\Delta T = qR_{\text{therm}}$$



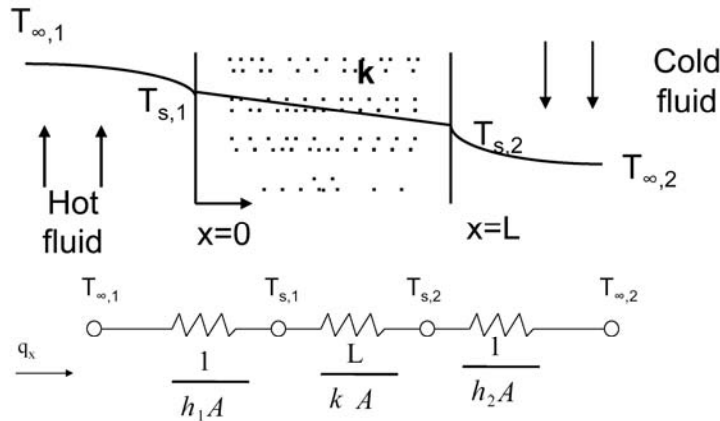
A thermal resistance may also be associated with heat transfer by convection at a surface. From Newton's law of cooling,

$$q = hA(T_s - T_\infty)$$

the thermal resistance for convection is then

$$R_{t,conv} = \frac{T_s - T_\infty}{q} = \frac{1}{hA}$$

Applying thermal resistance concept to the plane wall, the equivalent thermal circuit for the plane wall with convection boundary conditions is shown in the figure below



The heat transfer rate may be determined from separate consideration of each element in the network. Since q_x is constant throughout the network, it follows that

$$q_x = \frac{T_{\infty,1} - T_{s,1}}{1/h_1 A} = \frac{T_{s,1} - T_{s,2}}{L/kA} = \frac{T_{s,2} - T_{\infty,2}}{1/h_2 A}$$

In terms of the overall temperature difference $T_{\infty,1} - T_{\infty,2}$, and the total thermal resistance R_{tot} , the heat transfer rate may also be expressed as

$$q_x = \frac{T_{\infty,1} - T_{\infty,2}}{R_{tot}}$$

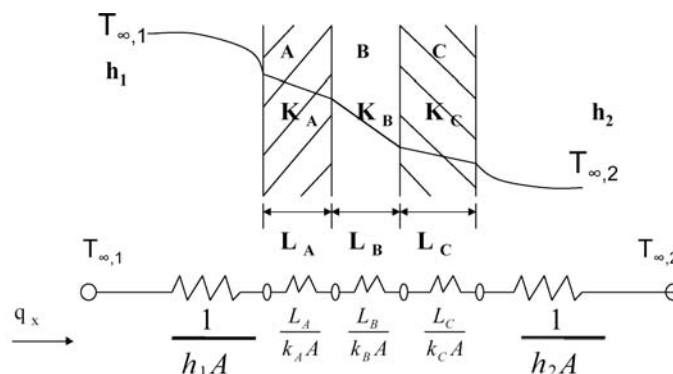
Since the resistance are in series, it follows that

$$R_{tot} = \sum R_t = \frac{1}{h_1 A} + \frac{L}{kA} + \frac{1}{h_2 A}$$

Composite walls:

Thermal Resistances in Series:

Consider three blocks, A, B and C, as shown. They are insulated on top, bottom, front and back. Since the energy will flow first through block A and then through blocks B and C, we say that these blocks are thermally in a series arrangement.



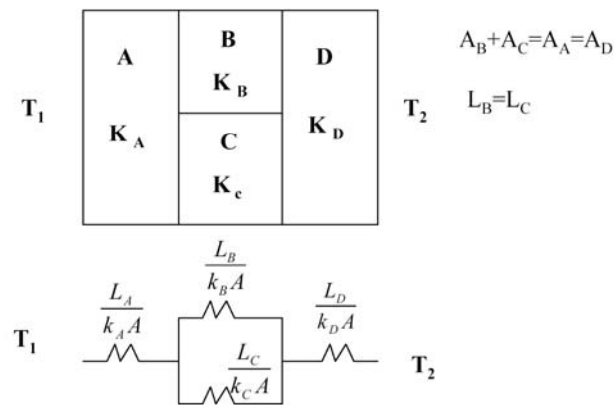
The steady state heat flow rate through the walls is given by:

$$q_x = \frac{T_{\infty,1} - T_{\infty,2}}{\sum R_t} = \frac{T_{\infty,1} - T_{\infty,2}}{\frac{1}{h_1 A} + \frac{L_A}{k_A} + \frac{L_B}{k_B} + \frac{L_C}{k_C} + \frac{1}{h_2 A}} = UA\Delta T$$

where $U = \frac{1}{R_{tot}A}$ is the overall heat transfer coefficient. In the above case, U is expressed as

$$U = \frac{1}{\frac{1}{h_1} + \frac{L_A}{k_A} + \frac{L_B}{k_B} + \frac{L_C}{k_C} + \frac{1}{h_2}}$$

Series-parallel arrangement:

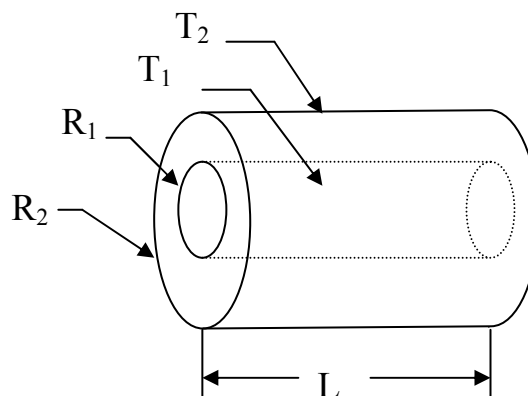


The following assumptions are made with regard to the above thermal resistance model:

- 1) Face between B and C is insulated.
- 2) Uniform temperature at any face normal to X.

1-D radial conduction through a cylinder:

One frequently encountered problem is that of heat flow through the walls of a pipe or through the insulation placed around a pipe. Consider the cylinder shown. The pipe is either insulated on the ends or is of sufficient length, L, that heat losses through the ends is negligible. Assume no heat sources within the wall of the tube. If $T_1 > T_2$, heat will flow outward, radially, from the inside radius, R_1 , to the outside radius, R_2 . The process will be described by the Fourier Law.



The differential equation governing heat diffusion is: $\frac{1}{r} \frac{d}{dr} \left(r \frac{dT}{dr} \right) = 0$

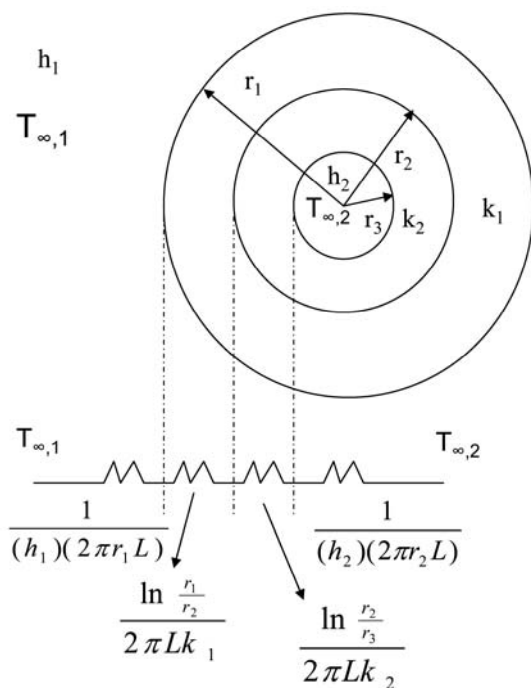
With constant k, the solution is

The heat flow rate across the wall is given by:

$$q_x = -kA \frac{dT}{dx} = \frac{kA}{L} (T_{s,1} - T_{s,2}) = \frac{T_{s,1} - T_{s,2}}{L/kA}$$

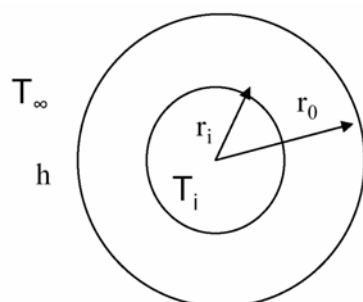
Hence, the thermal resistance in this case can be expressed as: $\frac{\ln \frac{r_1}{r_2}}{2\pi kL}$

Composite cylindrical walls:



$$q_r = \frac{T_{\infty,2} - T_{\infty,1}}{\sum R_t}$$

Critical Insulation Thickness :



$$R_{tot} = \frac{\ln(\frac{r_o}{r_i})}{2\pi kL} + \frac{1}{(2\pi r_o L)h}$$

Insulation thickness : $r_o - r_i$

Objective : decrease q , increase R_{tot}

Vary r_o ; as r_o increases, first term increases, second term decreases.

This is a maximum – minimum problem. The point of extrema can be found by setting

$$\frac{dR_{tot}}{dr_o} = 0$$

or,
$$\frac{1}{2\pi k r_o L} - \frac{1}{2\pi h L r_o^2} = 0$$

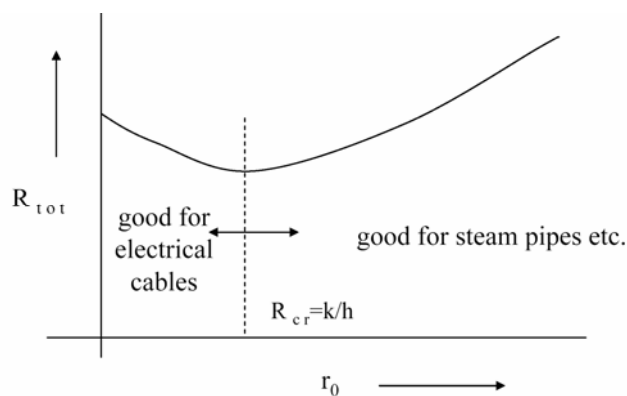
or,
$$r_o = \frac{k}{h}$$

In order to determine if it is a maxima or a minima, we make the second derivative zero:

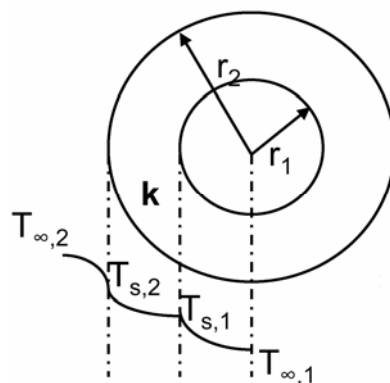
$$\frac{d^2 R_{tot}}{dr_o^2} = 0 \quad \text{at} \quad r_o = \frac{k}{h}$$

$$\frac{d^2 R_{tot}}{dr_o^2} = \frac{-1}{2\pi k r_o^2 L} + \frac{1}{\pi r_o^2 h L} \bigg|_{r_o = \frac{k}{h}} = \frac{h^2}{2\pi L k^3} > 0$$

Minimum q at $r_o = (k/h) = r_{cr}$ (critical radius)



1-D radial conduction in a sphere:



$$\frac{1}{r^2} \frac{d}{dr} \left(kr^2 \frac{dT}{dr} \right) = 0$$

$$\rightarrow T(r) = T_{s,1} - \{T_{s,1} - T_{s,2}\} \left[\frac{1 - (r_1/r)}{1 - (r_1/r_2)} \right]$$

$$\rightarrow q_r = -kA \frac{dT}{dr} = \frac{4\pi k (T_{s,1} - T_{s,2})}{(1/r_1 - 1/r_2)}$$

$$\rightarrow R_{t,cond} = \frac{1/r_1 - 1/r_2}{4\pi k}$$

2.6 Summary of Electrical Analogy

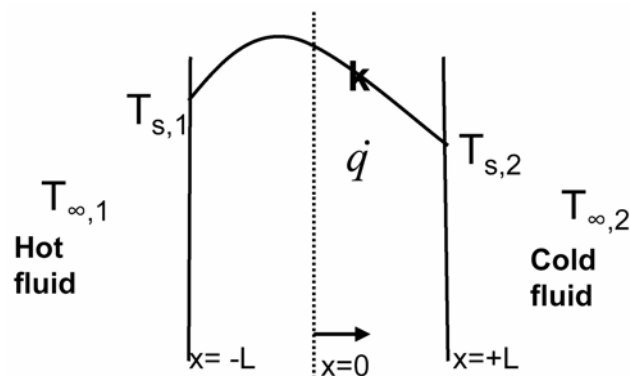
System	Current	Resistance	Potential Difference
Electrical	I	R	ΔV
Cartesian Conduction	q	$\frac{L}{kA}$	ΔT
Cylindrical Conduction	q	$\frac{\ln r_2/r_1}{2\pi kL}$	ΔT
Conduction through sphere	q	$\frac{1/r_1 - 1/r_2}{4\pi k}$	ΔT
Convection	q	$\frac{1}{h \cdot A_s}$	ΔT

2.7 One-Dimensional Steady State Conduction with Internal Heat Generation

Applications: current carrying conductor, chemically reacting systems, nuclear reactors.

Energy generated per unit volume is given by $\dot{q} = \frac{\dot{Q}}{V}$

Plane wall with heat source: Assumptions: 1D, steady state, constant k, uniform \dot{q}



$$\frac{d^2T}{dx^2} + \frac{\dot{q}}{k} = 0$$

Boundary cond.: $x = -L, \quad T = T_{s,1}$

$x = +L, \quad T = T_{s,2}$

Solution: $T = -\frac{\dot{q}}{2k}x^2 + C_1x + C_2$

Use boundary conditions to find C_1 and C_2

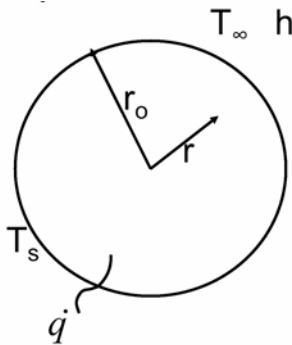
Final solution: $T = \frac{\dot{q}L^2}{2k} \left(1 - \frac{x^2}{L^2} \right) + \frac{T_{s,2} - T_{s,1}}{2} \frac{x}{L} + \frac{T_{s,2} + T_{s,1}}{2}$

Heat flux: $q_x'' = -k \frac{dT}{dx}$

Note: From the above expressions, it may be observed that the solution for temperature is no longer linear. As an exercise, show that the expression for heat flux is no longer independent of x . Hence *thermal resistance concept is not correct to use when there is internal heat generation*.

Cylinder with heat source: Assumptions: 1D, steady state, constant k , uniform \dot{q}

Start with 1D heat equation in cylindrical co-ordinates



$$\frac{1}{r} \frac{d}{dr} \left(r \frac{dT}{dr} \right) + \frac{\dot{q}}{k} = 0$$

Boundary cond.: $r = r_0, \quad T = T_s$

$$r = 0, \quad \frac{dT}{dr} = 0$$

Solution: $T(r) = \frac{\dot{q}}{4k} r_0^2 \left(1 - \frac{r^2}{r_0^2} \right) + T_s$

Exercise: T_s may not be known. Instead, T_∞ and h may be specified. Eliminate T_s , using T_∞ and h .

MODULE I

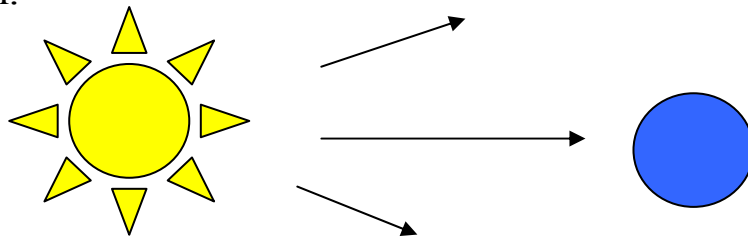
RADIATION HEAT TRANSFER

Radiation

Definition

Radiation, energy transfer across a system boundary due to a ΔT , by the mechanism of photon emission or electromagnetic wave emission.

