ATME College of Engineering

SIGNALS AND DIGITAL SIGNAL PROCESSING
BEE502

Module – 1 Signals, systems and signal processing, classification of signals, Basic Operations on Signals, Basic Elementary Signals, properties of systems. concept of frequency in continuous and Discrete time signals, sampling of analog signals, the sampling theorem, quantization of continuous amplitude and sinusoidal signals , coding of quantized samples, digital to analog conversion, **Time-domain representations for LTI systems:** Convolution, impulse response representation, Convolution Sum and Convolution Integral, properties of impulse response representation, solution of difference equations.	
Bloom's Taxonomy Level	L1 – Remembering, L2 – Understanding, L3 – Applying, L – 4 Analyzing,
Module – 2 **Discrete Fourier Transforms (DFT):** Introduction to DFT, definition of DFT and its inverse, matrix relation to find DFT and IDFT ,Properties of DFT, linearity, circular time shift, circular frequency shift, circular folding, symmetry of : real valued sequences, real even and odd sequences, DFT of complex conjugate sequence, multiplication of two DFTs- the circular convolution, Parseval's theorem, circular correlation, Digital linear filtering using DFT. Signal segmentation , overlap-save and overlap-add method	
Bloom's Taxonomy Level	L1 – Remembering, L2 – Understanding, L3 – Applying, L – 4 Analysing,
Module – 3 **Fast-Fourier-Transform (FFT) algorithms**: Direct computation of DFT, need for efficient computation of the DFT (FFT algorithms)., speed improvement factor, Radix-2 FFT algorithm for the computation of DFT and IDFT–decimation-in-time and Decimation-in-frequency algorithms, calculation of DFT when N is not a power of 2	
Bloom's Taxonomy Level	L1 – Remembering, L2 – Understanding, L3 – Applying, L – 4 Analysing,

Module – 4 **IIR filter design:** Classification of analog filters, generation of Butterworth polynomials, frequency transformations. Design of Butterworth filters, low pass, high pass, band pass and band stop filters, Generation of Chebyshev polynomials, design of Chebyshev filters, design of Butterworth and Chebyshev filters using bilinear transformation and Impulse invariance method, representation of IIR filters using direct form one and two, series form and parallel form	
Bloom's Taxonomy Level	L1 – Remembering, L2 – Understanding, L3 – Applying, L – 4 Analysing,
Module – 5 **FIR filter design:** Introduction to FIR filters, symmetric and antisymmetric FIR filters, design of linear phase FIR filters using - Rectangular, Bartlett, Hamming, Hanning and Blackman windows, design of FIR differentiators and Hilbert transformers, FIR filter design using frequency sampling Technique. Representation of FIR filters using direct form and lattice structure.	
Bloom's Taxonomy Level	L1 – Remembering, L2 – Understanding, L3 – Applying, L – 4 Analysing,

Sl. NO	Experiment
1.	Verification of Sampling Theorem in time and frequency domains
2.	Generation of different signals in both continuous and discrete time domains
3.	To perform basic operations on given sequences- Signal folding, evaluation of even and odd signals
4.	Evaluation of impulse response of a system.
5.	Solution of a difference equation.
6.	Evaluation of linear convolution and circular convolution of given sequences
7.	Computation of N- point DFT and IDFT of a given sequence by use of (a) Defining equation; (b) FFT method
8.	Evaluation of circular convolution of two sequences using DFT and IDFT approach.
9.	Design and implementation of IIR filters to meet given specification (Low pass, high pass, band pass and band reject filters).
10.	Design and implementation of FIR filters to meet given specification (Low pass, high pass, band pass and band reject filters) using different window functions.
11.	Design and implementation of FIR filters to meet given specification (Low pass, high pass, band pass and band reject filters) using frequency sampling technique.
12.	Realization of IIR and FIR filters.

List of Text Books
1. Introduction to Digital Signal Processing, Jhonny R. Jhonson, Pearson, 1 st Edition, 2016.
List of Reference Books
1. Digital Signal Processing – Principles, Algorithms, and Applications, Jhon G. Proakis Dimitris G. Manolakis, Pearson 4th Edition, 2007. 2. Digital Signal Processing, A.Nagoor Kani, McGraw Hill, 2nd Edition, 2012. 3. Digital Signal Processing, Shaila D. Apte, Wiley, 2nd Edition, 2009. 4. Digital Signal Processing, Ashok Amberdar, Cengage 1st Edition, 2007. 5. Digital Signal Processing, Tarun Kumar Rawat, Oxford, 1st Edition, 2015.
List of URLs, Text Books, Notes, Multimedia Content, etc
Web links and Video Lectures (e-Resources): 1. http://www.freebookcentre.net/Electronics/DSP-Books 2. https://www.electronicsforu.com/special/cool-stuff-misc/8-free-digital-signal-processing-ebooks 3. http://nptel.ac.in/courses/117104074/

4. https://ocw.mit.edu/resources/res-6-007-signals-and-systems-spring.../lecture-notes

MOOCs:

1. https://nptel.ac.in/courses/117102060
2. https://onlinecourses.nptel.ac.in/noc21_ee20/preview

Course Outcomes	**At the end of the course the student will be able to:** **CO1**: Perform elementary signal operations, **apply** convolution for both continuous & discrete time signals and to understand sampling theorem. [L3] **CO2**: **Evaluate** Discrete Fourier Transform of a sequence, to understand the various Properties of DFT and Signal segmentation using overlap save and add method. [L4] **CO3**: **Evaluate** Discrete Fourier Transform of a sequence using decimation in time and decimation in frequency methods. [L4] **CO4**: **Design** Butterworth and Chebyshev IIR digital filters and to represent the IIR filters using different methods. [L4] **CO5**: **Design** FIR filters using window method and frequency sampling method and to represent FIR filters using direct method and lattice method. [L4]

1.Introduction of Module-1

- Definition of Signals & Systems
- Elementary Continuous Time Signals
- Elementary Discrete Time Signals

Module-1 : Introduction

- Signal is function that conveys information
- Mathematically,
- Signals are represented as Function of one or more independent variable

Eg: Speech Signal - Amplitude varies with time

Image Signal – brightness fun of spatial variables

- Time is mathematically represented as Independent variable

Signal Processing

- Humans are the most advanced signal processors
  - speech and pattern recognition, speech synthesis,. . .
- We encounter many types of signals in various applications
  - Electrical signals: voltage, current, magnetic and electric fields,. . .
  - Mechanical signals: velocity, force, displacement,. . .
  - Acoustic signals: sound, vibration,. . .
  - Other signals: pressure, temperature,. . .
- Most real-world signals are analog
  - They are continuous in time and amplitude
  - Convert to voltage or currents using sensors and transducers
- Analog circuits process these signals using
  - Resistors, Capacitors, Inductors, Amplifiers,. . .
- Analog signal processing examples
  - Audio processing in FM radios
  - Video processing in traditional TV sets

Limitations of Analog Signal Processing

- Accuracy limitations due to
  - Component tolerances
  - Undesired nonlinearities
- Limited repeatability due to
  - Tolerances
  - Changes in environmental conditions
    - Temperature
    - Vibration
- Sensitivity to electrical noise
- Limited dynamic range for voltage and currents
- Inflexibility to changes
- Difficulty of implementing certain operations
  - Nonlinear operations
  - Time-varying operations
- Difficulty of storing information

Digital Signal Processing

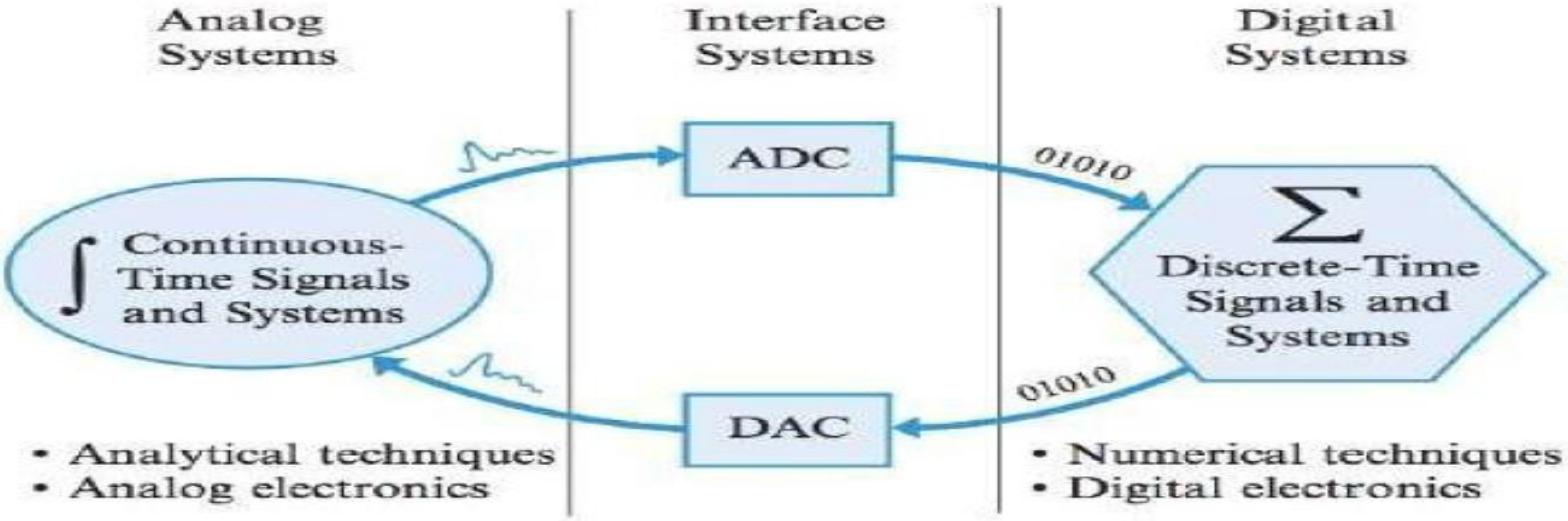
- Represent signals by a sequence of numbers
  - Sampling or analog-to-digital conversions
- Perform processing on these numbers with a digital processor
  - Digital signal processing
- Reconstruct analog signal from processed numbers
  - Reconstruction or digital-to-analog conversion

- Analog input – analog output
  - Digital recording of music
- Analog input – digital output
  - Touch tone phone dialing
- Digital input – analog output
  - Text to speech
- Digital input – digital output
  - Compression of a file on computer

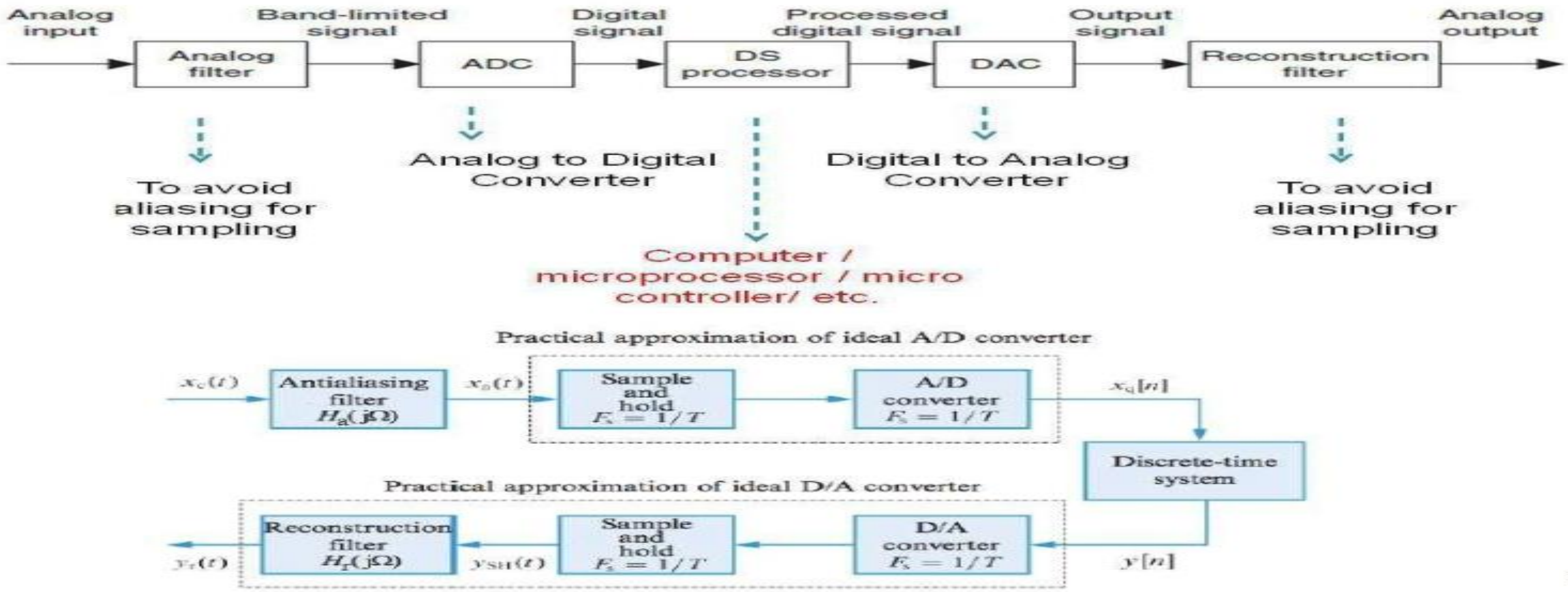
Pros and Cons of Digital Signal Processing