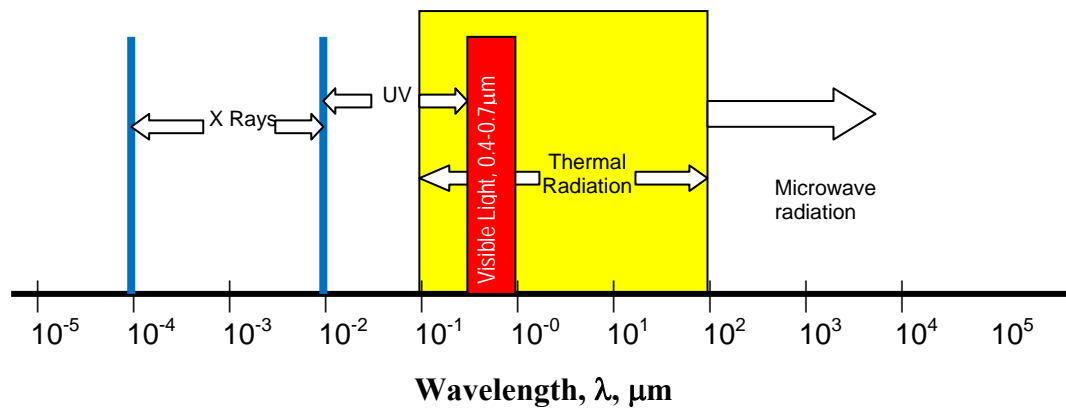
Because the mechanism of transmission is photon emission, unlike conduction and convection, there need be no intermediate matter to enable transmission.



The significance of this is that radiation will be the only mechanism for heat transfer whenever a vacuum is present.

Electromagnetic Phenomena.

We are well acquainted with a wide range of electromagnetic phenomena in modern life. These phenomena are sometimes thought of as wave phenomena and are, consequently, often described in terms of electromagnetic wave length, λ . Examples are given in terms of the wave distribution shown below:



One aspect of electromagnetic radiation is that the related topics are more closely associated with optics and electronics than with those normally found in mechanical engineering courses. Nevertheless, these are widely encountered topics and the student is familiar with them through every day life experiences.

From a viewpoint of previously studied topics students, particularly those with a background in mechanical or chemical engineering, will find the subject of Radiation Heat Transfer a little unusual. The physics background differs fundamentally from that found in the areas of continuum mechanics. Much of the related material is found in courses more closely identified with quantum physics or electrical engineering, i.e. Fields and Waves. At this point, it is important for us to recognize that since the subject arises from a different area of physics, it will be important that we study these concepts with extra care.

Stefan-Boltzman Law

Both Stefan and Boltzman were physicists; any student taking a course in quantum physics will become well acquainted with Boltzman's work as he made a number of important contributions to the field. Both were contemporaries of Einstein so we see that the subject is of fairly recent vintage. (Recall that the basic equation for convection heat transfer is attributed to Newton.)

$$E_b = \sigma \cdot T_{abs}^4$$

where: E_b = Emissive Power, the gross energy emitted from an ideal surface per unit area, time.

σ = Stefan Boltzman constant, $5.67 \cdot 10^{-8} \text{ W/m}^2 \cdot \text{K}^4$

T_{abs} = Absolute temperature of the emitting surface, K.

Take particular note of the fact that absolute temperatures are used in Radiation. It is suggested, as a matter of good practice, to convert all temperatures to the absolute scale as an initial step in all radiation problems.

You will notice that the equation does not include any heat flux term, q'' . Instead we have a term the emissive power. The relationship between these terms is as follows. Consider two infinite plane surfaces, both facing one another. Both surfaces are ideal surfaces. One surface is found to be at temperature, T_1 , the other at temperature, T_2 . Since both temperatures are at temperatures above absolute zero, both will radiate energy as described by the Stefan-Boltzman law. The heat flux will be the net radiant flow as given by:

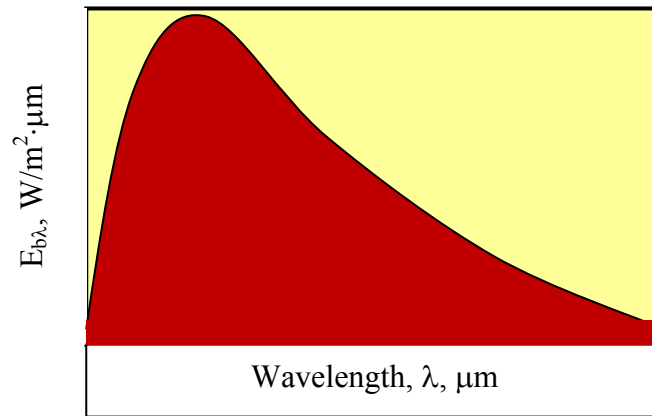
$$q'' = E_{b1} - E_{b2} = \sigma \cdot T_1^4 - \sigma \cdot T_2^4$$

Plank's Law

While the Stefan-Boltzman law is useful for studying overall energy emissions, it does not allow us to treat those interactions, which deal specifically with wavelength, λ . This problem was overcome by another of the modern physicists, Max Plank, who developed a relationship for wave-based emissions.

$$E_{b\lambda} = f(\lambda)$$

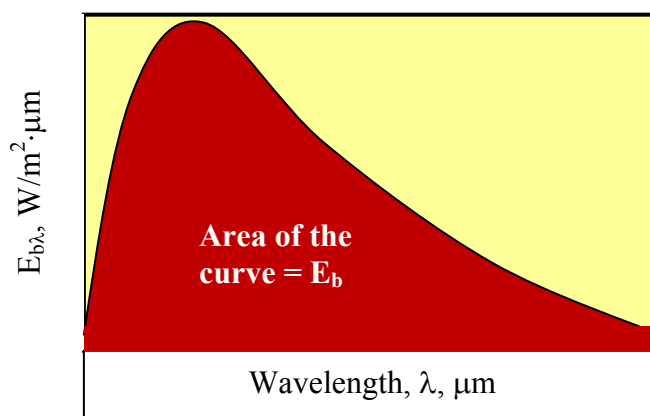
We plot a suitable functional relationship below:



We haven't yet defined the Monochromatic Emissive Power, $E_{b\lambda}$. An implicit definition is provided by the following equation:

$$E_b = \int_0^{\infty} E_{b\lambda} \cdot d\lambda$$

We may view this equation graphically as follows:



A definition of monochromatic Emissive Power would be obtained by differentiating the integral equation:

$$E_{b\lambda} \equiv \frac{dE_b}{d\lambda}$$

The actual form of Plank's law is:

$$E_{b\lambda} = \frac{C_1}{\lambda^5 \cdot \left[e^{c_2/\lambda \cdot T} - 1 \right]}$$

where: $C_1 = 2 \cdot \pi \cdot h \cdot c_0^2 = 3.742 \cdot 10^8 \text{ W} \cdot \mu\text{m}^4/\text{m}^2$
 $C_2 = h \cdot c_0/k = 1.439 \cdot 10^4 \mu\text{m} \cdot \text{K}$

Where: h , c_0 , k are all parameters from quantum physics. We need not worry about their precise definition here.

This equation may be solved at any T , λ to give the value of the monochromatic emissivity at that condition. Alternatively, the function may be substituted into the integral $E_b = \int_0^\infty E_{b\lambda} \cdot d\lambda$ to find the Emissive power for any temperature. While performing this integral by hand is difficult, students may readily evaluate the integral through one of several computer programs, i.e. MathCad, Maple, Mathematica, etc.

$$E_b = \int_0^\infty E_{b\lambda} \cdot d\lambda = \sigma \cdot T^4$$

Emission Over Specific Wave Length Bands

Consider the problem of designing a tanning machine. As a part of the machine, we will need to design a very powerful incandescent light source. We may wish to know how much energy is being emitted over the ultraviolet band (10^{-4} to $0.4 \mu\text{m}$), known to be particularly dangerous.

$$E_b(0.0001 \rightarrow 0.4) = \int_{0.001 \cdot \mu\text{m}}^{0.4 \cdot \mu\text{m}} E_{b\lambda} \cdot d\lambda$$

With a computer available, evaluation of this integral is rather trivial. Alternatively, the text books provide a table of integrals. The format used is as follows:

$$\frac{E_b(0.001 \rightarrow 0.4)}{E_b} = \frac{\int_{0.001 \mu m}^{0.4 \mu m} E_{b\lambda} \cdot d\lambda}{\int_0^{\infty} E_{b\lambda} \cdot d\lambda} = \frac{\int_0^{0.4 \mu m} E_{b\lambda} \cdot d\lambda}{\int_0^{\infty} E_{b\lambda} \cdot d\lambda} - \frac{\int_0^{0.0001 \mu m} E_{b\lambda} \cdot d\lambda}{\int_0^{\infty} E_{b\lambda} \cdot d\lambda} = F(0 \rightarrow 0.4) - F(0 \rightarrow 0.0001)$$

Referring to such tables, we see the last two functions listed in the second column. In the first column is a parameter, $\lambda \cdot T$. This is found by taking the product of the absolute temperature of the emitting surface, T , and the upper limit wave length, λ . In our example, suppose that the incandescent bulb is designed to operate at a temperature of 2000K. Reading from the table:

$\lambda, \mu m$	T, K	$\lambda \cdot T, \mu m \cdot K$	$F(0 \rightarrow \lambda)$
0.0001	2000	0.2	0
0.4	2000	600	0.000014
$F(0.4 \rightarrow 0.0001 \mu m) = F(0 \rightarrow 0.4 \mu m) - F(0 \rightarrow 0.0001 \mu m)$			0.000014

This is the fraction of the total energy emitted which falls within the IR band. To find the absolute energy emitted multiply this value times the total energy emitted:

$$E_{bIR} = F(0.4 \rightarrow 0.0001 \mu m) \cdot \sigma \cdot T^4 = 0.000014 \cdot 5.67 \cdot 10^{-8} \cdot 2000^4 = \mathbf{12.7 \text{ W/m}^2}$$

Solar Radiation

The magnitude of the energy leaving the Sun varies with time and is closely associated with such factors as solar flares and sunspots. Nevertheless, we often choose to work with an average value. The energy leaving the sun is emitted outward in all directions so that at any particular distance from the Sun we may imagine the energy being dispersed over an imaginary spherical area. Because this area increases with the distance squared, the solar flux also decreases with the distance squared. At the average distance between Earth and Sun this heat flux is 1353 W/m^2 , so that the average heat flux on any object in Earth orbit is found as:

$$G_{s,o} = S_c \cdot f \cdot \cos \theta$$

Where S_c = Solar Constant, 1353 W/m^2
 f = correction factor for eccentricity in Earth Orbit,
 $(0.97 < f < 1.03)$
 θ = Angle of surface from normal to Sun.

Because of reflection and absorption in the Earth's atmosphere, this number is significantly reduced at ground level. Nevertheless, this value gives us some opportunity to estimate the potential for using solar energy, such as in photovoltaic cells.

Some Definitions

In the previous section we introduced the Stefan-Boltzman Equation to describe radiation from an ideal surface.

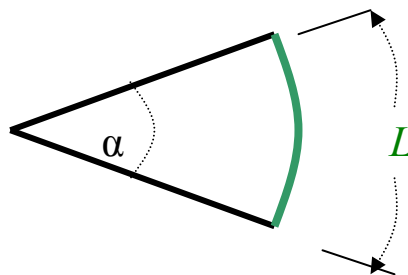
$$E_b = \sigma \cdot T_{\text{abs}}^4$$

This equation provides a method of determining the total energy leaving a surface, but gives no indication of the direction in which it travels. As we continue our study, we will want to be able to calculate how heat is distributed among various objects.

For this purpose, we will introduce the radiation intensity, I , defined as the energy emitted per unit area, per unit time, per unit solid angle. Before writing an equation for this new property, we will need to define some of the terms we will be using.

Angles and Arc Length

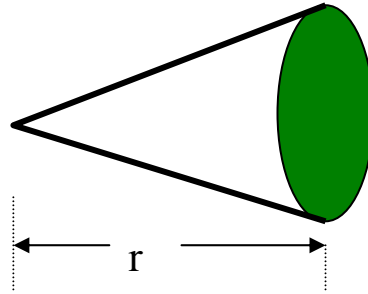
We are well accustomed to thinking of an angle as a two dimensional object. It may be used to find an arc length:



$$L = r \cdot \alpha$$

Solid Angle

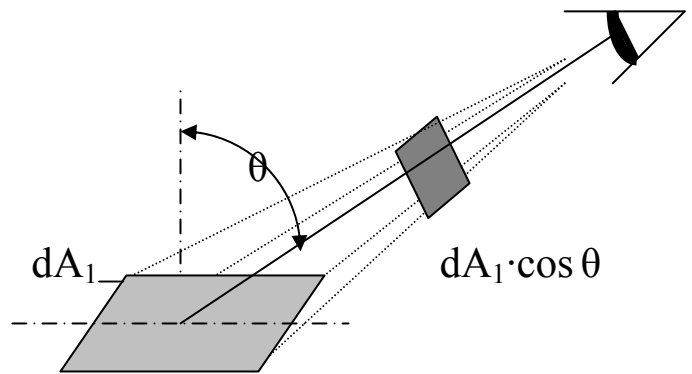
We generalize the idea of an angle and an arc length to three dimensions and define a solid angle, Ω , which like the standard angle has no dimensions. The solid angle, when multiplied by the radius squared will have dimensions of length squared, or area, and will have the magnitude of the encompassed area.



$$A = r^2 \cdot d\Omega$$

Projected Area

The area, dA_1 , as seen from the perspective of a viewer, situated at an angle θ from the normal to the surface, will appear somewhat smaller, as $\cos \theta \cdot dA_1$. This smaller area is termed the projected area.



$$A_{\text{projected}} = \cos \theta \cdot A_{\text{normal}}$$

Intensity

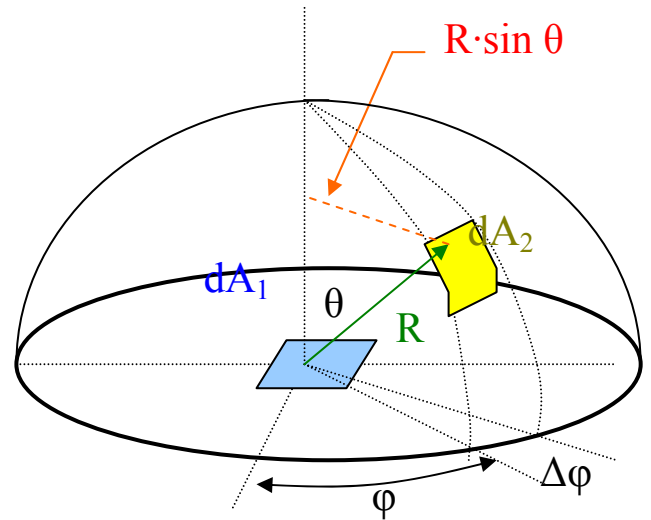
The ideal intensity, I_b , may now be defined as the energy emitted from an ideal body, per unit projected area, per unit time, per unit solid angle.

$$I = \frac{dq}{\cos \theta \cdot dA_1 \cdot d\Omega}$$

Spherical Geometry

Since any surface will emit radiation outward in all directions above the surface, the spherical coordinate system provides a convenient tool for analysis. The three basic coordinates shown are R , ϕ , and θ , representing the radial, azimuthal and zenith directions.

In general dA_1 will correspond to the emitting surface or the source. The surface dA_2 will correspond to the receiving surface or the target. Note that the area proscribed on the hemisphere, dA_2 , may be written as:



$$dA_2 = [(R \cdot \sin \theta) \cdot d\phi] \cdot [R \cdot d\theta]$$

or, more simply as:

$$dA_2 = R^2 \cdot \sin \theta \cdot d\phi \cdot d\theta$$

Recalling the definition of the solid angle,

$$dA = R^2 \cdot d\Omega$$

we find that:

$$d\Omega = \sin \theta \cdot d\theta \cdot d\phi$$

Real Surfaces

Thus far we have spoken of ideal surfaces, i.e. those that emit energy according to the Stefan-Boltzman law:

$$E_b = \sigma \cdot T_{\text{abs}}^4$$

Real surfaces have emissive powers, E , which are somewhat less than that obtained theoretically by Boltzman. To account for this reduction, we introduce the emissivity, ε .

$$\varepsilon \equiv \frac{E}{E_b}$$

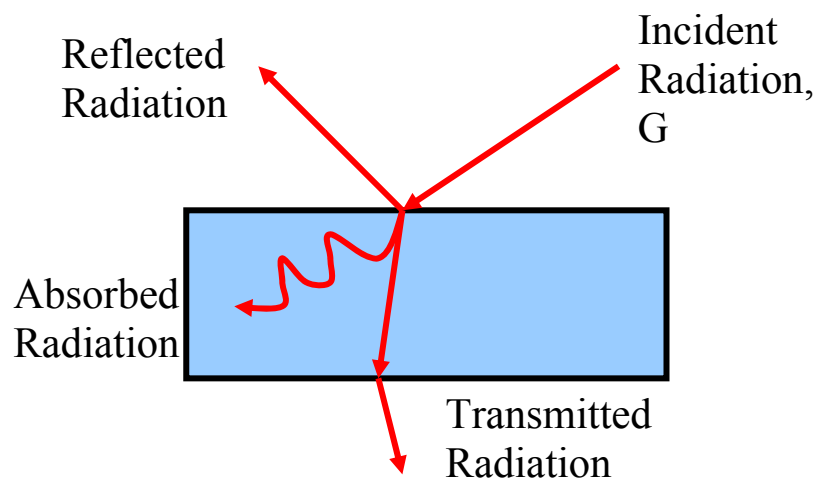
so that the emissive power from any real surface is given by:

$$E = \varepsilon \cdot \sigma \cdot T_{\text{abs}}^4$$

Receiving Properties

Targets receive radiation in one of three ways; they absorption, reflection or transmission. To account for these characteristics, we introduce three additional properties:

- Absorptivity, α , the fraction of incident radiation absorbed.
- Reflectivity, ρ , the fraction of incident radiation reflected.
- Transmissivity, τ , the fraction of incident radiation transmitted.



We see, from Conservation of Energy, that:

$$\alpha + \rho + \tau = 1$$

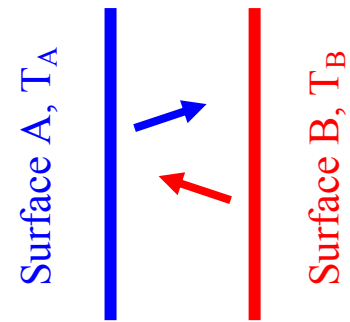
In this course, we will deal with only opaque surfaces, $\tau = 0$, so that:

$$\alpha + \rho = 1$$

Opaque Surfaces

Relationship Between Absorptivity, α , and Emissivity, ϵ

Consider two flat, infinite planes, surface A and surface B, both emitting radiation toward one another. Surface B is assumed to be an ideal emitter, i.e. $\epsilon_B = 1.0$. Surface A will emit radiation according to the Stefan-Boltzman law as:



$$E_A = \epsilon_A \cdot \sigma \cdot T_A^4$$

and will receive radiation as:

$$G_A = \alpha_A \cdot \sigma \cdot T_B^4$$

The net heat flow from surface A will be:

$$q'' = \epsilon_A \cdot \sigma \cdot T_A^4 - \alpha_A \cdot \sigma \cdot T_B^4$$

Now suppose that the two surfaces are at exactly the same temperature. The heat flow must be zero according to the 2nd law. It follows then that:

$$\alpha_A = \epsilon_A$$

Because of this close relation between emissivity, ϵ , and absorptivity, α , only one property is normally measured and this value may be used alternatively for either property.

Let's not lose sight of the fact that, as thermodynamic properties of the material, α and ϵ may depend on temperature. In general, this will be the case as radiative properties will depend on wavelength, λ . The wave length of radiation will, in turn, depend on the temperature of the source of radiation.

The emissivity, ϵ , of surface A will depend on the material of which surface A is composed, i.e. aluminum, brass, steel, etc. and on the temperature of surface A.

The absorptivity, α , of surface A will depend on the material of which surface A is composed, i.e. aluminum, brass, steel, etc. and on the temperature of surface B.

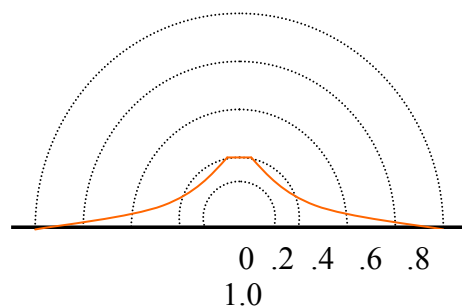
In the design of solar collectors, engineers have long sought a material which would absorb all solar radiation, ($\alpha = 1$, $T_{\text{sun}} \sim 5600\text{K}$) but would not re-radiate energy as it came to temperature ($\varepsilon \ll 1$, $T_{\text{collector}} \sim 400\text{K}$). NASA developed an anodized chrome, commonly called “black chrome” as a result of this research.

Black Surfaces

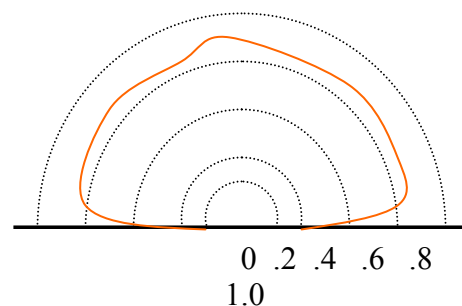
Within the visual band of radiation, any material, which absorbs all visible light, appears as black. Extending this concept to the much broader thermal band, we speak of surfaces with $\alpha = 1$ as also being “black” or “thermally black”. It follows that for such a surface, $\varepsilon = 1$ and the surface will behave as an ideal emitter. The terms ideal surface and black surface are used interchangeably.

Lambert’s Cosine Law:

A surface is said to obey Lambert’s cosine law if the intensity, I , is uniform in all directions. This is an idealization of real surfaces as seen by the emissivity at different zenith angles:



Dependence of Emissivity on Zenith Angle, Typical Metal.



Dependence of Emissivity on Zenith Angle, Typical Non-Metal.

The sketches shown are intended to show is that metals typically have a very low emissivity, ϵ , which also remain nearly constant, except at very high zenith angles, θ . Conversely, non-metals will have a relatively high emissivity, ϵ , except at very high zenith angles. Treating the emissivity as a constant over all angles is generally a good approximation and greatly simplifies engineering calculations.

Relationship Between Emissive Power and Intensity

By definition of the two terms, emissive power for an ideal surface, E_b , and intensity for an ideal surface, I_b .

$$E_b = \int_{\text{hemisphere}} I_b \cdot \cos \theta \cdot d\Omega$$

Replacing the solid angle by its equivalent in spherical angles:

$$E_b = \int_0^{2\pi} \int_0^{\pi/2} I_b \cdot \cos \theta \cdot \sin \theta \cdot d\theta \cdot d\phi$$

Integrate once, holding I_b constant:

$$E_b = 2 \cdot \pi \cdot I_b \cdot \int_0^{\pi/2} \cos \theta \cdot \sin \theta \cdot d\theta$$

Integrate a second time. (Note that the derivative of $\sin \theta$ is $\cos \theta \cdot d\theta$.)

$$E_b = 2 \cdot \pi \cdot I_b \cdot \left. \frac{\sin^2 \theta}{2} \right|_0^{\pi/2} = \pi \cdot I_b$$

$$E_b = \pi \cdot I_b$$

Radiation Exchange

During the previous lecture we introduced the intensity, I , to describe radiation within a particular solid angle.

$$I = \frac{dq}{\cos \theta \cdot dA_1 \cdot d\Omega}$$

This will now be used to determine the fraction of radiation leaving a given surface and striking a second surface.

Rearranging the above equation to express the heat radiated:

$$dq = I \cdot \cos \theta \cdot dA_1 \cdot d\Omega$$

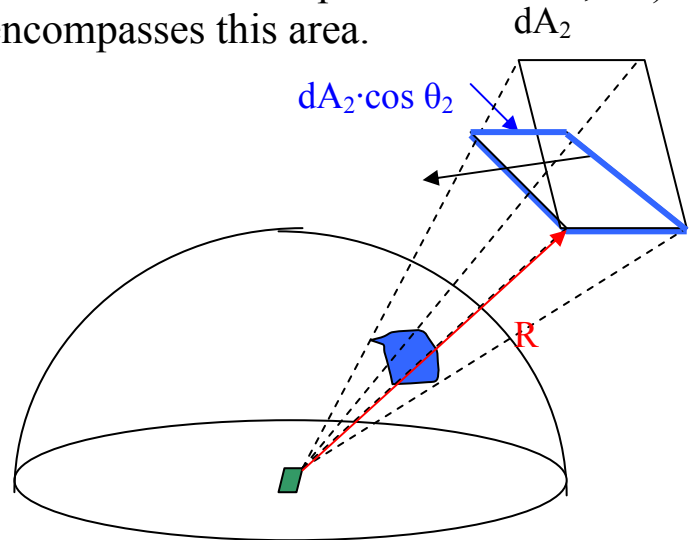
Next we will project the receiving surface onto the hemisphere surrounding the source. First find the projected area of surface dA_2 , $dA_2 \cdot \cos \theta_2$. (θ_2 is the angle between the normal to surface 2 and the position vector, R .) Then find the solid angle, Ω , which encompasses this area.

Substituting into the heat flow equation above:

$$dq = \frac{I \cdot \cos \theta_1 \cdot dA_1 \cdot \cos \theta_2 dA_2}{R^2}$$

To obtain the entire heat transferred from a finite area, dA_1 , to a finite area, dA_2 , we integrate over both surfaces:

$$q_{1 \rightarrow 2} = \int_{A_2} \int_{A_1} \frac{I \cdot \cos \theta_1 \cdot dA_1 \cdot \cos \theta_2 dA_2}{R^2}$$



To express the total energy emitted from surface 1, we recall the relation between emissive power, E , and intensity, I .

$$q_{\text{emitted}} = E_1 \cdot A_1 = \pi \cdot I_1 \cdot A_1$$

View Factors-Integral Method

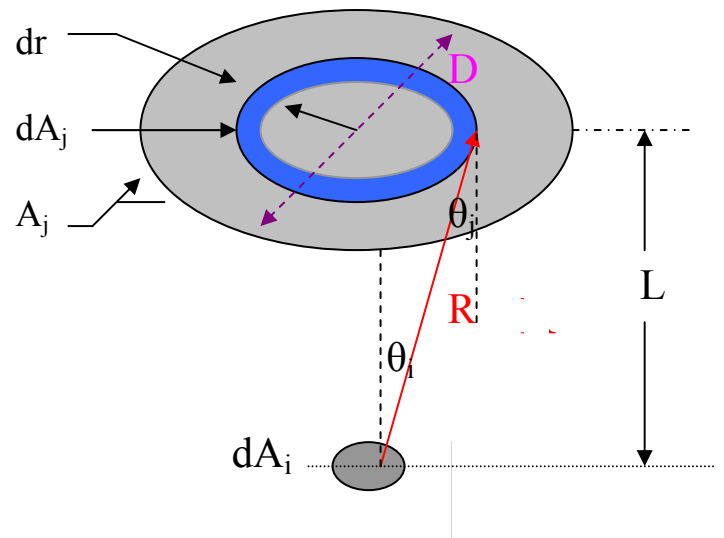
Define the view factor, $F_{1 \rightarrow 2}$, as the fraction of energy emitted from surface 1, which directly strikes surface 2.

$$F_{1 \rightarrow 2} = \frac{q_{1 \rightarrow 2}}{q_{\text{emitted}}} = \frac{\int_{A_2} \int_{A_1} \frac{I \cdot \cos \theta_1 \cdot dA_1 \cdot \cos \theta_2 dA_2}{R^2}}{\pi \cdot I \cdot A_1}$$

after algebraic simplification this becomes:

$$F_{1 \rightarrow 2} = \frac{1}{A_1} \cdot \int_{A_2} \int_{A_1} \frac{\cos \theta_1 \cdot \cos \theta_2 \cdot dA_1 \cdot dA_2}{\pi \cdot R^2}$$

Example Consider a diffuse circular disk of diameter D and area A_j and a plane diffuse surface of area $A_i \ll A_j$. The surfaces are parallel, and A_i is located at a distance L from the center of A_j . Obtain an expression for the view factor F_{ij} .



The view factor may be obtained from:

$$F_{1 \rightarrow 2} = \frac{1}{A_1} \cdot \int_{A_2} \int_{A_1} \frac{\cos \theta_1 \cdot \cos \theta_2 \cdot dA_1 \cdot dA_2}{\pi \cdot R^2}$$

Since dA_i is a differential area

$$F_{1 \rightarrow 2} = \int_{A_1} \frac{\cos \theta_1 \cdot \cos \theta_2 \cdot dA_1}{\pi \cdot R^2}$$

Substituting for the cosines and the differential area:

$$F_{1 \rightarrow 2} = \int_{A_1} \frac{\left(\frac{L}{R}\right)^2 \cdot 2\pi \cdot r \cdot dr}{\pi \cdot R^2}$$

After simplifying:

$$F_{1 \rightarrow 2} = \int_{A_1} \frac{L^2 \cdot 2 \cdot r \cdot dr}{R^4}$$

Let $\rho^2 \equiv L^2 + r^2 = R^2$. Then $2 \cdot \rho \cdot d\rho = 2 \cdot r \cdot dr$.

$$F_{1 \rightarrow 2} = \int_{A_1} \frac{L^2 \cdot 2 \cdot \rho \cdot d\rho}{\rho^4}$$

After integrating,

$$F_{1 \rightarrow 2} = -2 \cdot L^2 \cdot \frac{\rho^{-2}}{2} \Big|_{A_2} = -L^2 \cdot \left[\frac{1}{L^2 + \rho^2} \right]_0^{D/2}$$

Substituting the upper & lower limits

$$F_{1 \rightarrow 2} = -L^2 \cdot \left[\frac{4}{4 \cdot L^2 + D^2} - \frac{1}{L^2} \right]_0^{D/2} = \frac{D^2}{4 \cdot L^2 + D^2}$$

This is but one example of how the view factor may be evaluated using the integral method. The approach used here is conceptually quite straight forward; evaluating the integrals and algebraically simplifying the resulting equations can be quite lengthy.