- Pros
  - Accuracy can be controlled by choosing word length
  - Repeatable
  - Sensitivity to electrical noise is minimal
  - Dynamic range can be controlled using floating point numbers
  - Flexibility can be achieved with software implementations
  - Non-linear and time-varying operations are easier to implement
  - Digital storage is cheap
  - Digital information can be encrypted for security
  - Price/performance and reduced time-to-market
- Cons
  - Sampling causes loss of information
  - A/D and D/A requires mixed-signal hardware
  - Limited speed of processors
  - Quantization and round-off errors

Analog, digital, mixed signal processing



Digital Signal Processing



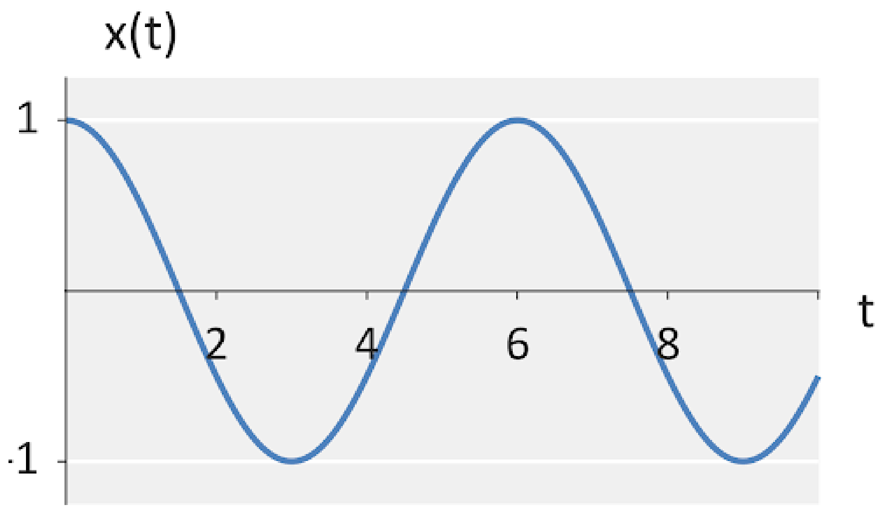
Basic types of Signals

Continuous Time Signal

Discrete Time Signal

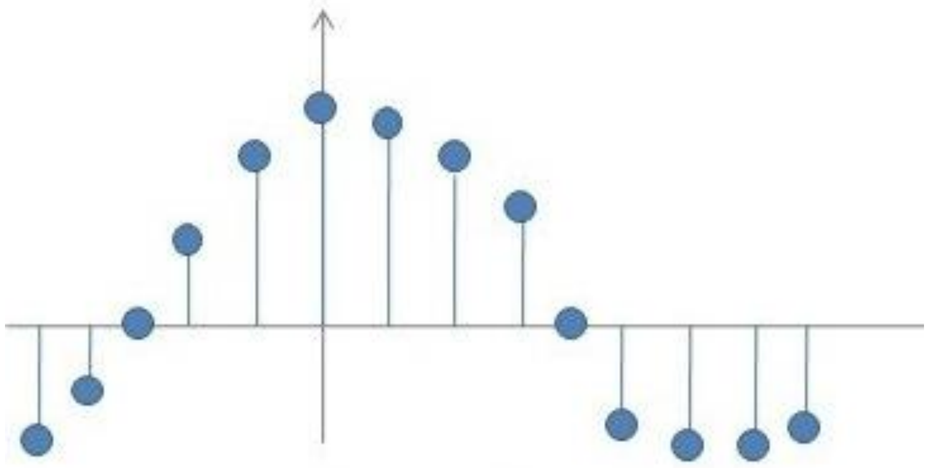
Signal x(t) is said to be continuous time signal if it has a value of amplitude for all time 't'

Denoted as x(t)

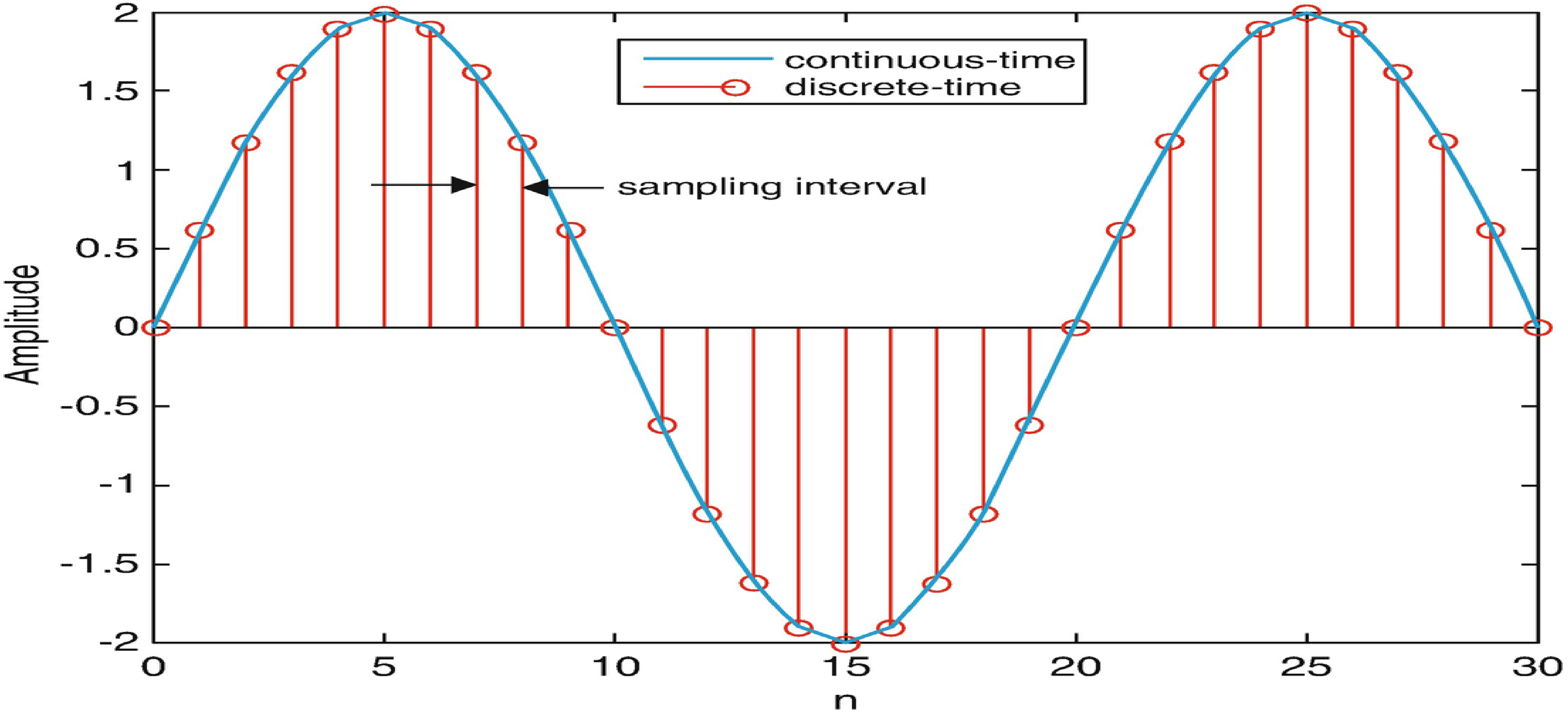


Discrete Time Signal

- A discrete time signal is defined only at discrete instants of time
- Denoted as x(n)



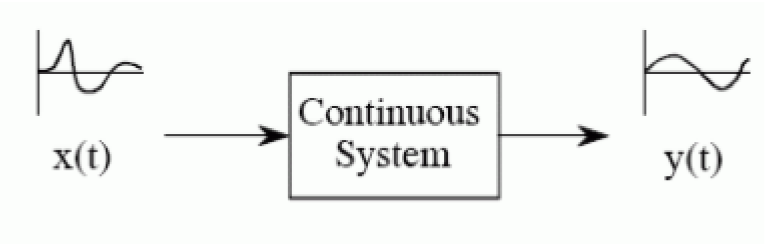
Discrete Signal

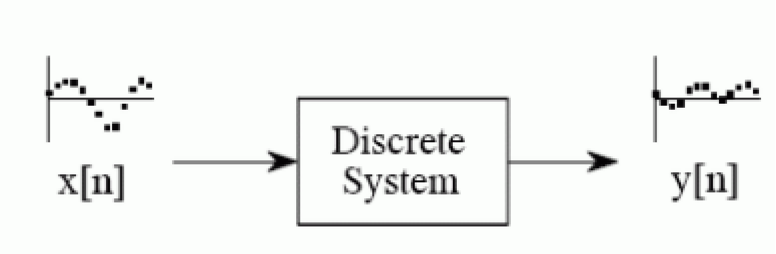

continuous-time
discrete-time
sampling interval
Amplitude
n
2
1.5
1
0.5
0
-0.5
-1
-1.5
-2
0
5
10
15
20
25
30

System

It is an interacting group of physical objects or physical conditions

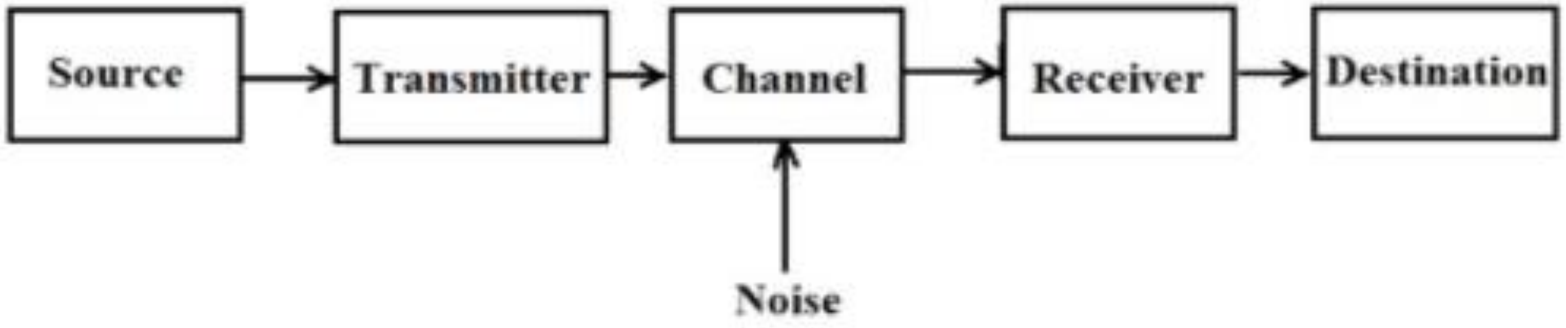
In signals and systems terminology, corresponding to every possible input signal, a system “produces” an output signal.