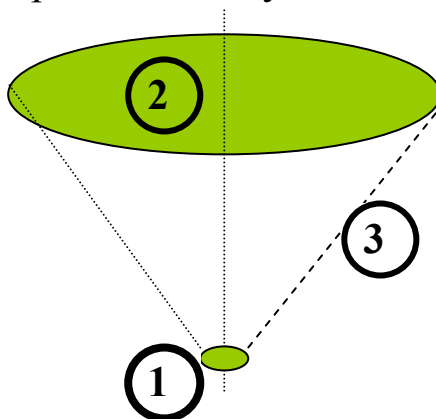
Enclosures

In order that we might apply conservation of energy to the radiation process, we must account for all energy leaving a surface. We imagine that the surrounding surfaces act as an enclosure about the heat source which receive all emitted energy. Should there be an opening in this enclosure through which energy might be lost, we place an imaginary surface across this opening to intercept this portion of the emitted energy. For an N surfaced enclosure, we can then see that:

$$\sum_{j=1}^N F_{i,j} = 1$$

This relationship is known as the “Conservation Rule”.

Example: Consider the previous problem of a small disk radiating to a larger disk placed directly above at a distance L.



The view factor was shown to be given by the relationship:

$$F_{1 \rightarrow 2} = \frac{D^2}{4 \cdot L^2 + D^2}$$

Here, in order to provide an enclosure, we will define an imaginary surface 3, a truncated cone intersecting circles 1 and 2.

From our conservation rule we have:

$$\sum_{j=1}^N F_{i,j} = F_{1,1} + F_{1,2} + F_{1,3}$$

Since surface 1 is not convex $F_{1,1} = 0$. Then:

$$F_{1 \rightarrow 3} = 1 - \frac{D^2}{4 \cdot L^2 + D^2}$$

Reciprocity

We may write the view factor from surface i to surface j as:

$$A_i \cdot F_{i \rightarrow j} = \int_{A_j} \int_{A_i} \frac{\cos \theta_i \cdot \cos \theta_j \cdot dA_i \cdot dA_j}{\pi \cdot R^2}$$

Similarly, between surfaces j and i:

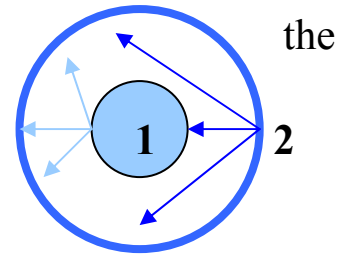
$$A_j \cdot F_{j \rightarrow i} = \int_{A_j} \int_{A_i} \frac{\cos \theta_j \cdot \cos \theta_i \cdot dA_j \cdot dA_i}{\pi \cdot R^2}$$

Comparing the integrals we see that they are identical so that:

$$A_i \cdot F_{i \rightarrow j} = A_j \cdot F_{j \rightarrow i}$$

This relationship
is known as
“Reciprocity”.

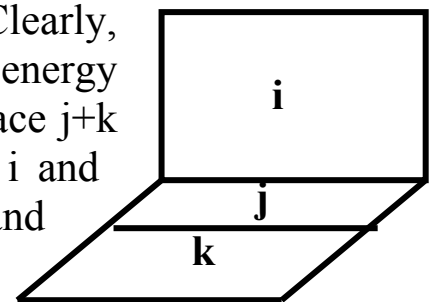
Example: Consider two concentric spheres shown to the right. All radiation leaving the outside of surface 1 will strike surface 2. Part of the radiant energy leaving the inside surface of object 2 will strike surface 1, part will return to surface 2. To find the fraction of energy leaving surface 2 which strikes surface 1, we apply reciprocity:



$$A_2 \cdot F_{2,1} = A_1 \cdot F_{1,2} \Rightarrow F_{2,1} = \frac{A_1}{A_2} \cdot F_{1,2} = \frac{A_1}{A_2} = \frac{D_1}{D_2}$$

Associative Rule

Consider the set of surfaces shown to the right: Clearly, from conservation of energy, the fraction of energy leaving surface i and striking the combined surface j+k will equal the fraction of energy emitted from i and striking j plus the fraction leaving surface i and striking k.



$$F_{i \Rightarrow (j+k)} = F_{i \Rightarrow j} + F_{i \Rightarrow k}$$

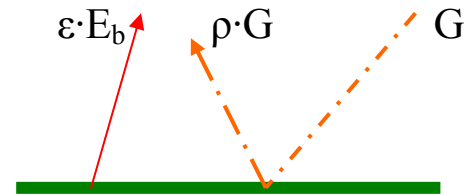
This relationship is known as the “Associative Rule”.

Radiosity

We have developed the concept of intensity, I , which led to the concept of the view factor. We have discussed various methods of finding view factors. There remains one additional concept to introduce before we can consider the solution of radiation problems.

Radiosity, J , is defined as the total energy leaving a surface per unit area and per unit time. This may initially sound much like the definition of emissive power, but the sketch below will help to clarify the concept.

$$J \equiv \varepsilon \cdot E_b + \rho \cdot G$$



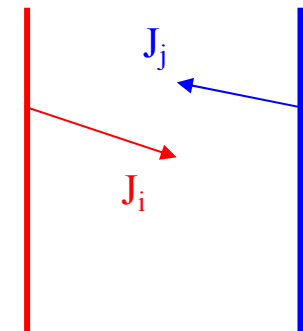
Net Exchange Between Surfaces

Consider the two surfaces shown. Radiation will travel from surface i to surface j and will also travel from j to i.

$$q_{i \rightarrow j} = J_i \cdot A_i \cdot F_{i \rightarrow j}$$

likewise,

$$q_{j \rightarrow i} = J_j \cdot A_j \cdot F_{j \rightarrow i}$$



The net heat transfer is then:

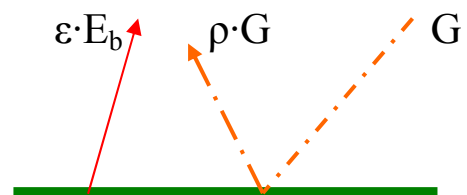
$$q_{j \rightarrow i \text{ (net)}} = J_i \cdot A_i \cdot F_{i \rightarrow j} - J_j \cdot A_j \cdot F_{j \rightarrow i}$$

From reciprocity we note that $F_{1 \rightarrow 2} \cdot A_1 = F_{2 \rightarrow 1} \cdot A_2$ so that

$$q_{j \rightarrow i \text{ (net)}} = J_i \cdot A_i \cdot F_{i \rightarrow j} - J_j \cdot A_i \cdot F_{i \rightarrow j} = A_i \cdot F_{i \rightarrow j} \cdot (J_i - J_j)$$

Net Energy Leaving a Surface

The net energy leaving a surface will be the difference between the energy leaving a surface and the energy received by a surface:



$$q_{l \rightarrow} = [\varepsilon \cdot E_b - \alpha \cdot G] \cdot A_1$$

Combine this relationship with the definition of Radiosity to eliminate G.

$$J \equiv \varepsilon \cdot E_b + \rho \cdot G \Rightarrow G = [J - \varepsilon \cdot E_b] / \rho$$

$$q_{1\rightarrow} = \{\varepsilon \cdot E_b - \alpha \cdot [J - \varepsilon \cdot E_b] / \rho\} \cdot A_1$$

Assume opaque surfaces so that $\alpha + \rho = 1 \rightarrow \rho = 1 - \alpha$, and substitute for ρ .

$$q_{1\rightarrow} = \{\varepsilon \cdot E_b - \alpha \cdot [J - \varepsilon \cdot E_b] / (1 - \alpha)\} \cdot A_1$$

Put the equation over a common denominator:

$$q_{1\rightarrow} = \left[\frac{(1 - \alpha) \cdot \varepsilon \cdot E_b - \alpha \cdot J + \alpha \cdot \varepsilon \cdot E_b}{1 - \alpha} \right] \cdot A_1 = \left[\frac{\varepsilon \cdot E_b - \alpha \cdot J}{1 - \alpha} \right] \cdot A_1$$

If we assume that $\alpha = \varepsilon$ then the equation reduces to:

$$q_{1\rightarrow} = \left[\frac{\varepsilon \cdot E_b - \varepsilon \cdot J}{1 - \varepsilon} \right] \cdot A_1 = \left[\frac{\varepsilon \cdot A_1}{1 - \varepsilon} \right] \cdot (E_b - J)$$

Electrical Analogy for Radiation

We may develop an electrical analogy for radiation, similar to that produced for conduction. **The two analogies should not be mixed: they have different dimensions on the potential differences, resistance and current flows.**

	Equivalent Current	Equivalent Resistance	Potential Difference
Ohms Law	I	R	ΔV
Net Energy Leaving Surface	$q_{1\rightarrow}$	$\left[\frac{1 - \varepsilon}{\varepsilon \cdot A} \right]$	$E_b - J$
Net Exchange Between Surfaces	$q_{i\rightarrow j}$	$\frac{1}{A_1 \cdot F_{1\rightarrow 2}}$	$J_1 - J_2$

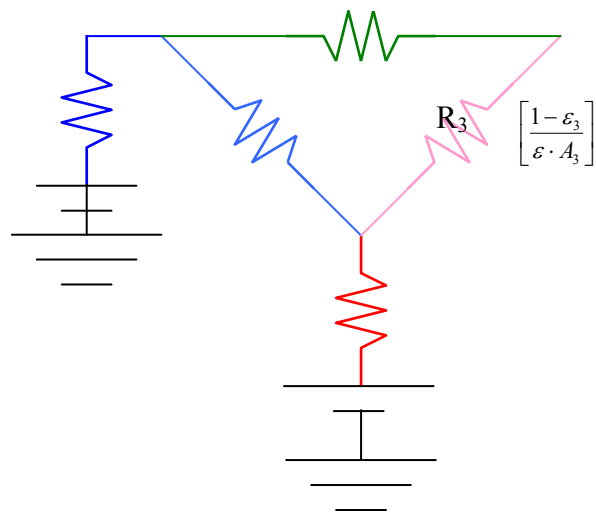
Alternate Procedure for Developing Networks

- Count the number of surfaces. (A surface must be at a “uniform” temperature and have uniform properties, i.e. ϵ , α , ρ .)
- Draw a radiosity node for each surface.
- Connect the Radiosity nodes using view factor resistances, $1/A_i \cdot F_{i \rightarrow j}$.
- Connect each Radiosity node to a grounded battery, through a surface resistance, $\left[\frac{1 - \epsilon}{\epsilon \cdot A} \right]$.

This procedure should lead to exactly the same circuit as we obtain previously.

Simplifications to the Electrical Network

- Insulated surfaces. In steady state heat transfer, a surface cannot receive net energy if it is insulated. Because the energy cannot be stored by a surface in steady state, all energy must be re-radiated back into the enclosure. *Insulated surfaces are often termed as re-radiating surfaces.*



Electrically cannot flow through a battery if it is not grounded.

Surface 3 is not grounded so that the battery and surface resistance serve no purpose and are removed from the drawing.

- Black surfaces: A black, or ideal surface, will have no surface resistance:

$$\left[\frac{1 - \varepsilon}{\varepsilon \cdot A} \right] = \left[\frac{1 - 1}{1 \cdot A} \right] = 0$$

In this case the nodal Radiosity and emissive power will be equal.

This result gives some insight into the physical meaning of a black surface. Ideal surfaces radiate at the maximum possible level. Non-black surfaces will have a reduced potential, somewhat like a battery with a corroded terminal. They therefore have a reduced potential to cause heat/current flow.

- Large surfaces: Surfaces having a large surface area will behave as black surfaces, irrespective of the actual surface properties:

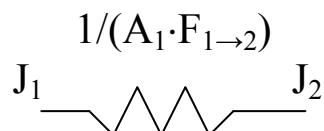
$$\left[\frac{1 - \varepsilon}{\varepsilon \cdot A} \right] = \left[\frac{1 - \varepsilon}{\varepsilon \cdot \infty} \right] = 0$$

Physically, this corresponds to the characteristic of large surfaces that as they reflect energy, there is very little chance that energy will strike the smaller surfaces; most of the energy is reflected back to another part of the same large surface. After several partial absorptions most of the energy received is absorbed.

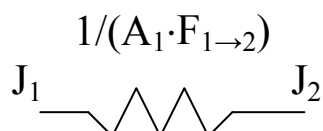
Solution of Analogous Electrical Circuits.

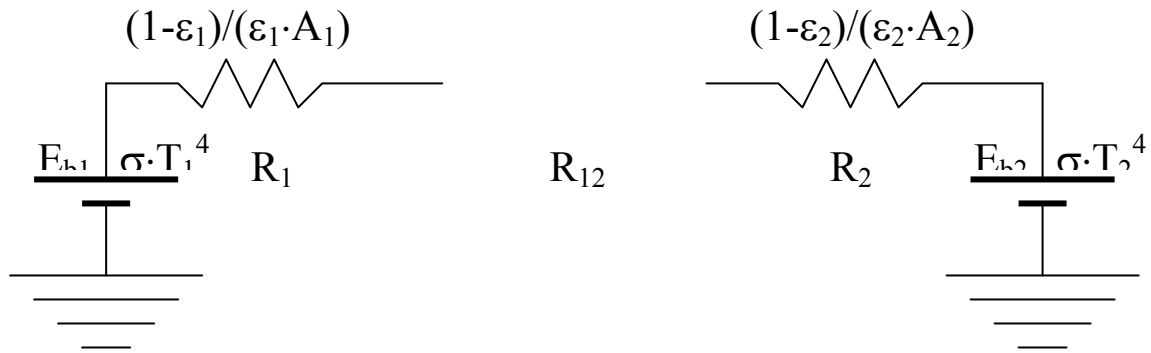
- Large Enclosures

Consider the case of an object, 1, placed inside a large enclosure, 2. The system will consist of two objects, so we proceed to construct a circuit with two radiosity nodes.

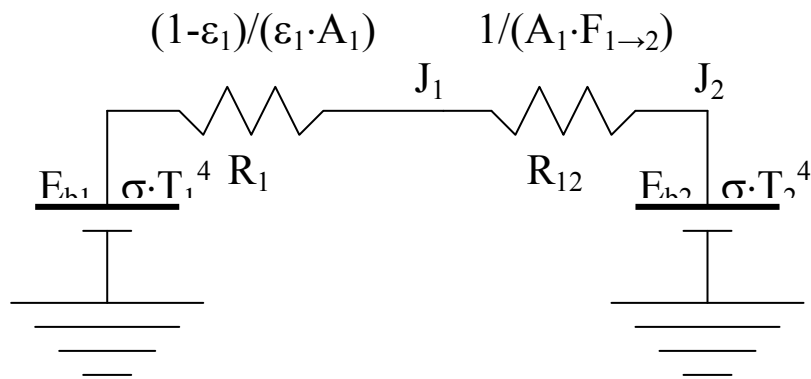


Now we ground both Radiosity nodes through a surface resistance.





Since A_2 is large, $R_2 = 0$. The view factor, $F_{1 \rightarrow 2} = 1$



Sum the series resistances:

$$R_{\text{Series}} = (1-\epsilon_1)/(\epsilon_1 \cdot A_1) + 1/A_1 = 1/(\epsilon_1 \cdot A_1)$$

Ohm's law:

$$i = \Delta V/R$$

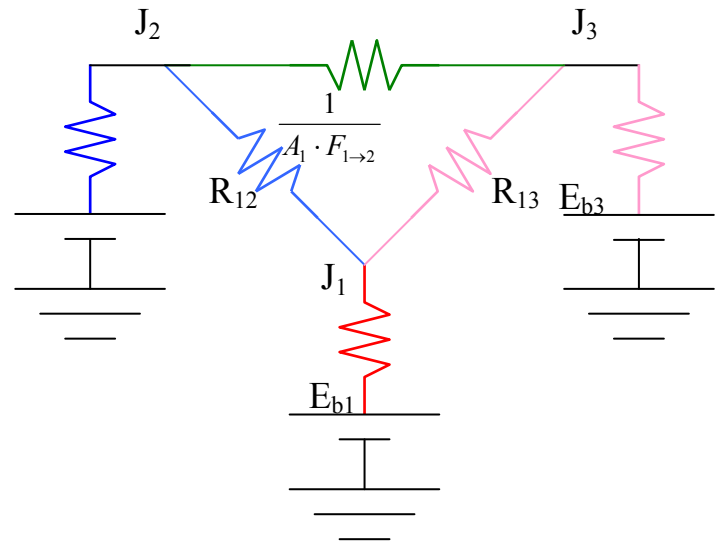
or by analogy:

$$q = \Delta E_b / R_{\text{Series}} = \epsilon_1 \cdot A_1 \cdot \sigma \cdot (T_1^4 - T_2^4)$$

You may recall this result from Thermo I, where it was introduced to solve this type of radiation problem.

- Networks with Multiple Potentials

Systems with 3 or more grounded potentials will require a slightly different solution, but one which students have previously encountered in the Circuits course.



The procedure will be to apply Kirchhoff's law to each of the Radiosity junctions.

$$\sum_{i=1}^3 q_i = 0$$

In this example there are three junctions, so we will obtain three equations. This will allow us to solve for three unknowns.

Radiation problems will generally be presented on one of two ways:

- The surface net heat flow is given and the surface temperature is to be found.
- The surface temperature is given and the net heat flow is to be found.

Returning for a moment to the coal grate furnace, let us assume that we know (a) the total heat being produced by the coal bed, (b) the temperatures of the water walls and (c) the temperature of the super heater sections.

Apply Kirchhoff's law about node 1, for the coal bed:

$$q_1 + q_{2 \rightarrow 1} + q_{3 \rightarrow 1} = q_1 + \frac{J_2 - J_1}{R_{12}} + \frac{J_3 - J_1}{R_{13}} = 0$$

Similarly, for node 2:

$$q_2 + q_{1 \rightarrow 2} + q_{3 \rightarrow 2} = \frac{E_{b2} - J_2}{R_2} + \frac{J_1 - J_2}{R_{12}} + \frac{J_3 - J_2}{R_{23}} = 0$$

(Note how node 1, with a specified heat input, is handled differently than node 2, with a specified temperature.

And for node 3:

$$q_3 + q_{1 \rightarrow 3} + q_{2 \rightarrow 3} = \frac{E_{b3} - J_3}{R_3} + \frac{J_1 - J_3}{R_{13}} + \frac{J_2 - J_3}{R_{23}} = 0$$

The three equations must be solved simultaneously. Since they are each linear in J, matrix methods may be used:

$$\begin{bmatrix} -\frac{1}{R_{12}} - \frac{1}{R_{13}} & \frac{1}{R_{12}} & \frac{1}{R_{13}} \\ \frac{1}{R_{12}} & -\frac{1}{R_2} - \frac{1}{R_{12}} - \frac{1}{R_{13}} & \frac{1}{R_{23}} \\ \frac{1}{R_{13}} & \frac{1}{R_{23}} & -\frac{1}{R_3} - \frac{1}{R_{13}} - \frac{1}{R_{23}} \end{bmatrix} \cdot \begin{bmatrix} J_1 \\ J_2 \\ J_3 \end{bmatrix} = \begin{bmatrix} -q_1 \\ -\frac{E_{b2}}{R_2} \\ -\frac{E_{b3}}{R_3} \end{bmatrix}$$

The matrix may be solved for the individual Radiosity. Once these are known, we return to the electrical analogy to find the temperature of surface 1, and the heat flows to surfaces 2 and 3.

Surface 1: Find the coal bed temperature, given the heat flow:

$$q_1 = \frac{E_{b1} - J_1}{R_1} = \frac{\sigma \cdot T_1^4 - J_1}{R_1} \Rightarrow T_1 = \left[\frac{q_1 \cdot R_1 + J_1}{\sigma} \right]^{0.25}$$

Surface 2: Find the water wall heat input, given the water wall temperature:

$$q_2 = \frac{E_{b2} - J_2}{R_2} = \frac{\sigma \cdot T_2^4 - J_2}{R_2}$$

Surface 3: (Similar to surface 2) Find the water wall heat input, given the water wall temperature:

$$q_3 = \frac{E_{b3} - J_3}{R_3} = \frac{\sigma \cdot T_3^4 - J_3}{R_3}$$

MODULE 5

UNSTEADY STATE HEAT CONDUCTION

5.1 Introduction

To this point, we have considered conductive heat transfer problems in which the temperatures are independent of time. In many applications, however, the temperatures are varying with time, and we require the understanding of the complete time history of the temperature variation. For example, in metallurgy, the heat treating process can be controlled to directly affect the characteristics of the processed materials. Annealing (slow cool) can soften metals and improve ductility. On the other hand, quenching (rapid cool) can harden the strain boundary and increase strength. In order to characterize this transient behavior, the full unsteady equation is needed:

$$\frac{1}{a} \cdot \frac{\partial T}{\partial \tau} = \frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2} + \frac{q}{k} \quad (5.1)$$

where $\alpha = \frac{k}{\rho c}$ is the thermal diffusivity. Without any heat generation and considering spatial variation of temperature only in x-direction, the above equation reduces to:

$$\frac{1}{a} \cdot \frac{\partial T}{\partial \tau} = \frac{\partial^2 T}{\partial x^2} \quad (5.2)$$

For the solution of equation (5.2), we need two boundary conditions in x-direction and one initial condition. Boundary conditions, as the name implies, are frequently specified along the physical boundary of an object; they can, however, also be internal – e.g. a known temperature gradient at an internal line of symmetry.

5.2 Biot and Fourier numbers

In some transient problems, the internal temperature gradients in the body may be quite small and insignificant. Yet the temperature at a given location, or the average temperature of the object, may be changing quite rapidly with time. From eq. (5.1) we can note that such could be the case for large thermal diffusivity α .

A more meaningful approach is to consider the general problem of transient cooling of an object, such as the hollow cylinder shown in figure 5.1.

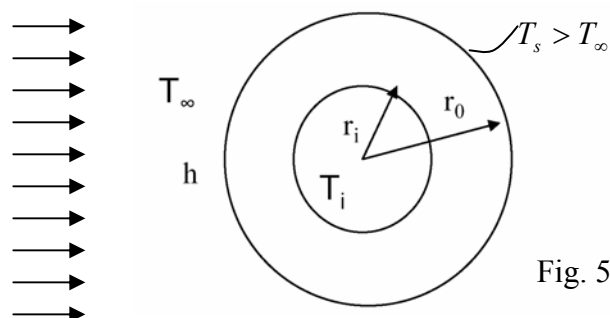


Fig. 5.1

For very large r_i , the heat transfer rate by conduction through the cylinder wall is approximately

$$q \approx -k(2\pi r_o l) \left(\frac{T_s - T_i}{r_o - r_i} \right) = k(2\pi r_o l) \left(\frac{T_i - T_s}{L} \right) \quad (5.3)$$

where l is the length of the cylinder and L is the material thickness. The rate of heat transfer away from the outer surface by convection is

$$q = \bar{h}(2\pi r_o l)(T_s - T_\infty) \quad (5.4)$$

where \bar{h} is the average heat transfer coefficient for convection from the entire surface. Equating (5.3) and (5.4) gives

$$\frac{T_i - T_s}{T_s - T_\infty} = \frac{\bar{h}L}{k} = \text{Biot number} \quad (5.5)$$

The Biot number is dimensionless, and it can be thought of as the ratio

$$\mathbf{Bi} = \frac{\text{resistance to internal heat flow}}{\text{resistance to external heat flow}}$$

Whenever the Biot number is small, the internal temperature gradients are also small and a transient problem can be treated by the “lumped thermal capacity” approach. The lumped capacity assumption implies that the object for analysis is considered to have a single mass-averaged temperature.

In the derivation shown above, the significant object dimension was the conduction path length, $L = r_o - r_i$. In general, a characteristic length scale may be obtained by dividing the volume of the solid by its surface area:

$$L = \frac{V}{A_s} \quad (5.6)$$

Using this method to determine the characteristic length scale, the corresponding Biot number may be evaluated for objects of any shape, for example a plate, a cylinder, or a sphere. As a thumb rule, if the Biot number turns out to be less than 0.1, lumped capacity assumption is applied.

In this context, a *dimensionless time*, known as the **Fourier number**, can be obtained by multiplying the dimensional time by the thermal diffusivity and dividing by the square of the characteristic length:

$$\text{dimensionless time} = \frac{\alpha t}{L^2} = \mathbf{Fo} \quad (5.7)$$

5.3 Lumped thermal capacity analysis

The simplest situation in an unsteady heat transfer process is to use the lumped capacity assumption, wherein we neglect the temperature distribution inside the solid and only deal with the heat transfer between the solid and the ambient fluids. In other words, we are assuming that the temperature inside the solid is constant and is equal to the surface temperature.

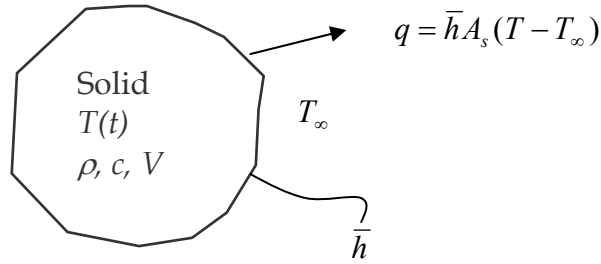


Fig. 5.2

The solid object shown in figure 5.2 is a metal piece which is being cooled in air after hot forming. Thermal energy is leaving the object from all elements of the surface, and this is shown for simplicity by a single arrow. The first law of thermodynamics applied to this problem is

$$\left(\begin{array}{c} \text{heat out of object} \\ \text{during time } dt \end{array} \right) = \left(\begin{array}{c} \text{decrease of internal thermal} \\ \text{energy of object during time } dt \end{array} \right)$$

Now, if Biot number is small and temperature of the object can be considered to be uniform, this equation can be written as

$$\bar{h}A_s[T(t) - T_\infty]dt = -\rho c V dT \quad (5.8)$$

or,

$$\frac{dT}{(T - T_\infty)} = -\frac{\bar{h}A_s}{\rho c V} dt \quad (5.9)$$

Integrating and applying the initial condition $T(0) = T_i$,

$$\ln \frac{T(t) - T_\infty}{T_i - T_\infty} = -\frac{\bar{h}A_s}{\rho c V} t \quad (5.10)$$

Taking the exponents of both sides and rearranging,

$$\frac{T(t) - T_\infty}{T_i - T_\infty} = e^{-bt} \quad (5.11)$$

where

$$b = \frac{\bar{h}A_s}{\rho c V} \quad (1/s) \quad (5.12)$$

Note: In eq. 5.12, b is a positive quantity having dimension $(\text{time})^{-1}$. The reciprocal of b is usually called *time constant*, which has the dimension of time.

Question: What is the significance of b ?

Answer: According to eq. 5.11, the temperature of a body approaches the ambient temperature T_∞ exponentially. In other words, the temperature changes rapidly in the beginning, and then slowly. A larger value of b indicates that the body will approach the surrounding temperature in a shorter time. You can visualize this if you note the variables in the numerator and denominator of the expression for b . As an exercise, plot T vs. t for various values of b and note the behaviour.

Rate of convection heat transfer at any given time t :

$$\dot{Q}(t) = hA_s [T(t) - T_\infty]$$

Total amount of heat transfer between the body and the surrounding from $t=0$ to t :

$$Q = mc [T(t) - T_i]$$

Maximum heat transfer (limit reached when body temperature equals that of the surrounding):

$$Q = mc [T_\infty - T_i]$$

5.4 Numerical methods in transient heat transfer: The Finite Volume Method

Consider, now, unsteady state diffusion in the context of heat transfer, in which the temperature, T , is the scalar. The corresponding partial differential equation is:

$$\rho c \frac{\partial T}{\partial t} = \frac{\partial}{\partial x} \left(k \frac{\partial T}{\partial x} \right) + S \quad (5.13)$$

The term on the left hand side of eq. (5.13) is the storage term, arising out of accumulation/depletion of heat in the domain under consideration. Note that eq. (5.13) is a partial differential equation as a result of an extra independent variable, time (t). The corresponding grid system is shown in fig. 5.3.

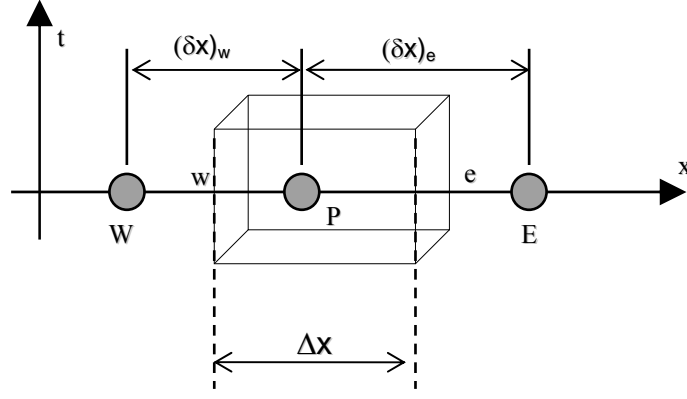


Fig. 5.3: Grid system of an unsteady one-dimensional computational domain

In order to obtain a discretized equation at the nodal point P of the control volume, integration of the governing eq. (5.13) is required to be performed with respect to time as well as space. Integration over the control volume and over a time interval gives

$$\int_t^{t+\Delta t} \left(\int_{CV} \left(\rho c \frac{\partial T}{\partial t} \right) dV \right) dt = \int_t^{t+\Delta t} \left(\int_{cv} \frac{\partial}{\partial x} \left(k \frac{\partial T}{\partial x} \right) dV \right) dt + \int_t^{t+\Delta t} \left(\int_{CV} S dV \right) dt \quad (5.14)$$

Rewritten,

$$\int_w^e \left(\int_t^{t+\Delta t} \rho c \frac{\partial T}{\partial t} dt \right) dV = \int_t^{t+\Delta t} \left(\left(kA \frac{\partial T}{\partial x} \right)_e - \left(kA \frac{\partial T}{\partial x} \right)_w \right) dt + \int_t^{t+\Delta t} (\bar{S} \Delta V) dt \quad (5.15)$$

If the temperature at a node is assumed to prevail over the whole control volume, applying the central differencing scheme, one obtains:

$$\rho c (T_P^{new} - T_P^{old}) \Delta V = \int_t^{t+\Delta t} \left(\left(k_e A \frac{T_E - T_P}{\delta x_e} \right) - \left(k_w A \frac{T_P - T_W}{\delta x_w} \right) \right) dt + \int_t^{t+\Delta t} (\bar{S} \Delta V) dt \quad (5.16)$$

Now, an assumption is made about the variation of T_P , T_E and T_W with time. By generalizing the approach by means of a weighting parameter f between 0 and 1:

$$\int_t^{t+\Delta t} \phi_P dt = \phi_P \Delta t = [f \phi_P^{new} - (1-f) \phi_P^{old}] \Delta t \quad (5.17)$$

Repeating the same operation for points E and W,

$$\begin{aligned} \rho c \left(\frac{T_P^{new} - T_P^{old}}{\Delta t} \right) \Delta x = & f \left[\left(k_e \frac{T_E^{new} - T_P^{new}}{\delta x_e} \right) - \left(k_w \frac{T_P^{new} - T_W^{new}}{\delta x_w} \right) \right] \\ & + (1-f) \left[\left(k_e \frac{T_E^{old} - T_P^{old}}{\delta x_e} \right) - \left(k_w \frac{T_P^{old} - T_W^{old}}{\delta x_w} \right) \right] + \bar{S} \Delta x \end{aligned} \quad (5.18)$$

Upon re-arranging, dropping the superscript “new”, and casting the equation into the standard form:

$$a_p T_p = a_w [f T_w + (1-f) T_w^{old}] + a_e [f T_e + (1-f) T_e^{old}] + [a_p^0 - (1-f)a_w - (1-f)a_e] T_p^{old} + b \quad (5.19)$$

where

$$a_p = \theta(a_w + a_e) + a_p^0; \quad a_p^0 = \rho c \frac{\Delta x}{\Delta t}; \quad a_w = \frac{k_w}{\delta x_w}; \quad a_e = \frac{k_e}{\delta x_e}; \quad b = \bar{S} \Delta x \quad (5.20)$$

The time integration scheme would depend on the choice of the parameter f . When $f=0$, the resulting scheme is “explicit”; when $0 < f \leq 1$, the resulting scheme is “implicit”; when $f=1$, the resulting scheme is “fully implicit”, when $f = 1/2$, the resulting scheme is “Crank-Nicolson” (Crank and Nicolson, 1947). The variation of T within the time interval Δt for the different schemes is shown in fig. 5.4.