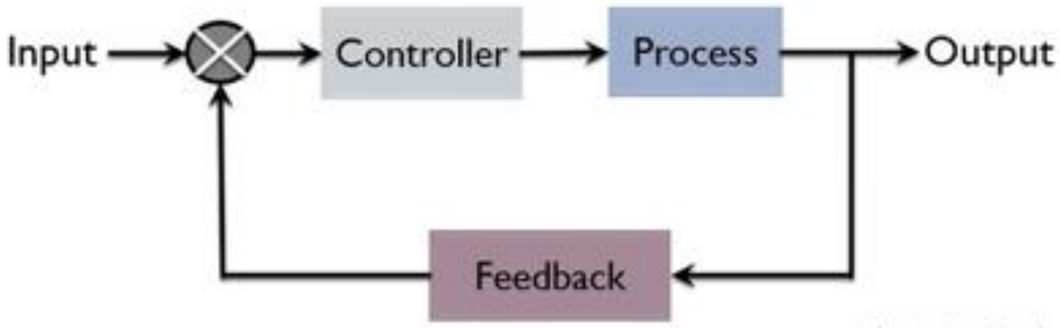


Real Time Examples

1. Communication System



2. Control Systems



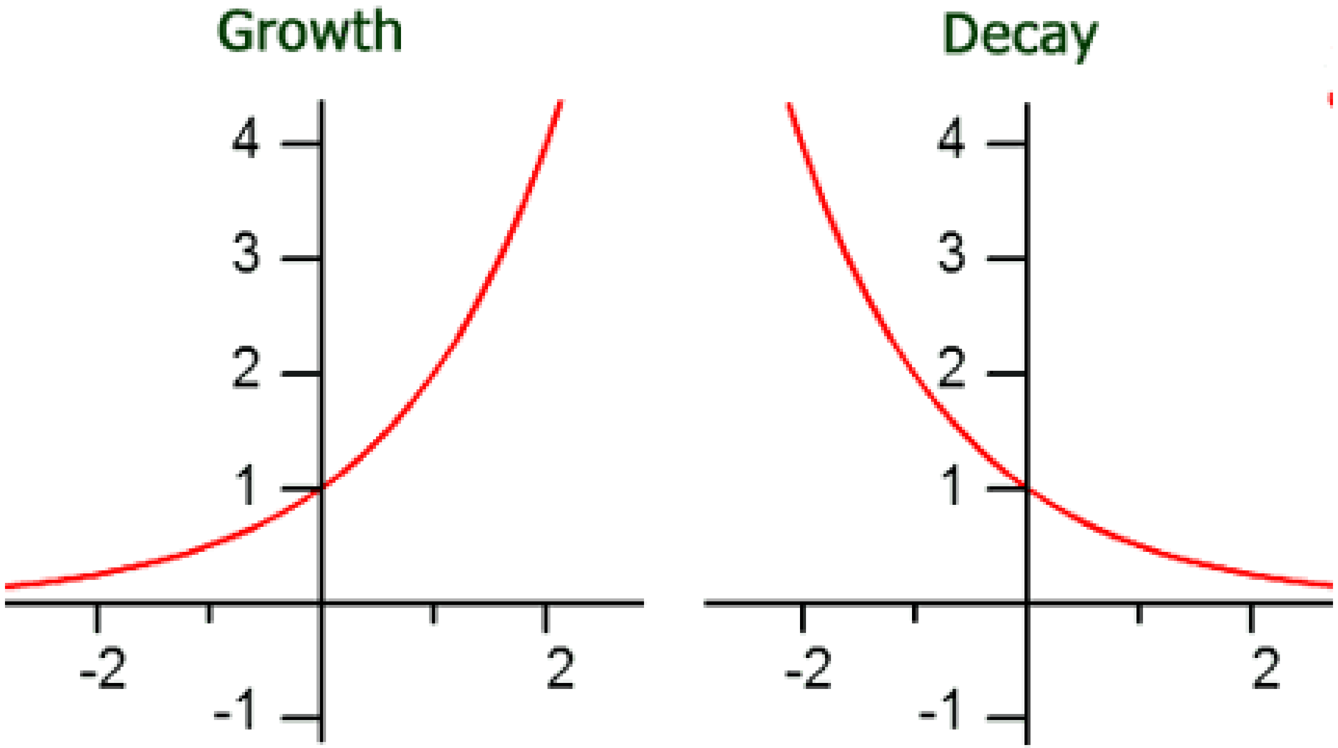
Elementary Continuous Signals

- Exponential
- Sinusoidal
- Exponential damped sinusoidal signal
- Unit Step function
- Unit Impulse function
- Unit Ramp function