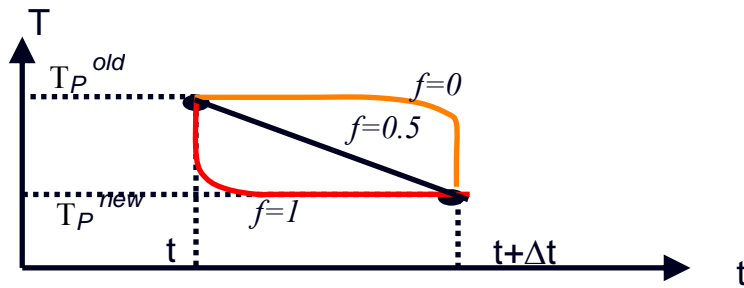


Fig. 5.4: Variation of T within the time interval Δt for different schemes

Explicit scheme

Linearizing the source term as $b = S_u + S_p T_p^{old}$ and setting $f = 0$ in eq. (5.19), the explicit discretisation becomes:

$$a_p T_p = a_w T_w^{old} + a_e T_e^{old} + [a_p^0 - (a_w + a_e)] T_p^{old} + S_u \quad (5.21)$$

where

$$a_p = a_p^0; \quad a_p^0 = \rho c \frac{\Delta x}{\Delta t}; \quad a_w = \frac{k_w}{\delta x_w}; \quad a_e = \frac{k_e}{\delta x_e} \quad (5.22)$$

The above scheme is based on backward differencing and its Taylor series truncation error accuracy is first-order with respect to time. For stability, all coefficients must be positive in the discretized equation. Hence,

$$a_p^0 - (a_w + a_e - S_p) > 0$$

$$\text{or, } \rho c \frac{\Delta x}{\Delta t} - \left(\frac{k_w}{\delta x_w} + \frac{k_e}{\delta x_e} \right) > 0$$

$$\text{or, } \rho c \frac{\Delta x}{\Delta t} > \frac{2k}{\Delta x}$$

$$\text{or, } \Delta t < \rho c \frac{(\Delta x)^2}{2k} \quad (5.23)$$

The above limitation on time step suggests that the explicit scheme becomes very expensive to improve spatial accuracy. Hence, this method is generally not recommended for general transient problems. Nevertheless, provided that the time step size is chosen with care, the explicit scheme described above is efficient for simple conduction calculations.

Crank-Nicolson scheme

Setting $f=0.5$ in eq. (5.19), the Crank-Nicolson discretisation becomes:

$$a_p T_p = a_E \left(\frac{T_E + T_E^{old}}{2} \right) + a_W \left(\frac{T_W + T_W^{old}}{2} \right) + \left[a_p^0 - \frac{a_E}{2} - \frac{a_W}{2} \right] T_p^0 + b \quad (5.24)$$

where

$$a_p = \frac{1}{2}(a_E + a_W) + a_p^0 - \frac{1}{2}S_p; \quad a_p^0 = \rho c \frac{\Delta x}{\Delta t}; \quad a_W = \frac{k_w}{\delta x_w}; \quad a_E = \frac{k_e}{\delta x_e}; \quad b = S_u + \frac{1}{2}S_p T_p^{old} \quad (5.25)$$

The above method is implicit and simultaneous equations for all node points need to be solved at each time step. For stability, all coefficient must be positive in the discretized equation, requiring

$$a_p^0 > \frac{a_E + a_W}{2}$$

$$\text{or, } \Delta t < \rho c \frac{(\Delta x)^2}{k} \quad (5.26)$$

The Crank-Nicolson scheme only slightly less restrictive than the explicit method. It is based on central differencing and hence it is second-order accurate in time.

The fully implicit scheme

Setting $f=1$ in eq. (5.19), the fully implicit discretisation becomes:

$$a_p T_p = a_E T_E + a_W T_W + a_p^0 T_p^{old} \quad (5.27)$$

$$\text{where } a_p = a_p^0 + a_E + a_W - S_p; \quad a_p^0 = \rho c \frac{\Delta x}{\Delta t}; \quad a_W = \frac{k_w}{\delta x_w}; \quad a_E = \frac{k_e}{\delta x_e} \quad (5.28)$$

A system of algebraic equations must be solved at each time level. The accuracy of the scheme is first-order in time. The time marching procedure starts with a given initial field of the scalar ϕ^0 . The system is solved after selecting time step Δt . For the implicit scheme, all coefficients are positive, which makes it unconditionally stable for any size of time step. Hence, the implicit method is recommended for general purpose transient calculations because of its robustness and unconditional stability.

MODULE 8

BOILING AND CONDENSATION

8.1 Boiling: General considerations

- Boiling is associated with transformation of liquid to vapor at a solid/liquid interface due to convection heat transfer from the solid.
- Agitation of fluid by vapor bubbles provides for large convection coefficients and hence large heat fluxes at low-to-moderate surface-to-fluid temperature differences

- Special form of Newton's law of cooling:

$$q_s'' = h(T_s - T_{sat}) = h\Delta T_e$$

where T_{sat} is the saturation temperature of the liquid, and $\Delta T_e = T_s - T_{sat}$ is the excess temperature.

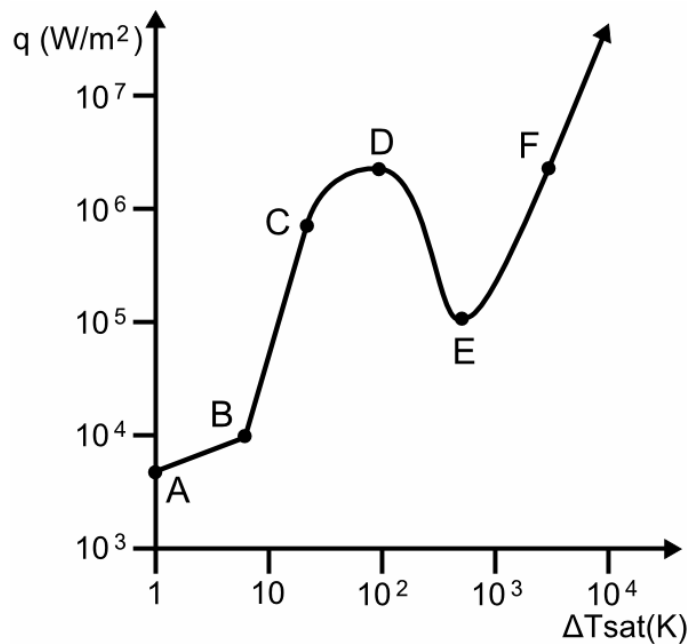
8.2 Special cases

- Pool Boiling: Liquid motion is due to natural convection and bubble-induced mixing.
- Forced Convection Boiling: Fluid motion is induced by external means, as well as by bubble-induced mixing.
- Saturated Boiling: Liquid temperature is slightly larger than saturation temperature
- Subcooled Boiling: Liquid temperature is less than saturation temperature

8.3 The boiling curve

The boiling curve reveals range of conditions associated with saturated pool boiling on a q_s'' vs. ΔT_e plot.

Water at Atmospheric Pressure



Pool boiling regimes:

A-B: Pure convection with liquid rising to surface for evaporation
 B-C: Nucleate boiling with bubbles condensing in liquid
 C-D: Nucleate boiling with bubbles rising to surface
 D: Peak temperature
 D-E: Partial nucleate boiling and unstable film boiling
 E: Film boiling is stabilized
 E-F: Radiation becomes a dominant mechanism for heat transfer

Free Convection Boiling ($\Delta T_e < 5^\circ\text{C}$)

- Little vapor formation.
- Liquid motion is due principally to single-phase natural convection.

Onset of Nucleate Boiling – ONB ($\Delta T_e \approx 5^\circ\text{C}$)

Nucleate boiling ($5^\circ\text{C} < \Delta T_e < 30^\circ\text{C}$)

- Isolated Vapor Bubbles ($5^\circ\text{C} < \Delta T_e < 10^\circ\text{C}$)

Liquid motion is strongly influenced by nucleation of bubbles at the surface.

h and q_s'' rise sharply with increasing ΔT_e

Heat transfer is principally due to contact of liquid with the surface (single-phase convection) and not to vaporization

- Jets and Columns ($10^\circ\text{C} < \Delta T_e < 30^\circ\text{C}$)

Increasing number of nucleation sites causes bubble interactions and coalescence into jets and slugs.

Liquid/surface contact is impaired.

q_s'' continues to increase with ΔT_e while h begins to decrease

Critical Heat Flux - CHF, ($\Delta T_e \approx 30^\circ\text{C}$)

- Maximum attainable heat flux in nucleate boiling.

$$q_{\max}'' \approx 1 \text{ MW/m}^2 \text{ for water at atmospheric pressure.}$$

Potential Burnout for Power-Controlled Heating

- An increase in q_s'' beyond q_{\max}'' causes the surface to be blanketed by vapor and its temperature to spontaneously achieve a value that can exceed its melting point
- If the surface survives the temperature shock, conditions are characterized by film boiling

Film Boiling

- Heat transfer is by conduction and radiation across the vapor blanket
- A reduction in q_s'' follows the cooling the cooling curve continuously to the Leidenfrost point corresponding to the minimum heat flux q_{\min}'' for film boiling.
- A reduction in q_s'' below q_{\min}'' causes an abrupt reduction in surface temperature to the nucleate boiling regime

Transition Boiling for Temperature-Controlled Heating

- Characterised by continuous decay of q_s'' (from q_{\max}'' to q_{\min}'') with increasing ΔT_e

- Surface conditions oscillate between nucleate and film boiling, but portion of surface experiencing film boiling increases with ΔT_e
- Also termed unstable or partial film boiling.

8.4 Pool boiling correlations

Nucleate Boiling

- Rohsenow Correlation, clean surfaces only, $\pm 100\%$ errors

$$q_s'' = \mu_l h_{fg} \left[\frac{g(\rho_l - \rho_v)}{\sigma} \right]^{1/2} \left(\frac{c_{p,l} \Delta T_e}{C_{s,f} h_{fg} \text{Pr}_l^n} \right)^3$$

$C_{s,f}, n \rightarrow$ Surface/Fluid Combination

Critical heat flux:

$$q_{\max}'' = 0.149 h_{fg} \rho_v \left[\frac{\sigma g(\rho_l - \rho_v)}{\rho_v^2} \right]^{1/4}$$

Film Boiling

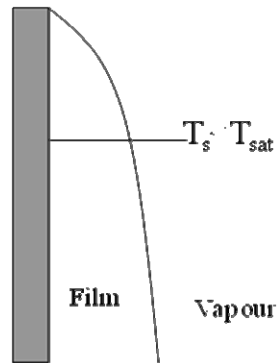
$$\overline{Nu}_D = \frac{\bar{h}_{conv} D}{k_v} = C \left[\frac{g(\rho_l - \rho_v) h'_{fg} D^3}{\nu_v k_v (T_s - T_{sat})} \right]^{1/4}$$

Geometry	C
Cylinder(Hor.)	0.62
Sphere	0.67

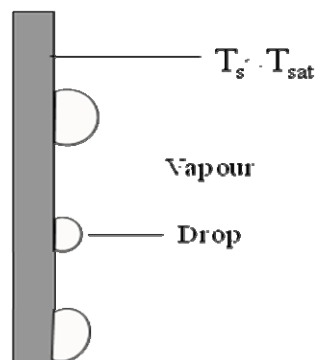
8.5 Condensation: General considerations

- Condensation occurs when the temperature of a vapour is reduced below its saturation temperature
- Condensation heat transfer

Film condensation

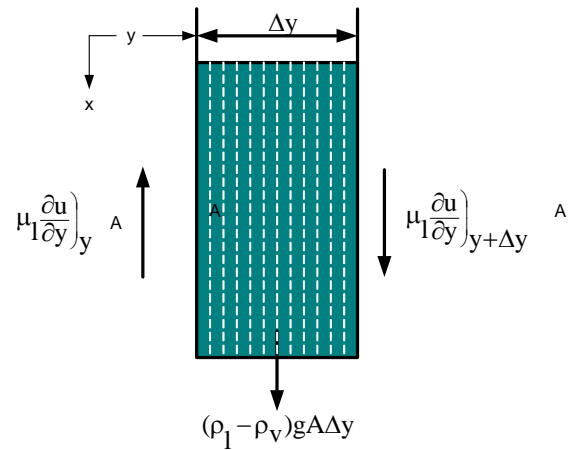
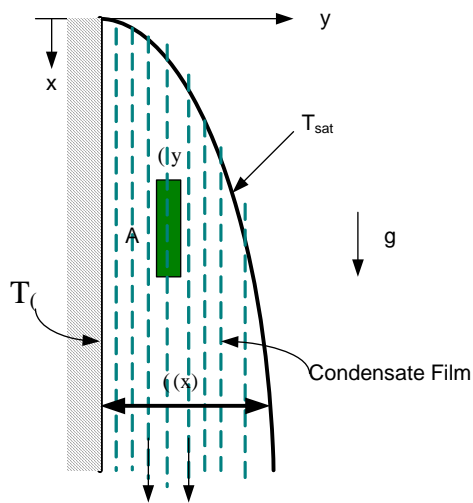


Dropwise condensation



- Heat transfer rates in dropwise condensation *may be as much as 10 times higher* than in film condensation

8.6 Laminar film condensation on a vertical wall



$$\delta(x) = \left[\frac{4xk_l(T_{sat} - T_w)\nu_l}{h_{fg}g(\rho_l - \rho_v)} \right]^{1/4}$$

$$h(x) = \left[\frac{h_{fg}g(\rho_l - \rho_v)k_l^3}{4x(T_{sat} - T_w)\nu_l} \right]^{1/4}$$

Average coeff. $\bar{h}_L = 0.943 \left[\frac{h_{fg}g(\rho_l - \rho_v)k_l^3}{L(T_{sat} - T_w)\nu_l} \right]^{1/4}$

where L is the plate length.

Total heat transfer rate : $q = \bar{h}_L A (T_{sat} - T_w)$

Condensation rate : $\dot{m} = \frac{q}{h_{fg}} = \frac{\bar{h}_L A (T_{sat} - T_w)}{h_{fg}}$

MODULE 6

CONVECTION

6.1 Objectives of convection analysis:

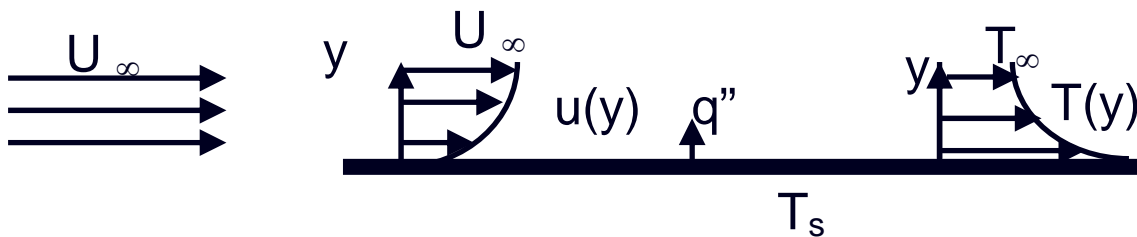
Main purpose of convective heat transfer analysis is to determine:

- flow field
- temperature field in fluid
- heat transfer coefficient, h

How do we determine h ?

Consider the process of convective cooling, as we pass a cool fluid past a heated wall. This process is described by Newton's law of Cooling:

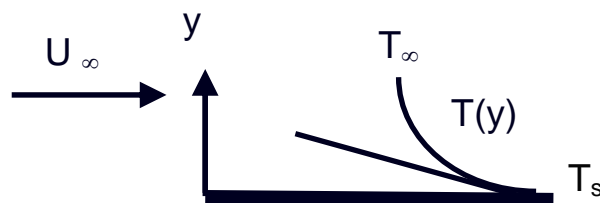
$$q = h \cdot A \cdot (T_s - T_\infty)$$



Near any wall a fluid is subject to the no slip condition; that is, there is a stagnant sub layer. Since there is no fluid motion in this layer, heat transfer is by conduction in this region. Above the sub layer is a region where viscous forces retard fluid motion; in this region some convection may occur, but conduction may well predominate. A careful analysis of this region allows us to use our conductive analysis in analyzing heat transfer. This is the basis of our convective theory.

At the wall, the convective heat transfer rate can be expressed as the heat flux.

$$q''_{conv} = -k_f \left. \frac{\partial T}{\partial y} \right|_{y=0} = h(T_s - T_\infty)$$

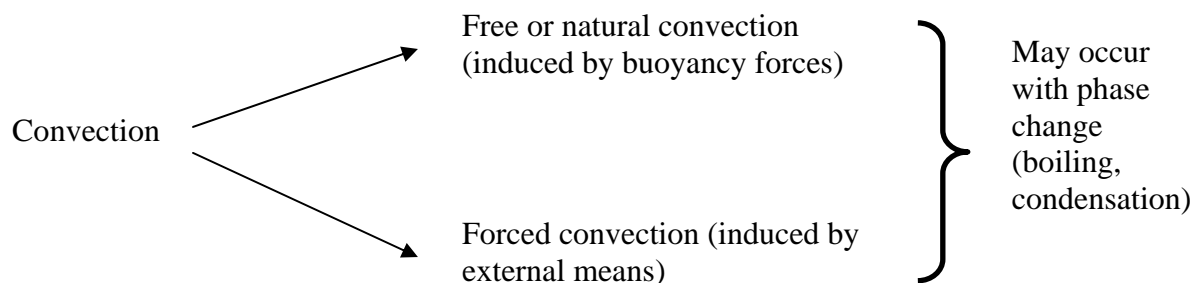


Hence, $h = \frac{-k_f \left. \frac{\partial T}{\partial y} \right|_{y=0}}{(T_s - T_\infty)}$

But $\left. \frac{\partial T}{\partial y} \right|_{y=0}$ depends on the whole fluid motion, and both fluid flow and heat transfer equations are needed

The expression shows that in order to determine h , we must first determine the temperature distribution in the thin fluid layer that coats the wall.

2.2 Classes of Convective Flows



- extremely diverse
- several parameters involved (fluid properties, geometry, nature of flow, phases etc)
- systematic approach required
- classify flows into certain types, based on certain parameters
- identify parameters governing the flow, and group them into **meaningful non-dimensional numbers**
- need to understand the physics behind each phenomenon

Common classifications:

A. Based on geometry:

External flow / Internal flow

B. Based on driving mechanism

Natural convection / forced convection / mixed convection

C. Based on number of phases

Single phase / multiple phase

D. Based on nature of flow

Laminar / turbulent

Table 6.1. Typical values of h ($\text{W/m}^2\text{K}$)

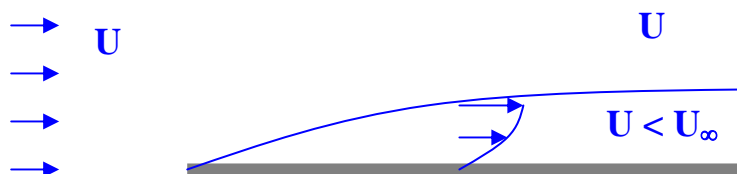
Free convection	gases: 2 - 25 liquid: 50 - 100
Forced convection	gases: 25 - 250 liquid: 50 - 20,000
Boiling/Condensation	2500 - 100,000

2.3 How to solve a convection problem ?

- Solve governing equations along with boundary conditions
- Governing equations include
 1. conservation of mass
 2. conservation of momentum
 3. conservation of energy
- In Conduction problems, only (3) is needed to be solved. Hence, only *few parameters* are involved
- In Convection, all the governing equations need to be solved.
 \Rightarrow large number of parameters can be involved

2.4 FORCED CONVECTION: external flow (over flat plate)

An internal flow is surrounded by solid boundaries that can restrict the development of its boundary layer, for example, a pipe flow. An external flow, on the other hand, are flows over bodies immersed in an unbounded fluid so that the flow boundary layer can grow freely in one direction. Examples include the flows over airfoils, ship hulls, turbine blades, etc



- Fluid particle adjacent to the solid surface is at rest
- These particles act to retard the motion of adjoining layers
- \Rightarrow boundary layer effect

Inside the boundary layer, we can apply the following conservation principles:

Momentum balance: inertia forces, pressure gradient, viscous forces, body forces

Energy balance: convective flux, diffusive flux, heat generation, energy storage

2.5 Forced Convection Correlations

Since the heat transfer coefficient is a direct function of the temperature gradient next to the wall, the physical variables on which it depends can be expressed as follows:

$h = f(\text{fluid properties, velocity field, geometry, temperature etc.})$

As the function is dependent on several parameters, the heat transfer coefficient is usually expressed in terms of **correlations involving pertinent non-dimensional numbers**.

Forced convection: **Non-dimensional groupings**

- **Nusselt No.** $Nu = hx / k = (\text{convection heat transfer strength}) / (\text{conduction heat transfer strength})$
- **Prandtl No.** $Pr = \nu / \alpha = (\text{momentum diffusivity}) / (\text{thermal diffusivity})$
- **Reynolds No.** $Re = U x / \nu = (\text{inertia force}) / (\text{viscous force})$

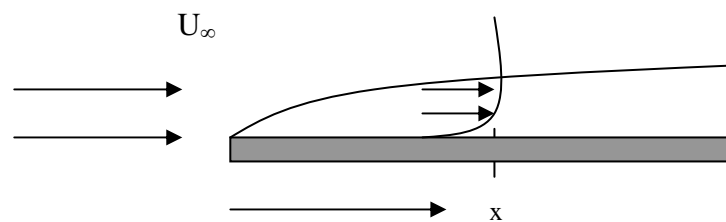
Viscous force provides the dampening effect for disturbances in the fluid. If dampening is strong enough \Rightarrow **laminar flow**

Otherwise, instability \Rightarrow **turbulent flow** \Rightarrow **critical Reynolds number**

For forced convection, the heat transfer correlation can be expressed as

$$Nu = f(Re, Pr)$$

The convective correlation for laminar flow across a flat plate heated to a constant wall temperature is:



$$Nu_x = 0.323 \cdot Re_x^{1/2} \cdot Pr^{1/3}$$

where

$$Nu_x \equiv h \cdot x / k$$

$$Re_x \equiv (U_\infty \cdot x \cdot \rho) / \mu$$

$$Pr \equiv c_p \cdot \mu / k$$

Physical Interpretation of Convective Correlation

The Reynolds number is a familiar term to all of us, but we may benefit by considering what the ratio tells us. Recall that the thickness of the dynamic boundary layer, δ , is proportional to the distance along the plate, x .

$$\text{Re}_x \equiv (U_\infty \cdot x \cdot \rho) / \mu \propto (U_\infty \cdot \delta \cdot \rho) / \mu = (\rho \cdot U_\infty^2) / (\mu \cdot U_\infty / \delta)$$

The numerator is a mass flow per unit area times a velocity; i.e. a momentum flow per unit area. The denominator is a viscous stress, i.e. a viscous force per unit area. The ratio represents the ratio of momentum to viscous forces. If viscous forces dominate, the flow will be laminar; if momentum dominates, the flow will be turbulent.

Physical Meaning of Prandtl Number

The Prandtl number was introduced earlier.

If we multiply and divide the equation by the fluid density, ρ , we obtain:

$$\text{Pr} \equiv (\mu/\rho)/(k/\rho \cdot c_p) = \nu/\alpha$$

The Prandtl number may be seen to be a ratio reflecting the ratio of the rate that viscous forces penetrate the material to the rate that thermal energy penetrates the material. As a consequence the Prandtl number is proportional to the rate of growth of the two boundary layers:

$$\delta/\delta_t = \text{Pr}^{1/3}$$

Physical Meaning of Nusselt Number

The Nusselt number may be physically described as well.

$$\text{Nu}_x \equiv h \cdot x / k$$

If we recall that the thickness of the boundary layer at any point along the surface, δ , is also a function of x then

$$\text{Nu}_x \propto h \cdot \delta / k \propto (\delta/k \cdot A) / (1/h \cdot A)$$

We see that the Nusselt may be viewed as the ratio of the conduction resistance of a material to the convection resistance of the same material.

Students, recalling the Biot number, may wish to compare the two so that they may distinguish the two.

$$\text{Nu}_x \equiv h \cdot x / k_{\text{fluid}}$$

$$\text{Bi}_x \equiv h \cdot x / k_{\text{solid}}$$

The denominator of the Nusselt number involves the thermal conductivity of the **fluid** at the solid-fluid convective interface; The denominator of the Biot number involves the thermal conductivity of the **solid** at the solid-fluid convective interface.

Local Nature of Convective Correlation

Consider again the correlation that we have developed for laminar flow over a flat plate at constant wall temperature

$$\text{Nu}_x = 0.323 \cdot \text{Re}_x^{1/2} \cdot \text{Pr}^{1/3}$$

To put this back into dimensional form, we replace the Nusselt number by its equivalent, hx/k and take the x/k to the other side:

$$h = 0.323 \cdot (k/x) \cdot \text{Re}_x^{1/2} \cdot \text{Pr}^{1/3}$$

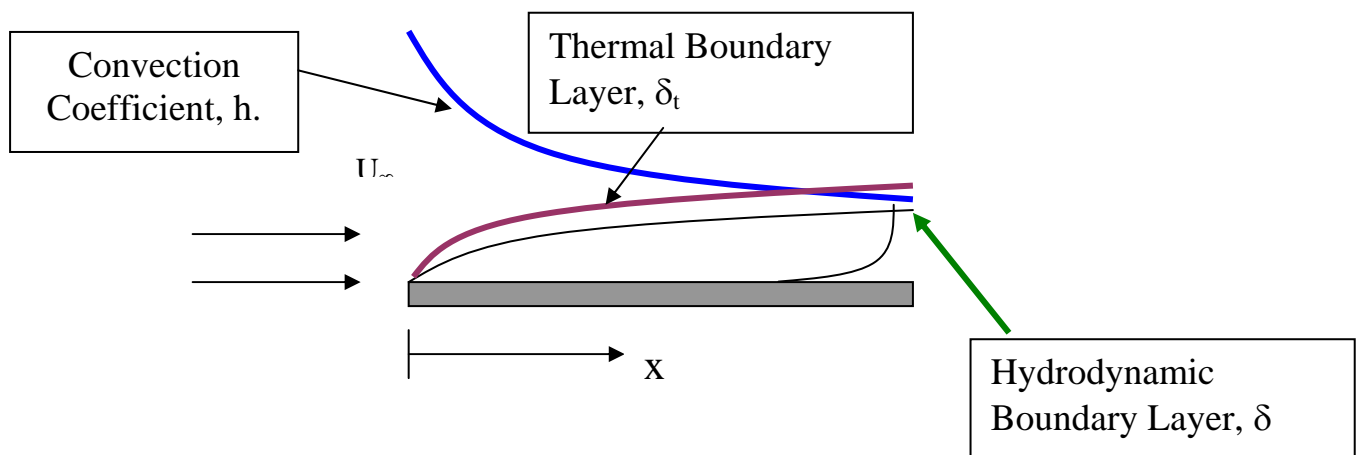
Now expand the Reynolds number

$$h = 0.323 \cdot (k/x) \cdot [(U_\infty \cdot x \cdot \rho) / \mu]^{1/2} \cdot \text{Pr}^{1/3}$$

We proceed to combine the x terms:

$$h = 0.323 \cdot k \cdot [(U_\infty \cdot \rho) / (x \cdot \mu)]^{1/2} \cdot \text{Pr}^{1/3}$$

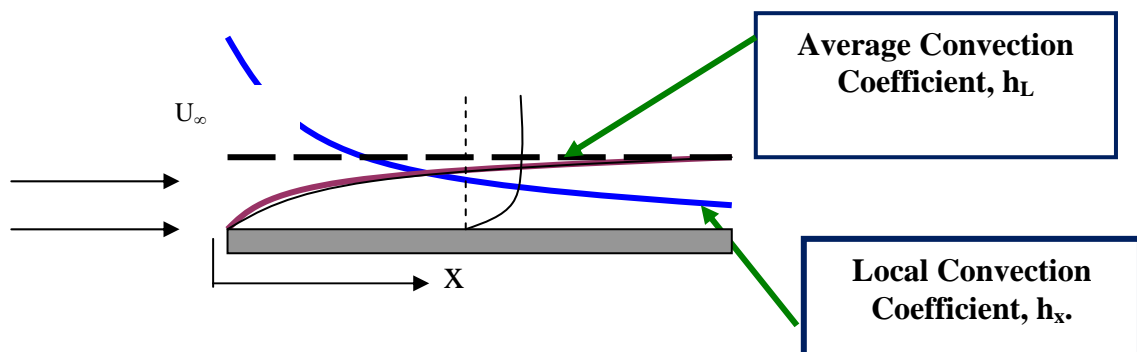
And see that the convective coefficient decreases with $x^{1/2}$.



We see that as the boundary layer thickens, the convection coefficient decreases. Some designers will introduce a series of “trip wires”, i.e. devices to disrupt the boundary layer, so that the buildup of the insulating layer must begin anew. This will result in regular “thinning” of the boundary layer so that the convection coefficient will remain high.

Averaged Correlations

If one were interested in the total heat loss from a surface, rather than the temperature at a point, then they may well want to know something about average convective coefficients.



The desire is to find a correlation that provides an overall heat transfer rate:

$$Q = h_L \cdot A \cdot [T_{wall} - T_\infty] = \int h_x \cdot [T_{wall} - T_\infty] \cdot dA = \int_0^L h_x \cdot [T_{wall} - T_\infty] \cdot dx$$

where h_x and h_L , refer to local and average convective coefficients, respectively.

Compare the second and fourth equations where the area is assumed to be equal to $A = (1 \cdot L)$:

$$h_L \cdot L \cdot [T_{wall} - T_\infty] = \int_0^L h_x \cdot [T_{wall} - T_\infty] \cdot dx$$

Since the temperature difference is constant, it may be taken outside of the integral and cancelled:

$$h_L \cdot L = \int_0^L h_x \cdot dx$$

This is a general definition of an integrated average.