Elementary Continuous Signals

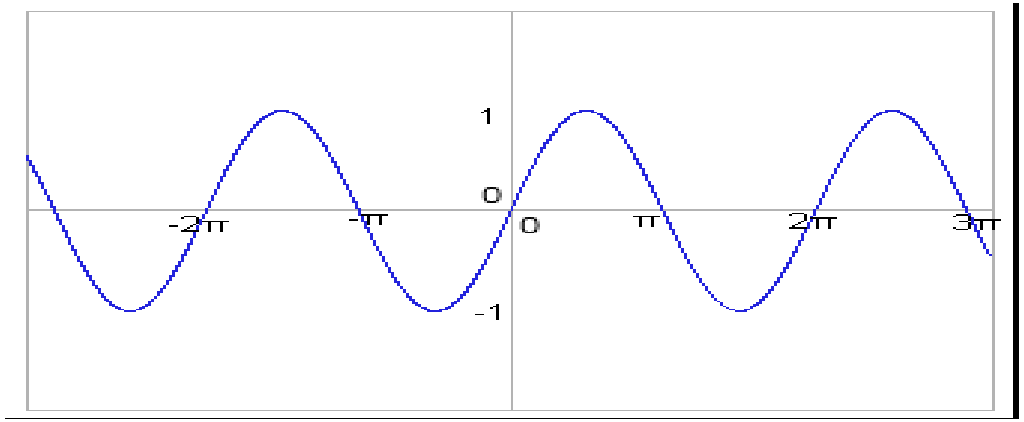
- Exponential: $x(t) = C e^{at}$

a > 0 **a < 0**

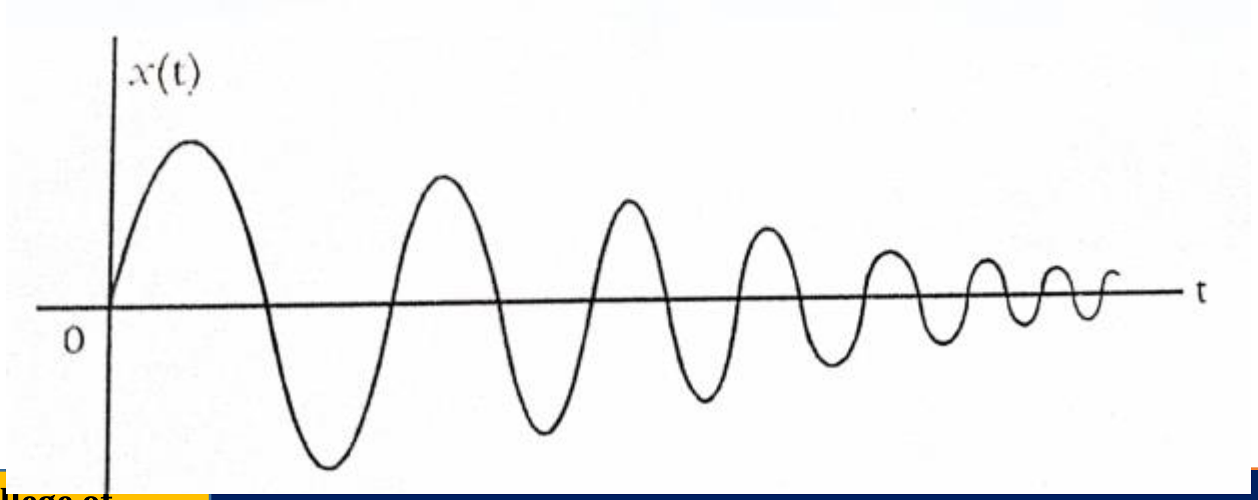


Elementary Continuous Signals

- Sinusoidal



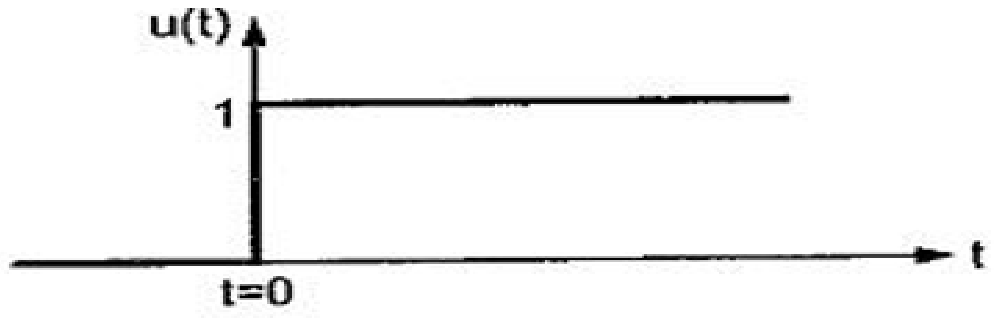
- Exponential damped sinusoidal signal



Elementary Continuous Signals

- Unit Step function

$$u(t) = 1 \text{ for } t \geq 0$$
$$\quad\quad 0 \text{ for } t < 0$$

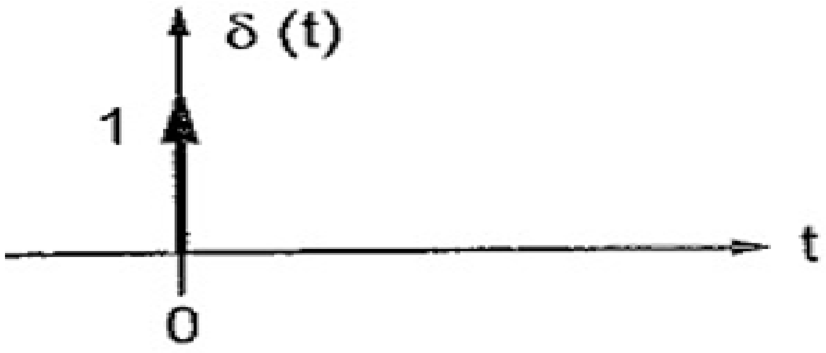


- Unit Impulse function

$$\delta(t) = 0, \quad t \neq 0$$

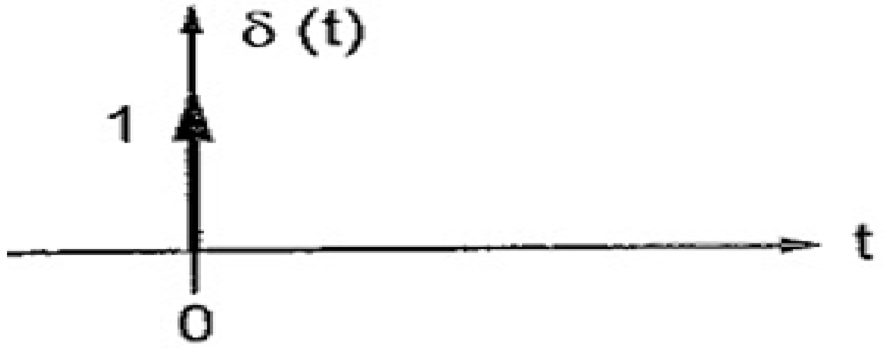
&

$$\int_{-\infty}^{\infty} \delta(t)\, dt = 1, \qquad t = 0$$



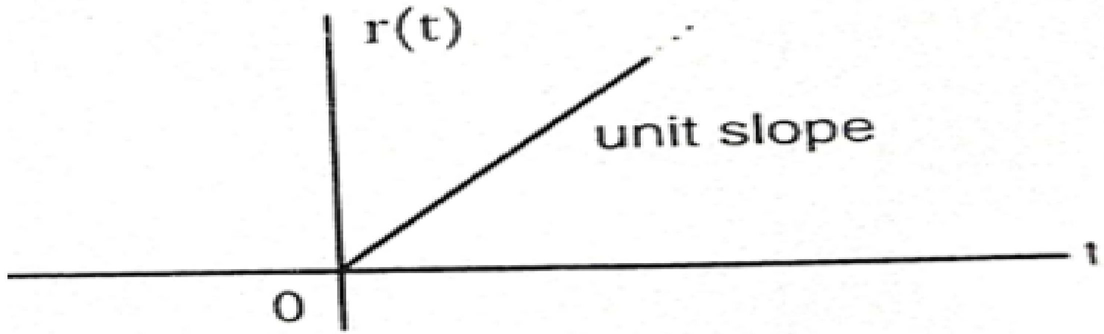
Elementary Continuous Signals

- Zero width
- Infinite height
- Unit area or unit strength



- Unit Ramp function

$$r(t) = t\,, \quad t \geq 0$$
$$= 0, \quad t < 0$$

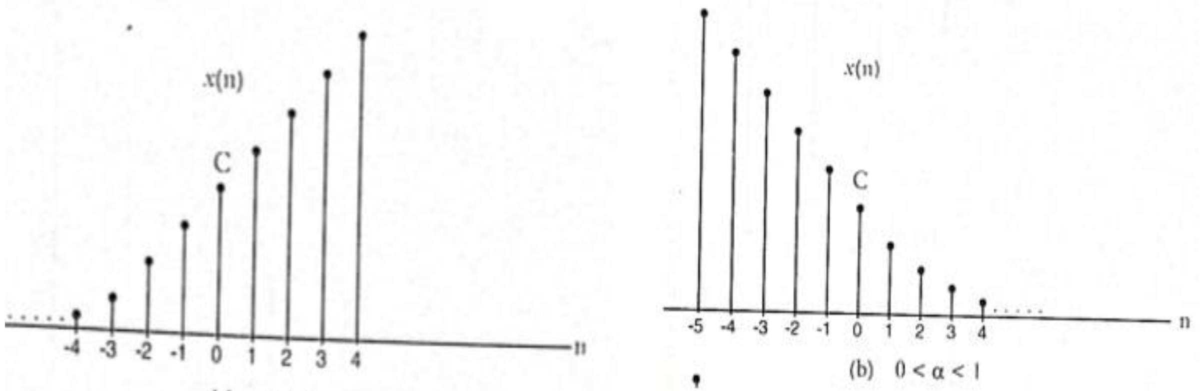


Elementary Discrete Time Signals

- Exponential
- Sinusoidal
- Exponential damped sinusoidal signal
- Unit Step function
- Unit Impulse function

Elementary Discrete Signals

- Exponential: $x(n) = C\,\alpha^n$
- $\alpha > 1$ $\qquad\qquad$ $0 < \alpha < 1$

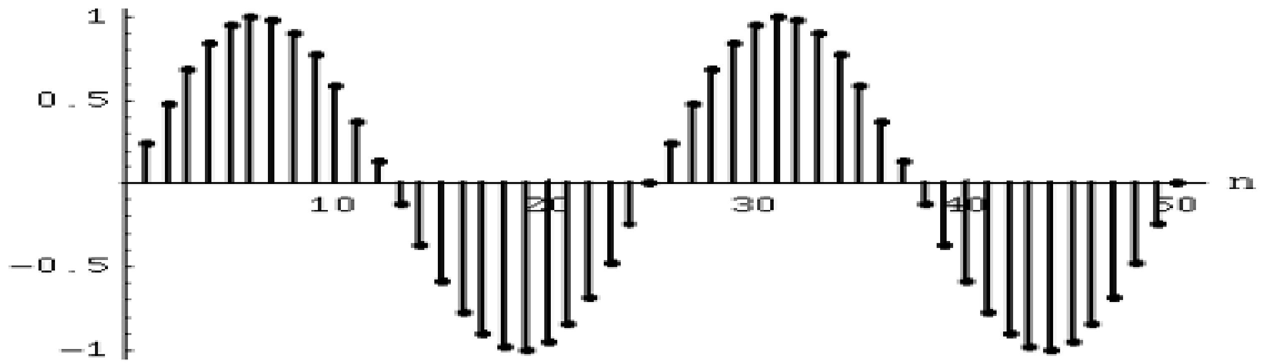


(a) $\alpha > 1$ $\qquad$ (b) $0 < \alpha < 1$

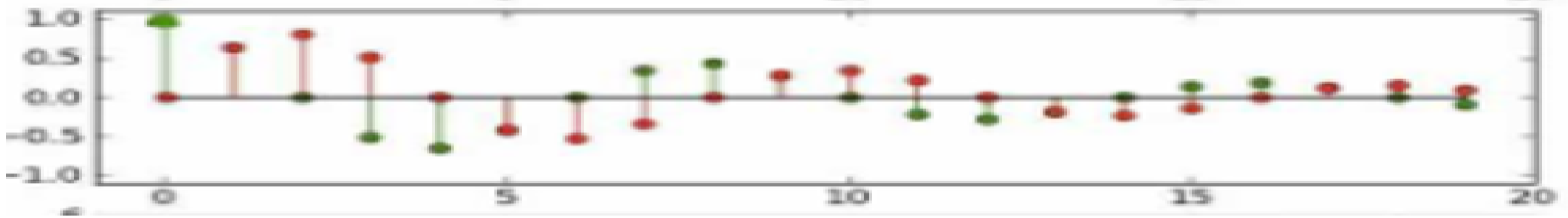
Elementary Discrete Signals

- Sinusoidal

$$X(n) = \cos\left(\frac{\pi}{4} n\right)$$



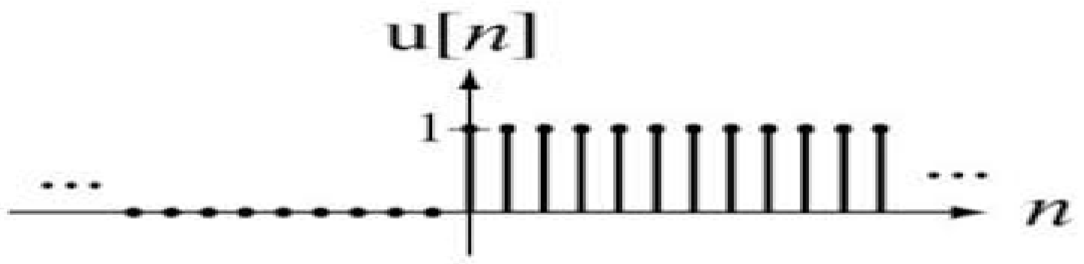
- Exponential damped sinusoidal signal



Elementary Discrete Signals

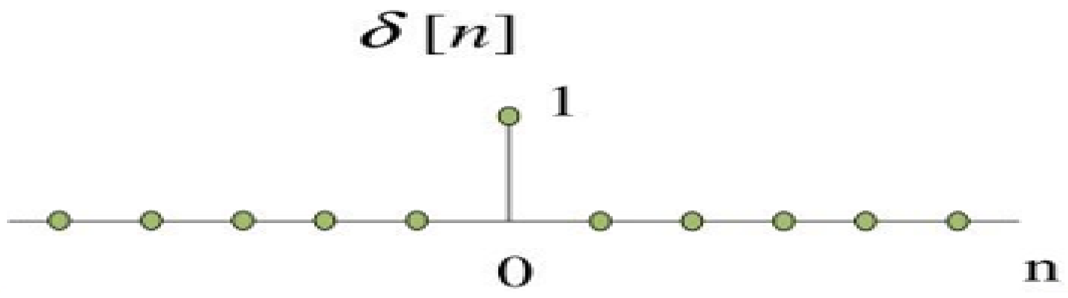
- Unit Step function

$$u(n) = \begin{cases} 1 & \text{for } n \geq 0 \\ 0 & \text{for } n < 0 \end{cases}$$



- Unit Impulse function
-

$$\delta(n) = \begin{cases} 0, & n \neq 0 \\ 1, & n=0 \end{cases}$$



Basic Operations on Signals

One dimensional signal can be defined as two variables

1. Dependent Variable
2. Independent Variable

- Operations performed on dependent Variable

1. Amplitude Scaling
2. Addition
3. Multiplication
4. Differentiation
5. Intergration

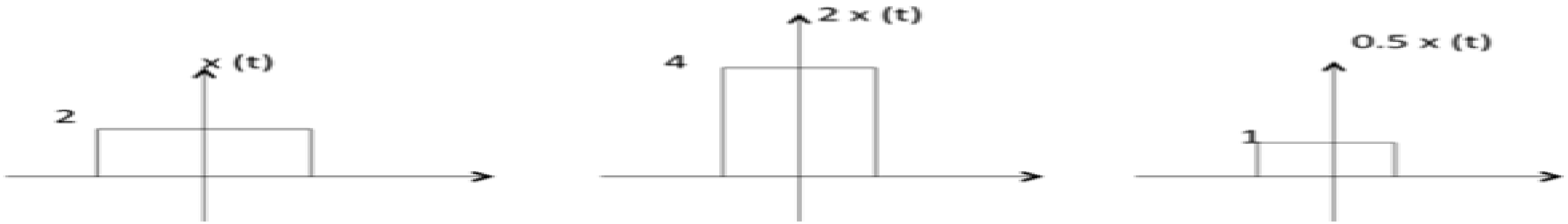
Basic Operations on dependent Variable

1. Amplitude Scaling:

y(t) = c x(t) or y(n) = c x(n)

Where, x(t) & x(n) are input signal 'c' is the scaling factor

Ex: y(t) = 2 x(t) & y(t) = ½ x(t)



Basic Operations on Signals

2. Addition:

y(t) = x1(t) + x2(t)

3. Multiplication:

y(t) = x1(t) . x2(t)

4. Differentiation

y(t) = dx(t)/dt

5. Intergration

Y(t) = $\int x(t)dt$

Basic Operations on Signals

Operations performed on Independent Variable

1. Time Scaling
2. Time Shifting
3. Reflection
4. Precedence Rule

Basic Operations on Signals

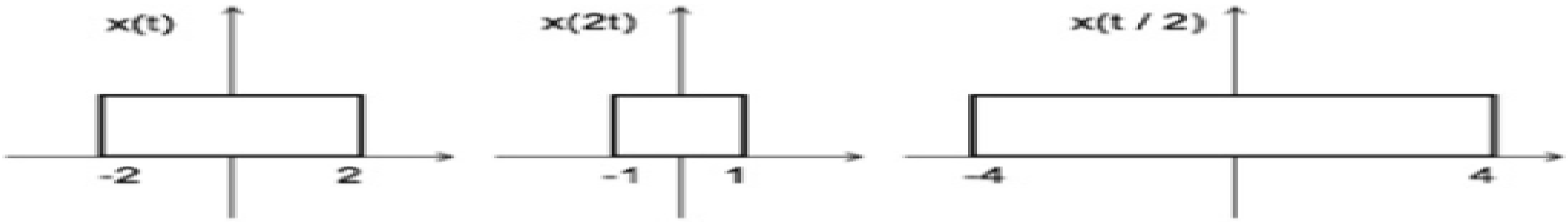
1. Time Scaling: if $x(t)$ is a CT signal and $y(t)$ is obtained by scaling the independent variable 't' is

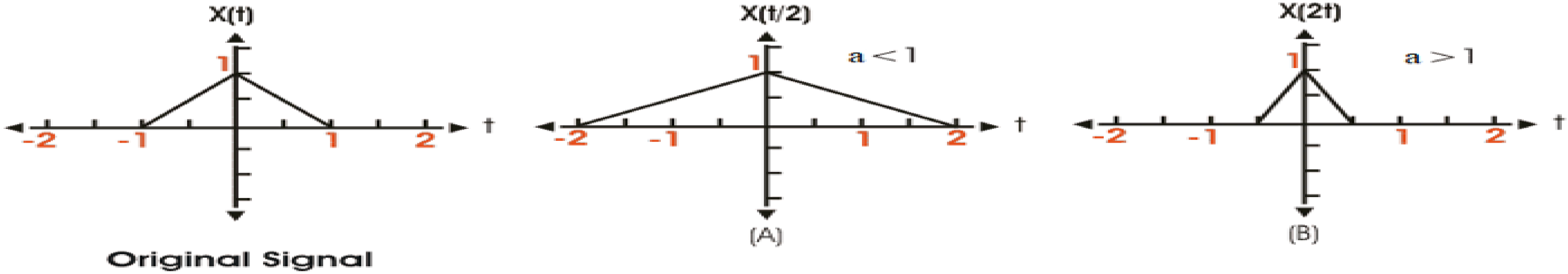
$$y(t) = x(at)$$

'a' is a scaling factor

If $a > 1$, Signal $y(t)$ is compressed

$0 < a < 1$, Signal $y(t)$ is expanded





W.r.t Discrete Signal

$y(n) = x(kn)$

'k' is a scaling factor

Basic Operations on Signals

1. Time Shifting: If $x(t)$ is a CT signal then

$y(t) = x(t-t_0)$, t_0 is the time shift

If $t_0 > 0$, Waveforms shift towards right

$t_0 < 0$, Waveforms shift towards left

Ex: $y(t) = x(t-2)$

Contd..

- $y(t) = x(t+2)$

$y(n) = x(n-n_0)$, n_0 is the shifting factor

Fig. 1 $x(n)$

Fig. 2 $y(n) = x(n-2)$

Summary

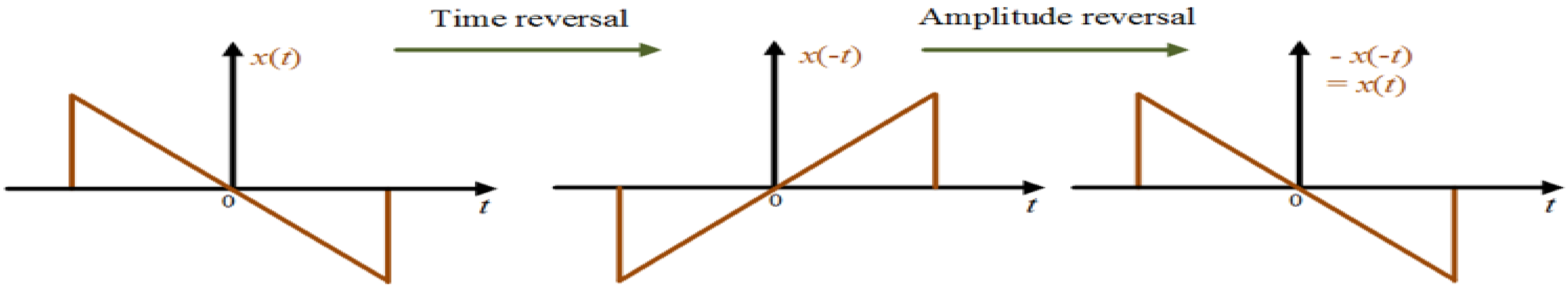
- DT Elementary signals
- Basic Operations on Signals
- Operations on Signals w.r.t Dependent and Independent Variable

Basic Operations on Signals

3. Reflection: If x(t) be the CT signal and then signal

y(t) = x(-t) is known as Time reversal

y(t) = - x(t) is known as Amplitude reversal version of the x(t)

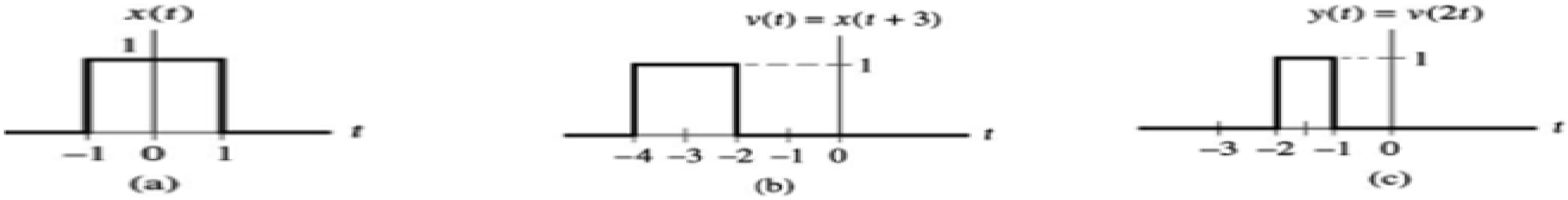


Basic Operations on Signals

4. Precedence Rule: It is a combination of Time shifting and Time Scaling

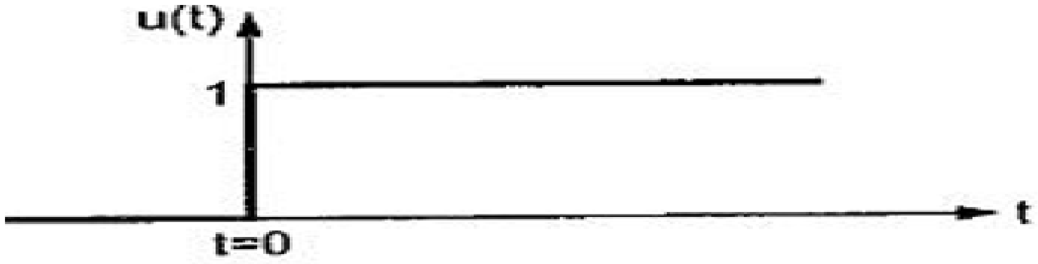
Let x(t) be the CT signal and y(t) be the output of the x(t) related with the following relation

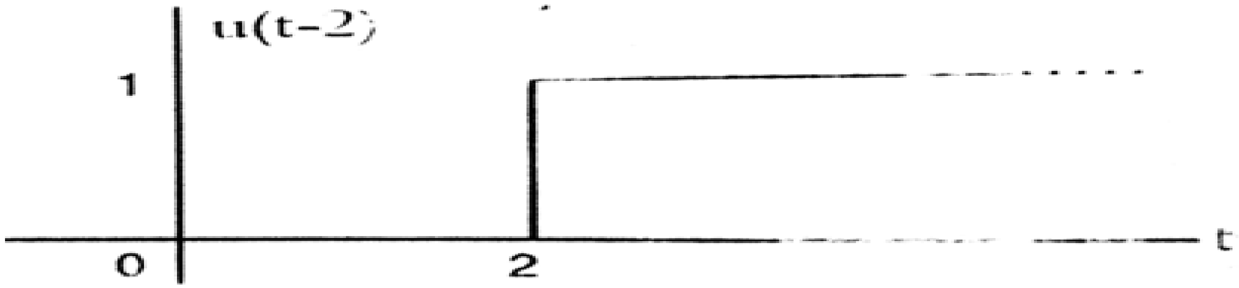
$y(t) = x(at - t_0)$ Ex: $y(t) = x(2t+3)$



Problems on Basic Operations on Signals

Sketch the Signal 1. $x(t) = u(t) - u(t-2)$





$$u(t) = 1 \text{ for } t \geq 0$$
$$\quad\;\; 0 \text{ for } t < 0$$

$$u(t-2) = 1 \quad ; t \geq 2$$
$$\quad\quad\;\; = 0 \quad ; t < 2$$

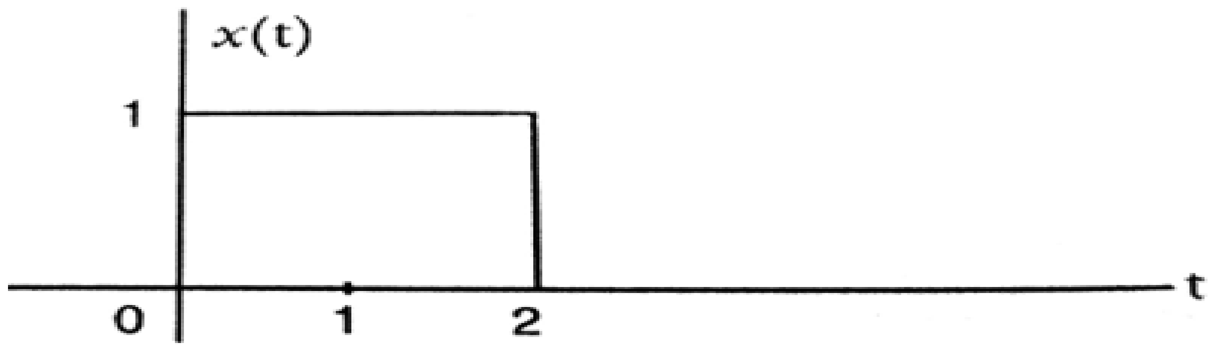
For $t<0$; $u(t) = u(t-2) = 0$ $\quad\therefore x(t) = 0 - 0 = 0$

For $0 < t < 2$; $u(t) = 1$ & $u(t-2) = 0$ $\quad\therefore x(t) = 1 - 0 = 1$

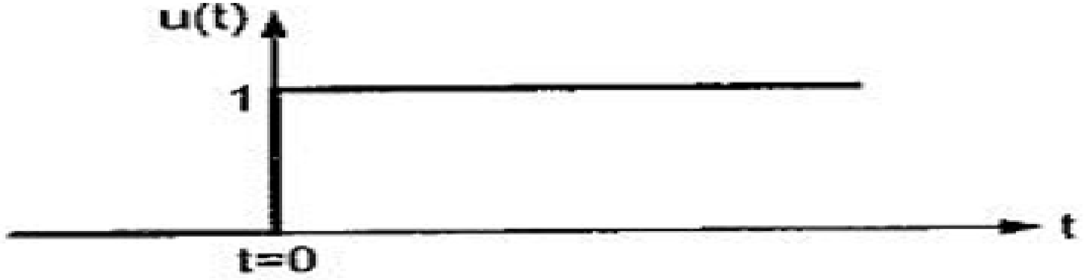
For $t > 2$; $u(t) = 1$ & $u(t-2) = 1$ $\quad\therefore x(t) = 1 - 1 = 0$