Proceed to substitute the correlation for the local coefficient.

$$h_L \cdot L = \int_0^L 0.323 \cdot \frac{k}{x} \cdot \left[\frac{U_\infty \cdot x \cdot \rho}{\mu} \right]^{0.5} \cdot \text{Pr}^{1/3} \cdot dx$$

Take the constant terms from outside the integral, and divide both sides by k .

$$h_L \cdot L / k = 0.323 \cdot \left[\frac{U_\infty \cdot \rho}{\mu} \right]^{0.5} \cdot \text{Pr}^{1/3} \cdot \int_0^L \left[\frac{1}{x} \right]^{0.5} \cdot dx$$

Integrate the right side.

$$h_L \cdot L / k = 0.323 \cdot \left[\frac{U_\infty \cdot \rho}{\mu} \right]^{0.5} \cdot \text{Pr}^{1/3} \cdot \left. \frac{x^{0.5}}{0.5} \right|_0^L$$

The left side is defined as the average Nusselt number, Nu_L . Algebraically rearrange the right side.

$$\text{Nu}_L = \frac{0.323}{0.5} \cdot \left[\frac{U_\infty \cdot \rho}{\mu} \right]^{0.5} \cdot \text{Pr}^{1/3} \cdot L^{0.5} = 0.646 \cdot \left[\frac{U_\infty \cdot L \cdot \rho}{\mu} \right]^{0.5} \cdot \text{Pr}^{1/3}$$

The term in the brackets may be recognized as the Reynolds number, evaluated at the end of the convective section. Finally,

$$Nu_L = 0.646 \cdot Re_L^{0.5} \cdot Pr^{1/3}$$

This is our average correlation for laminar flow over a flat plate with constant wall temperature.

Reynolds Analogy

In the development of the boundary layer theory, one may notice the strong relationship between the dynamic boundary layer and the thermal boundary layer. Reynold's noted the strong correlation and found that fluid friction and convection coefficient could be related. This is known as the Reynolds Analogy.

Conclusion from Reynold's analogy: Knowing the frictional drag, we know the Nusselt Number. If the drag coefficient is increased, say through increased wall roughness, then the convective coefficient will also increase.

Turbulent Flow

We could develop a turbulent heat transfer correlation in a manner similar to the von Karman analysis. It is probably easier, having developed the Reynolds analogy, to follow that course. The local fluid friction factor, C_f , associated with turbulent flow over a flat plate is given as:

$$C_f = 0.0592/Re_x^{0.2}$$

Substitute into the Reynolds analogy:

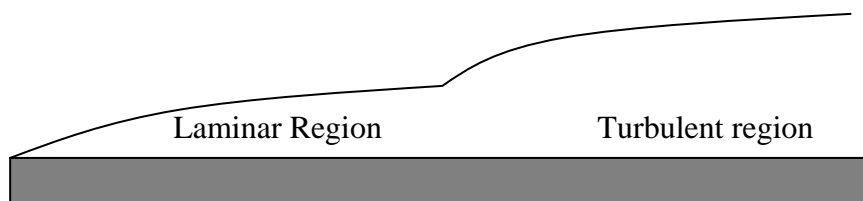
$$(0.0592/Re_x^{0.2})/2 = Nu_x/Re_x Pr^{1/3}$$

Rearrange to find

$$Nu_x = 0.0296 \cdot Re_x^{0.8} \cdot Pr^{1/3}$$

**Local Correlation
Turbulent Flow Flat Plate.**

In order to develop an average correlation, one would evaluate an integral along the plate similar to that used in a laminar flow:



$$h_L \cdot L = \int_0^L h_x dx = \int_0^{L_{crit}} h_{x,laminar} \cdot dx + \int_{L_{crit}}^L h_{x,turbulent} \cdot dx$$

Note: The critical Reynolds number for flow over a flat plate is $5 \cdot 10^5$; the critical Reynolds number for flow through a round tube is 2000.

The result of the above integration is:

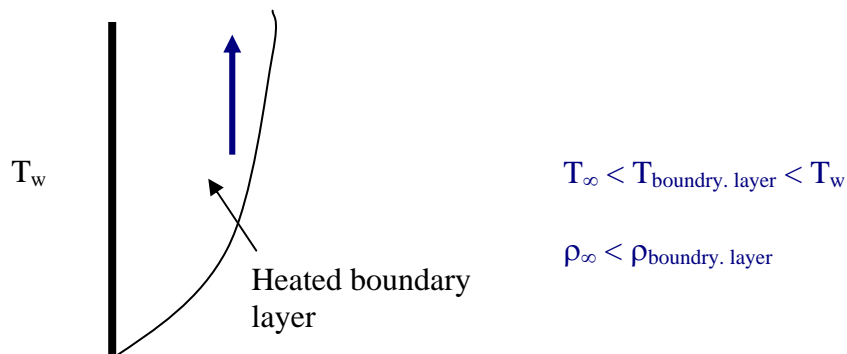
$$\text{Nu}_x = 0.037 \cdot (\text{Re}_x^{0.8} - 871) \cdot \text{Pr}^{1/3}$$

Note: Fluid properties should be evaluated at the average temperature in the boundary layer, i.e. at an average between the wall and free stream temperature.

$$T_{\text{prop}} = 0.5 \cdot (T_{\text{wall}} + T_{\infty})$$

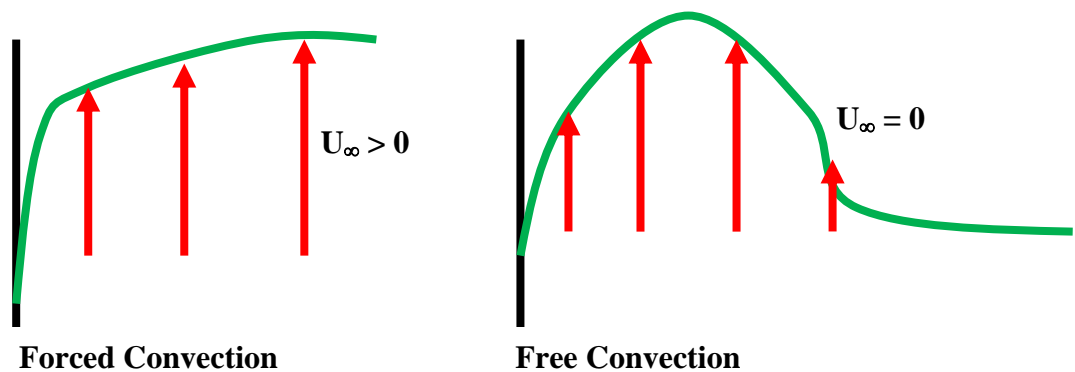
2.6 Free convection

Free convection is sometimes defined as a convective process in which fluid motion is caused by buoyancy effects.



Velocity Profiles

Compare the velocity profiles for forced and natural convection shown below:



Coefficient of Volumetric Expansion

The thermodynamic property which describes the change in density leading to buoyancy in the Coefficient of Volumetric Expansion, β .

$$\beta \equiv -\frac{1}{\rho} \cdot \frac{\partial \rho}{\partial T} \bigg|_{P=\text{Const.}}$$

Evaluation of β

- Liquids and Solids: β is a thermodynamic property and should be found from Property Tables. Values of β are found for a number of engineering fluids in Tables given in Handbooks and Text Books.
- Ideal Gases: We may develop a general expression for β for an ideal gas from the ideal gas law:

$$P = \rho \cdot R \cdot T$$

Then,

$$\rho = P/R \cdot T$$

Differentiating while holding P constant:

$$\left. \frac{d\rho}{dT} \right|_{P=Const.} = - \frac{P}{R \cdot T^2} = - \frac{\rho \cdot R \cdot T}{R \cdot T^2} = - \frac{\rho}{T}$$

Substitute into the definition of β

$$\beta = \frac{1}{T_{abs}}$$

Ideal Gas

Grashof Number

Because U_∞ is always zero, the Reynolds number, $[\rho \cdot U_\infty \cdot D]/\mu$, is also zero and is no longer suitable to describe the flow in the system. Instead, we introduce a new parameter for natural convection, the Grashof Number. Here we will be most concerned with flow across a vertical surface, so that we use the vertical distance, z or L , as the characteristic length.

$$Gr \equiv \frac{g \cdot \beta \cdot \Delta T \cdot L^3}{\nu^2}$$

Just as we have looked at the Reynolds number for a physical meaning, we may consider the Grashof number:

$$Gr \equiv \frac{\rho^2 \cdot g \cdot \beta \cdot \Delta T \cdot L^3}{\mu^2} = \frac{\left(\frac{\rho \cdot g \cdot \beta \cdot \Delta T \cdot L^3}{L^2} \right) \cdot (\rho \cdot U_{max}^2)}{\mu^2 \cdot \frac{U_{max}^2}{L^2}} = \frac{\left(\frac{Buoyant Force}{Area} \right) \cdot \left(\frac{Momentum}{Area} \right)}{\left(\frac{Viscous Force}{Area} \right)^2}$$

Free Convection Heat Transfer Correlations

The standard form for free, or natural, convection correlations will appear much like those for forced convection except that (1) the Reynolds number is replaced with a Grashof number and (2) the exponent on Prandtl number is not generally 1/3 (The von Karman boundary layer analysis from which we developed the 1/3 exponent was for forced convection flows):

$$Nu_x = C \cdot Gr_x^m \cdot Pr^n$$

Local Correlation

$$Nu_L = C \cdot Gr_L^m \cdot Pr^n$$

Average Correlation

Quite often experimentalists find that the exponent on the Grashof and Prandtl numbers are equal so that the general correlations may be written in the form:

$$Nu_x = C \cdot [Gr_x \cdot Pr]^m$$

Local Correlation

$$Nu_L = C \cdot [Gr_L \cdot Pr]^m$$

Average Correlation

This leads to the introduction of the new, dimensionless parameter, the Rayleigh number, Ra:

$$Ra_x = Gr_x \cdot Pr$$

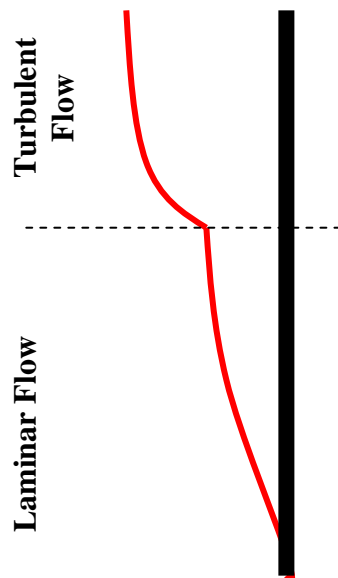
$$Ra_L = Gr_L \cdot Pr$$

So that the general correlation for free convection becomes:

$Nu_x = C \cdot Ra_x^m$	Local Correlation
$Nu_L = C \cdot Ra_L^m$	Average Correlation

Laminar to Turbulent Transition

Just as for forced convection, a boundary layer will form for free convection. The boundary layer, which acts as a thermal resistance, will be relatively thin toward the leading edge of the surface resulting in a relatively high convection coefficient. At a Rayleigh number of about 10^9 the flow over a flat plate will become transitional and finally become turbulent. The increased turbulence inside the boundary layer will enhance heat transfer leading to relative high convection coefficients because of better mixing.



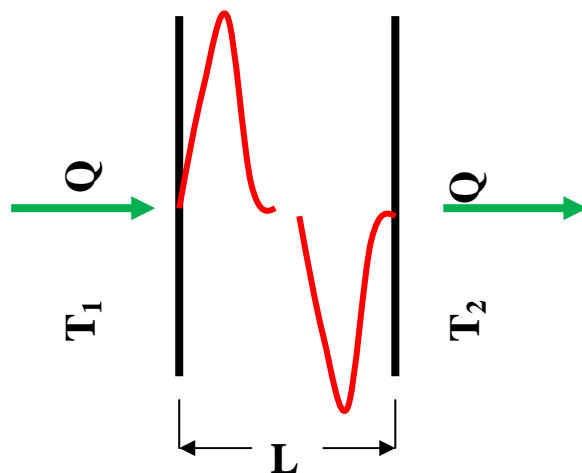
$Ra < 10^9$ Laminar flow. [Vertical Flat Plate]

$Ra > 10^9$ Turbulent flow. [Vertical Flat Plate]

Generally the characteristic length used in the correlation relates to the distance over which the boundary layer is allowed to grow. In the case of a vertical flat plate this will be x or L , in the case of a vertical cylinder this will also be x or L ; in the case of a horizontal cylinder, the length will be d .

Critical Rayleigh Number

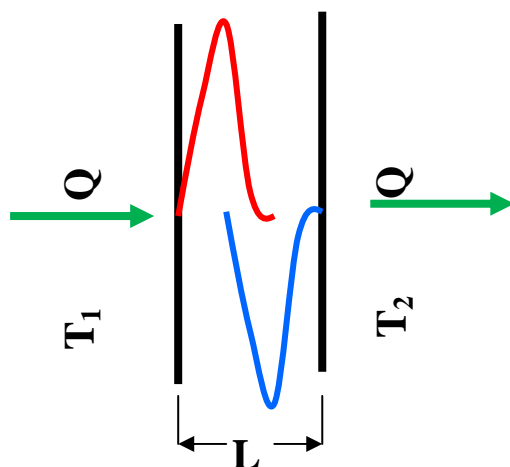
Consider the flow between two surfaces, each at different temperatures. Under developed flow conditions, the interstitial fluid will reach a temperature between the temperatures of the two surfaces and will develop free convection flow patterns. The fluid will be heated by one surface, resulting in an upward buoyant flow, and will be cooled by the other, resulting in a downward flow.



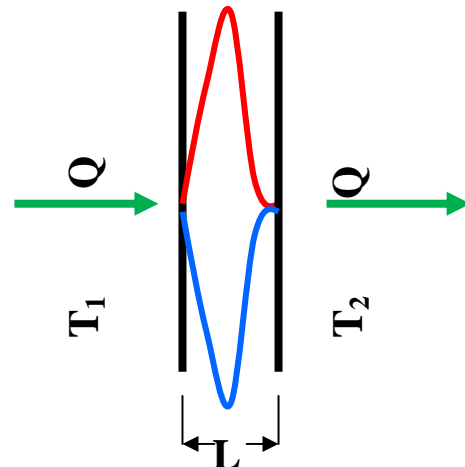
Note that for enclosures it is customary to develop correlations which describe the overall (both heated and cooled surfaces) within a single correlation.

**Free Convection Inside an Enclosure
(boundary layer limit)**

If the surfaces are placed closer together, the flow patterns will begin to interfere:



**Free Convection Inside an
Enclosure With Partial Flow
Interference**



**Free Convection Inside an
Enclosure With Complete Flow
Interference (Channel flow
limit)**

In the case of complete flow interference, the upward and downward forces will cancel, canceling circulation forces. This case would be treated as a pure convection problem since no bulk transport occurs.

The transition in enclosures from convection heat transfer to conduction heat transfer occurs at what is termed the “**Critical Rayleigh Number**”. Note that this terminology is in clear contrast to forced convection where the critical Reynolds number refers to the transition from laminar to turbulent flow.

$$Ra_{crit} = 1000 \quad (\text{Enclosures With Horizontal Heat Flow})$$

$$Ra_{crit} = 1728 \quad (\text{Enclosures With Vertical Heat Flow})$$

The existence of a Critical Rayleigh number suggests that there are now three flow regimes: (1) No flow, (2) Laminar Flow and (3) Turbulent Flow. In all enclosure problems the Rayleigh number will be calculated to determine the proper flow regime before a correlation is chosen.