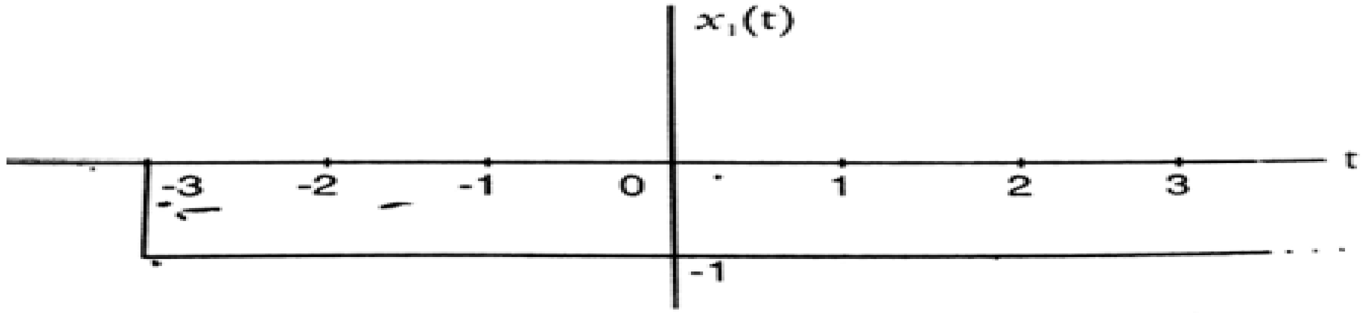
Problems on Basic Operations on Signals

Sketch the Signal 1. $x(t) = u(t) - u(t-2)$

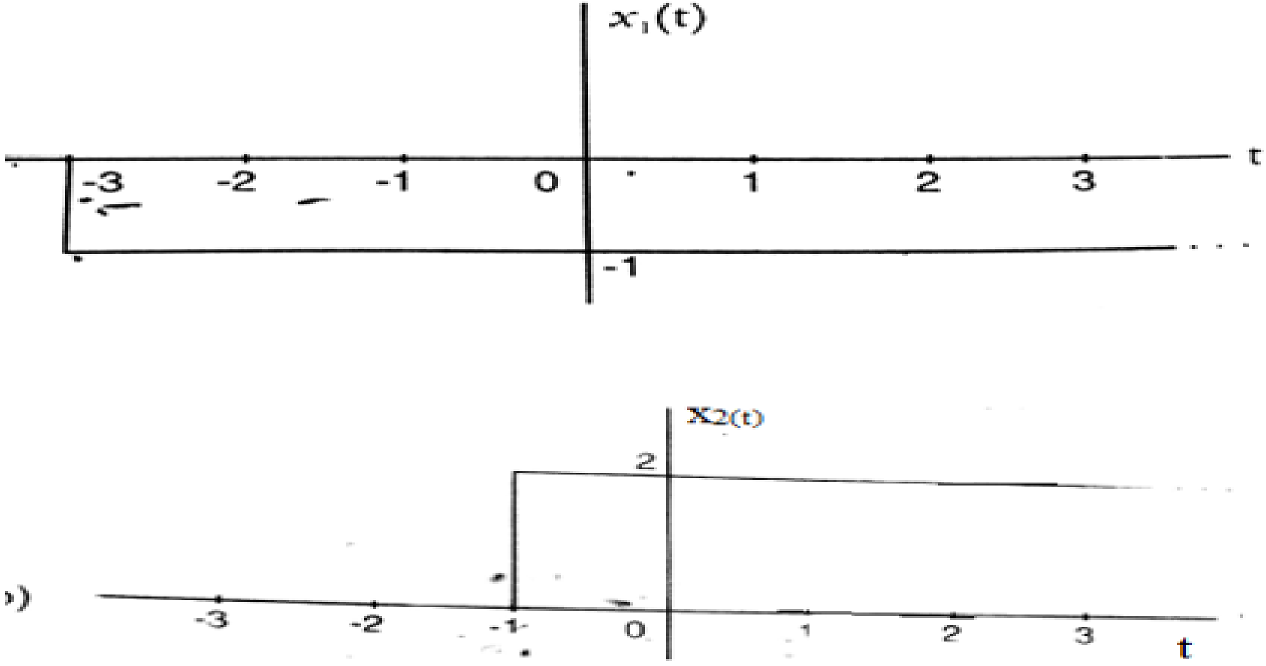


2. Sketch the signal x(t) = - u(t+3)+2 u(t+1)+ (-2u(t-1))+u(t-3)





2. Sketch the signal $x(t) = -u(t+3)+2u(t+1)+(-2u(t-1))+u(t-3)$



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$$x(t) = x_1(t)+x_2(t)+x_3(t)+x_4(t)$$

For $t<-3$; $x_1(t) = x_2(t) = x_3(t) = x_4(t) = 0$ $\therefore x(t) = 0+0+0+0 = 0$

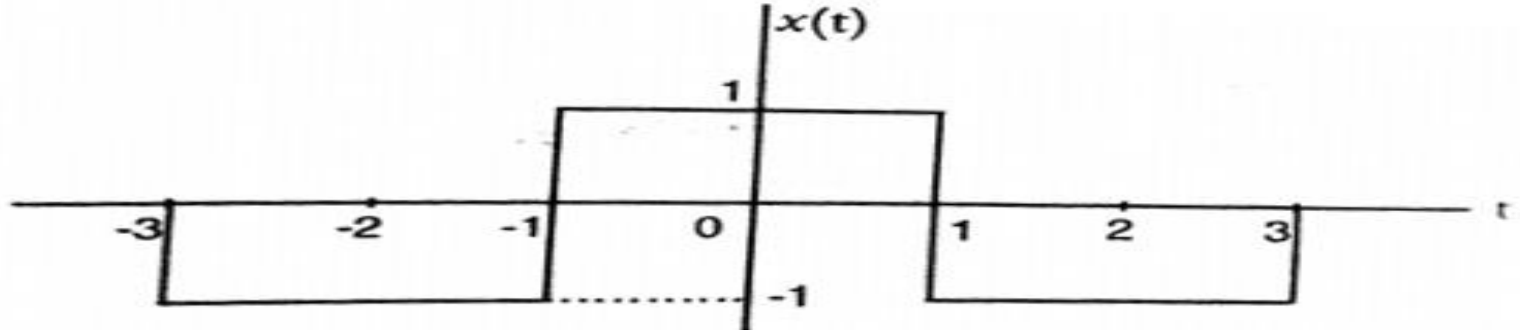
For $-3<t<-1$; $x_1(t) = -1$; $x_2(t) = x_3(t) = x_4(t) = 0$ $\therefore x(t) = -1+0+0+0 = -1$

For $-1<t<1$; $x_1(t) = -1$; $x_2(t) = 2$; $x_3(t) = x_4(t) = 0$ $\therefore x(t) = -1+2+0+0 = 1$

For $1<t<3$; $x_1(t) = -1$; $x_2(t) = 2$; $x_3(t) = -2$; $x_4(t)=0$ $\therefore x(t) = -1+2-2+0 = -1$

For $t>3$; $x_1(t) = -1$; $x_2(t) = 2$; $x_3(t) = -2$; $x_4(t) = 1$ $\therefore x(t) = -1+2-2+1 = 0$

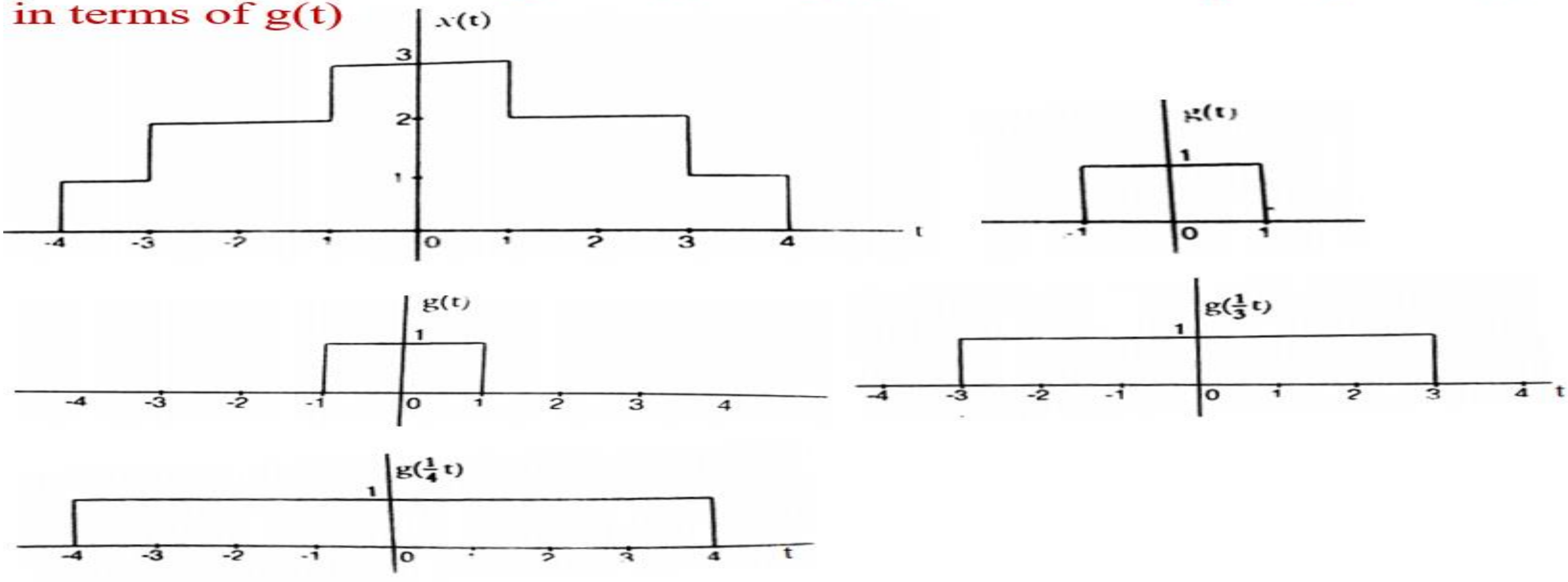
(c)



Numerical on operations on signals

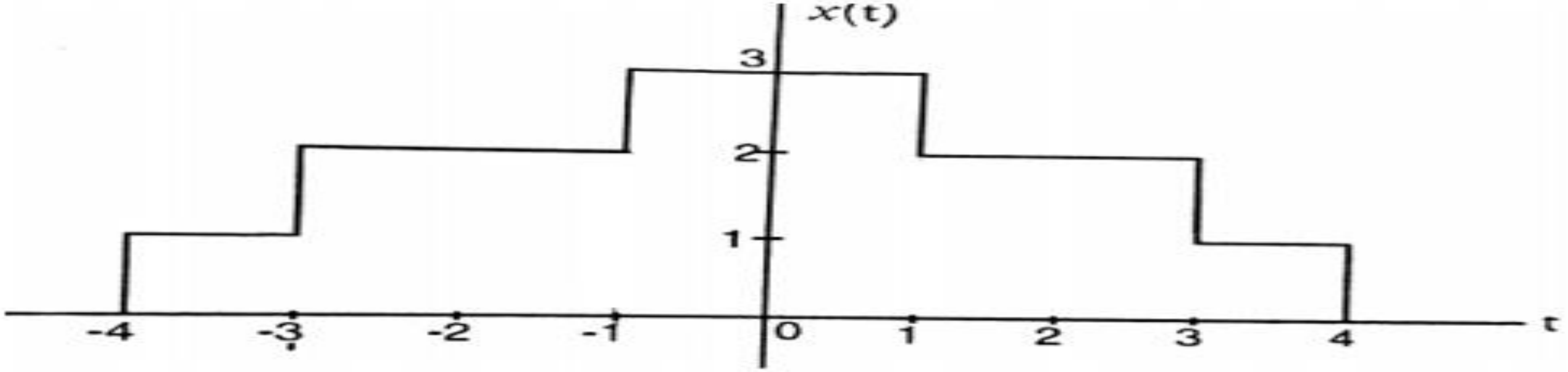
3. $y(t) = [x(t) + x(2-t)]\, u(1-t)$

3. A continuous time signal x(t) and g(t) is shown in fig. Express x(t) in terms of g(t)



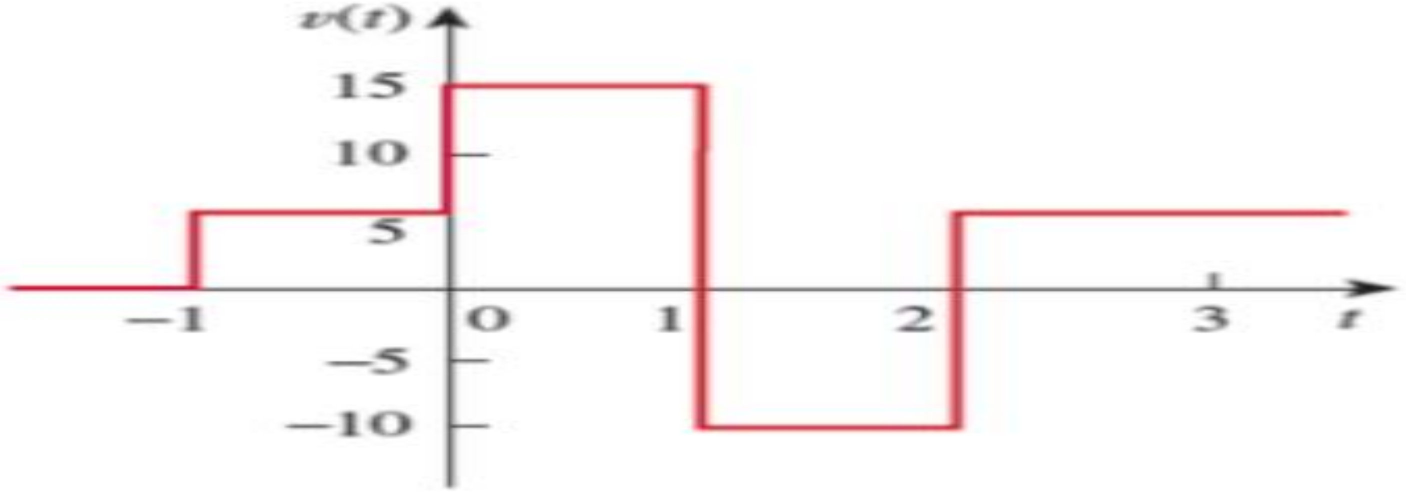
Adding g(t), g(1/3t), g(1/4*t) yields x(t) i.e.

$$\therefore x(t) = g(t) + g(\tfrac{1}{3}t) + g(\tfrac{1}{4}t)$$



Assignment Question

Express $v(t)$ in the given figure in terms of step functions.



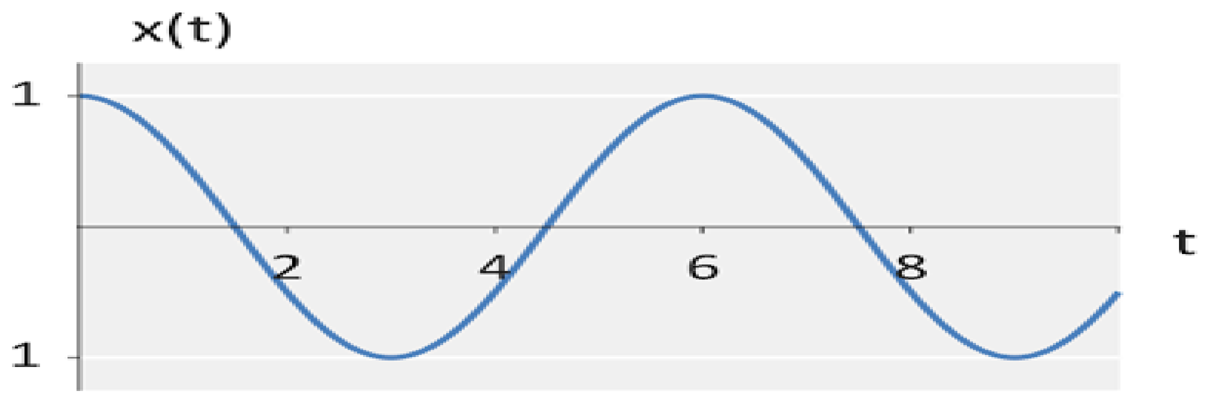
- $v(t) = (5u(t+1) + 10(t-1) - 25u(t) + 15u(t-2))$ V
- $v(t) = (5u(t-1) + 10u(t) - 25u(t+2) + 15u(t+2))$ V
- $v(t) = (5u(t-2) + 10u(t-1) - 25u(t) + 15u(t+1))$ V
- $v(t) = (5u(t+1) + 10u(t) - 25u(t-1) + 15u(t-2))$ V

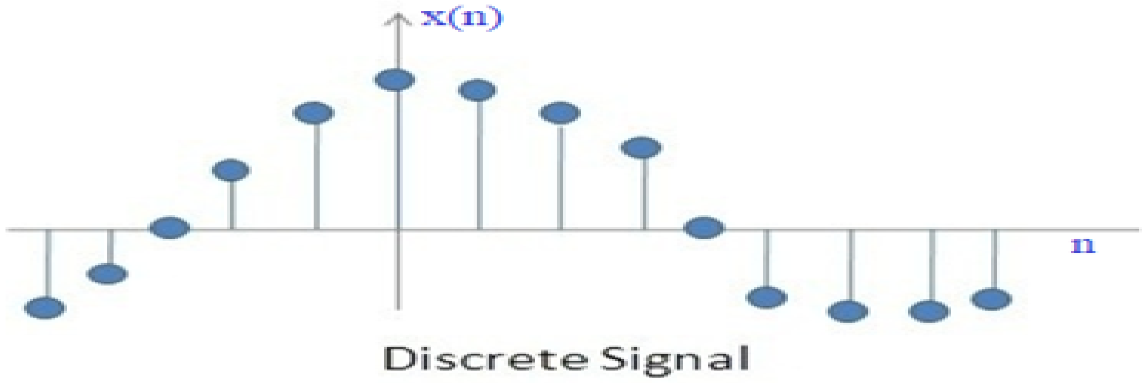
Classification of Signals

- Continuous Time and Discrete Time Signals
- Even and ODD Signals
- Periodic and Non Periodic Signals
- Deterministic and Random Signals
- Energy and Power Signals

Continuous Time and Discrete Time Signals

- Signal x(t) is said to be continuous time signal if it has a value of amplitude for all time ‘t’
- A discrete time signal x(n) is defined only at discrete instants of time





Even and ODD Signals

- A signal is said to be an Even signal, if it satisfies the condition $x(t) = x(-t)$ for a continuous signal

$x(n) = x(-n)$ for a Discrete signal

Eg: Cosine signal

i.e. $\text{Cos}(30°) = \frac{\sqrt{3}}{2}$ & $\text{Cos}(-30°) = \frac{\sqrt{3}}{2}$

- A signal is said to be an ODD signal, if it satisfies the condition $x(-t) = -x(t)$ for a continuous signal

$x(-n) = -x(n)$ for a Discrete signal

Eg: Sine Signal

$\text{Sin}(30°) = \frac{1}{2}$ & $\text{Sin}(-30°) = -\frac{1}{2}$

Decomposition of Signals

- A CT signal is decomposed into sum of two signals (even and odd)
- $x_e(t) = \frac{1}{2}\,[\, x(t) + x(-t) \,]$
- $x_o(t) = \frac{1}{2}\,[\, x(t) - x(-t) \,]$

Numericals on Even and ODD Signal

Find the even and ODD component of the signal

1) $x(t) = \cos(t) + \sin(t) + \sin(t)\cos(t)$

2) $x(t) = (1+t^3)\cos^3(10t)$

3) $x(t) = 1 + t + 3t^2 + 5t^3 + 9t^4$

4) Determine and sketch the even and Odd parts of the signal shown in Fig

Numerical on Even and ODD Signal

4) Determine and sketch the even and Odd parts of the signal shown in Fig

Properties of Systems

- Linearity

- System should satisfy super position principle

i.e. if the input is the weighted sum of the several signals, then the output is the weighted sum of the responses of the system then system is said to Linear

$$\text{If } x_1(t) \rightarrow y_1(t) \;\&\; x_2(t) \rightarrow y_2(t) \text{ then}$$

$$a x_1(t) + b x_2(t) \rightarrow a\, y_1(t) + b\, y_2(t)$$

Properties of Systems

- Time Invariance

If time shift in the input signal causes corresponding time shift in the output signal

$$\text{If } x(t) \rightarrow y(t)$$

$$x(t-t_0) \rightarrow y(t-t_0)$$

Similarly, for a discrete time signal

$$\text{If } x(n) \rightarrow y(n)$$

$$x(n-n_0) \rightarrow y(n-n_0)$$

Memory:

System is referred as Memoryless when o/p at any point of time depends only on the present input

Properties of Systems

- Linearity
- Time Invariance
- Memory
- Causality
- Stability
- Invertibility

Properties of Systems

Memory:

System is referred as Memoryless when o/p at any point of time depends only on the present input

Properties of Systems

- Causality

- A system is referred as causal if the o/p y(t) depends only on past and /or present value of the input x(t)

- Stability

- A system is said to be Bounded Input Bounded Output (BIBO) stable if and only if Bounded Input results in a Bounded Output

$$|x(t)| \leq B_x < \infty \text{ ; for all 't'}$$

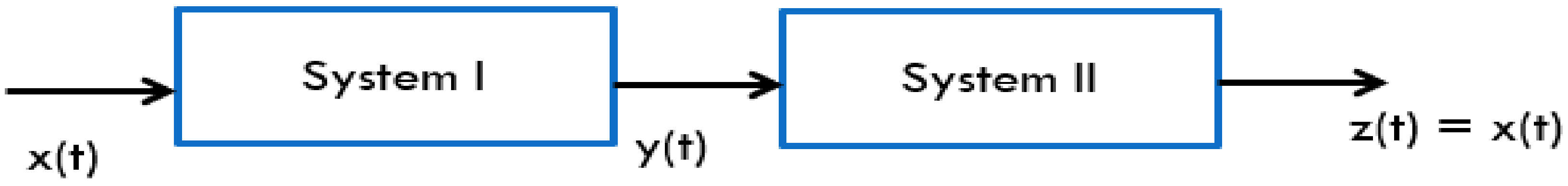
$$|y(t)| \leq B_y < \infty \text{ ; for all 't'}$$

Where B_x and B_y are infinite positive numbers

Properties of Systems

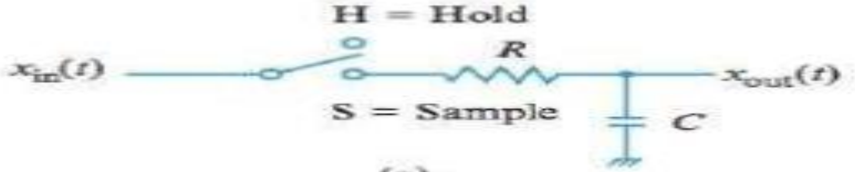
- Invertibility

A system is said to be invertible if the input of the system is recovered from its system output

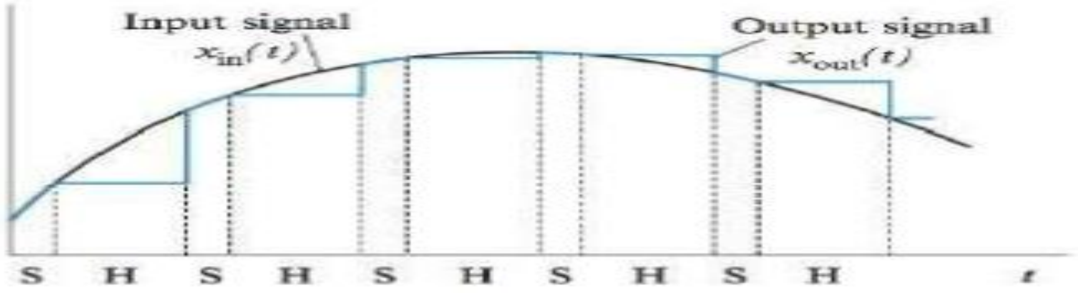


Sampling and reconstruction

- The main function of the *low-pass antialiasing filter* is to band-limit the input signal to the folding frequency without distortion.
- It should be noted that even if the signal is band-limited, there is always wide- band additive noise which will be folded back to create aliasing.
- When an analog voltage is connected directly to an ADC, the conversion process can be adversely affected if the voltage is changing during the conversion time.
- The quality of the conversion process can be improved by using a *sample-and- hold (S/H) circuit.*



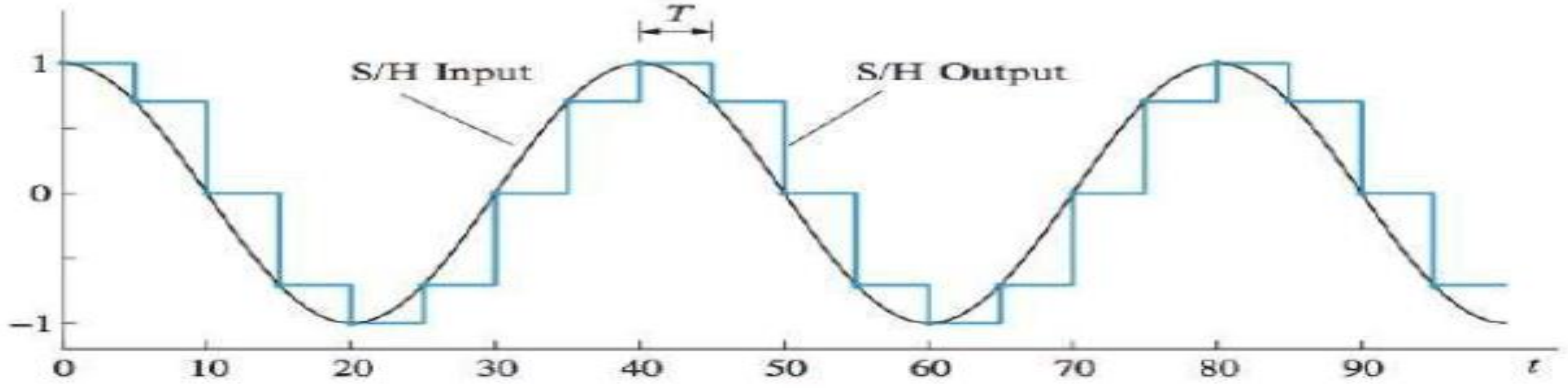
(a)



(b)

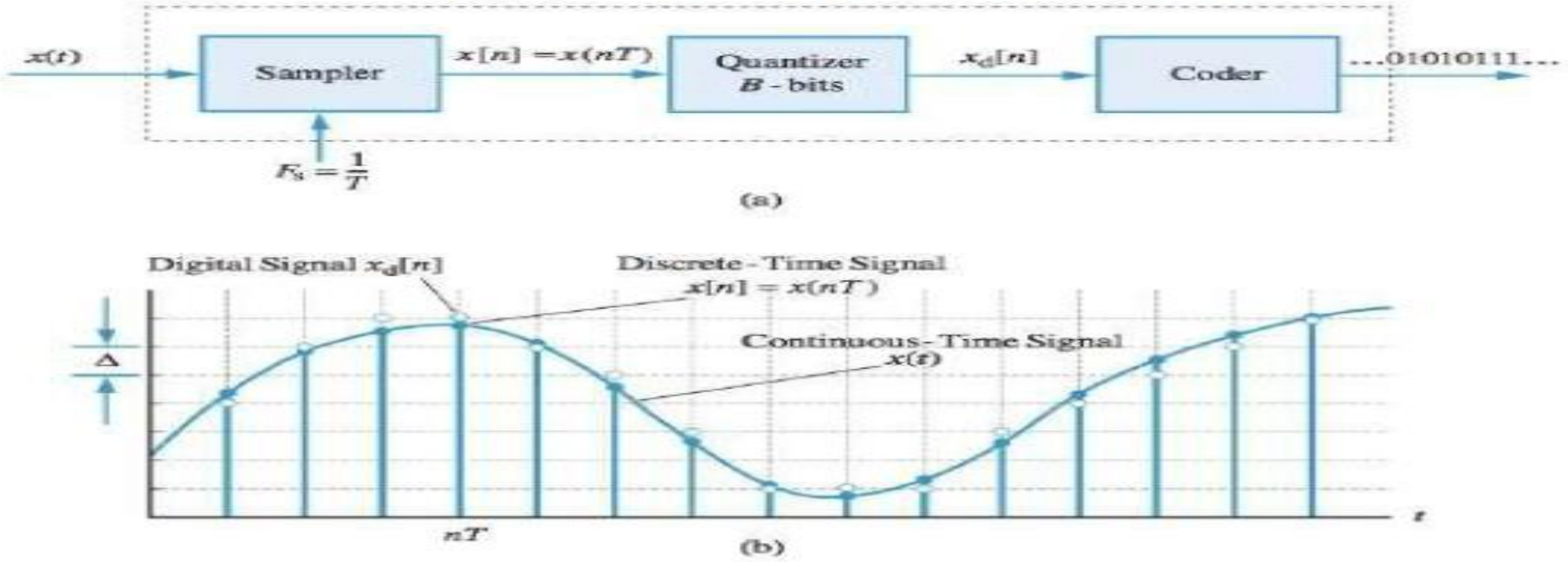
Sample and hold (S/H)circuit

- Since the sampling operation is performed by the S/H circuit, the role of S/H is to sample $x_c(t)$ as instantaneously as possible and to hold the sample value as constant as possible until the next sample.
- Thus, the output of the S/H circuit can be modelled as a staircase waveform where each sample value is held constant until the acquisition of the next sample.



- Note that the S/H system is linear but time-varying.

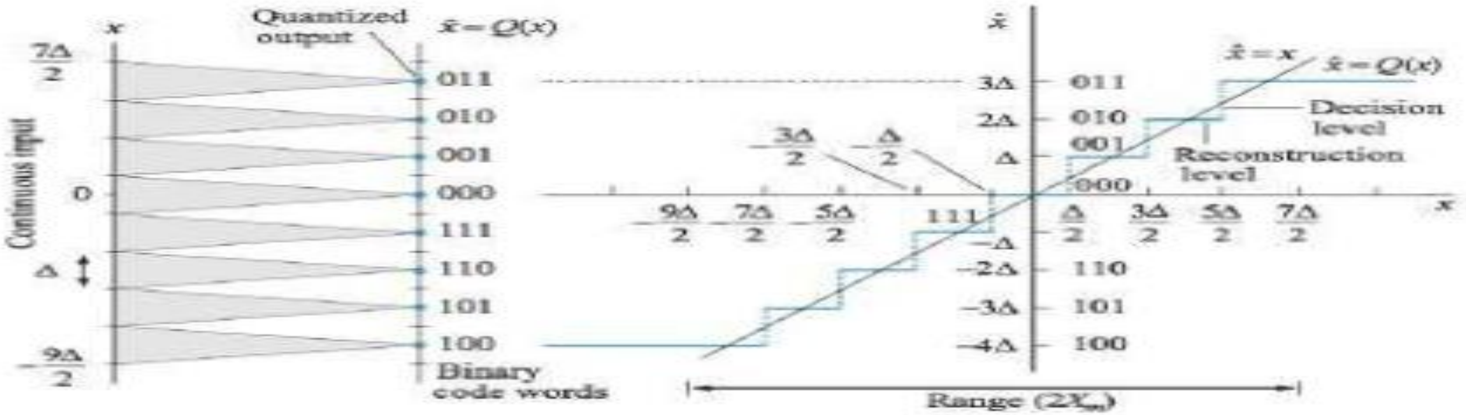
A/D converter



- *Quantization* converts a continuous-amplitude signal $x(t)$ into a discrete-amplitude signal $x_d[n]$.
- In theory, we are dealing with discrete-time signals; in practice, we are dealing with digital signals.

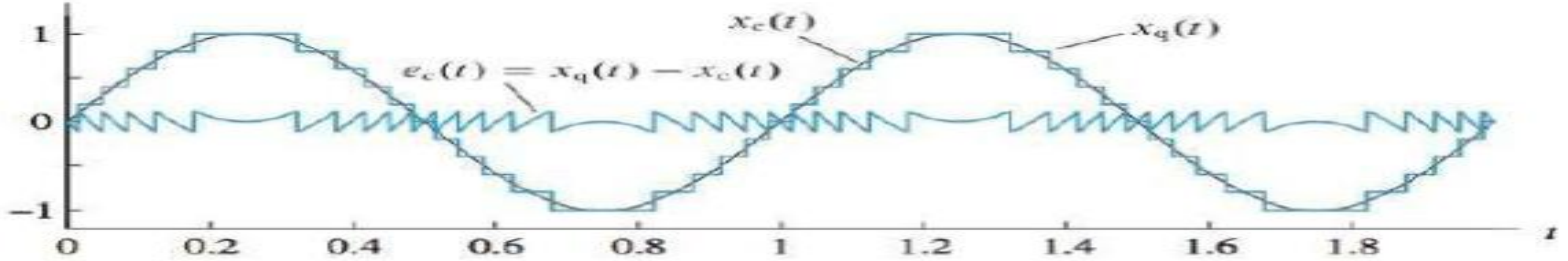
A/D converter

- The major difference between ideal and practical conversion is that an ADC generates sample values that are known with finite precision.
- The ADC is the device in which both quantization and binary coding of the sampled signal take place.
- A B-bit quantizer can represent 2^B different numbers.
- If the input amplitude range is divided into K quantization intervals of equal width Δ (*quantization step*) and the output levels are uniformly spaced, the resulting quantizer is called *uniform*.



Quantization noise

- The two major types of error introduced by an ADC are aliasing error and *quantization error*.



- Since quantization is a nonlinear operation, analysis of quantization error is done using statistical techniques.
- If there is a large number of small quantization intervals, the signal $x_c(t)$ can be assumed to be approximately linear between quantization levels. In this case:

$$e_c(t) \triangleq x_q(t) - x_c(t) = \frac{\Delta}{2\tau}t, \quad -\tau \leq t \leq \tau$$

- Then the mean squared quantization error power is

$$P_Q = \frac{1}{2\tau}\int_{-\tau}^{\tau} |e_c(t)|^2 dt = \frac{\Delta^2}{12}$$

D/A convertion

- A band-limited signal can be reconstructed from a sequence of samples using the ideal DAC described by

$$x_r(t) = \sum_n x[n]\, g_{BL}(t - nT) = \sum_n x[n]\, \mathrm{sinc}(t/T - n)$$

- A system that implements the above formula, for an arbitrary function $g_r(t)$, is known as a *practical digital-to-analog converter (DAC)*.
- The function $g_r(t)$ is also known as the *characteristic pulse* of a DAC. At each sample time $t = nT$, the converter generates a pulse $g_r(t - nT)$ scaled by $x[n]$.
- In particular, the *switch-and-hold* DAC performs the following operation

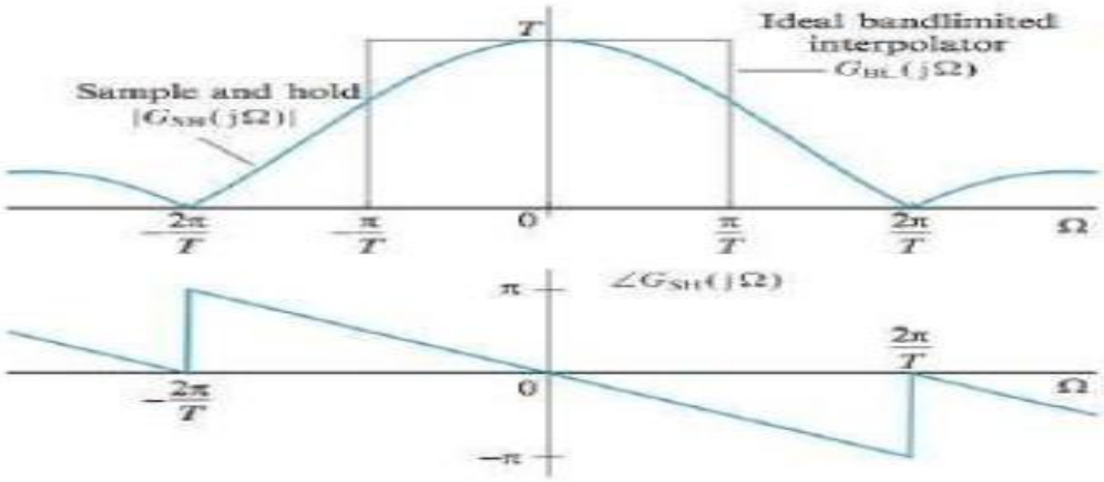
$$x_{SH}(t) = \sum_n x_q[n]\, g_{SH}(t - nT)$$

where

$$g_{SH}(t) = \begin{cases} 1, & 0 \leq t \leq T \\ 0, & \text{otherwise} \end{cases} \quad \Longleftrightarrow \quad G_{SH}(j\Omega) = \frac{2\sin(\Omega T/2)}{\Omega} e^{-j\Omega T/2}$$

D/A convertion

- The S/H circuit cannot completely eliminate the spectral replicas introduced by the sampling process.
- Moreover, it introduces amplitude distortion in the Nyquist band $|F_s| < F_s/2$.



- To compensate for the effects of the S/H circuit, we can use an analog post-filter $H_r(j\Omega)$ so that $G_{SH}(j\Omega)\,H_r(j\Omega) = G_{BL}(j\Omega)$:

$$H_r(j\Omega) = \begin{cases} \dfrac{\Omega T/2}{\sin(\Omega T/2)} e^{j\Omega T/2}, & |\Omega| < \pi/T \\ 0, & \text{otherwise} \end{cases}$$

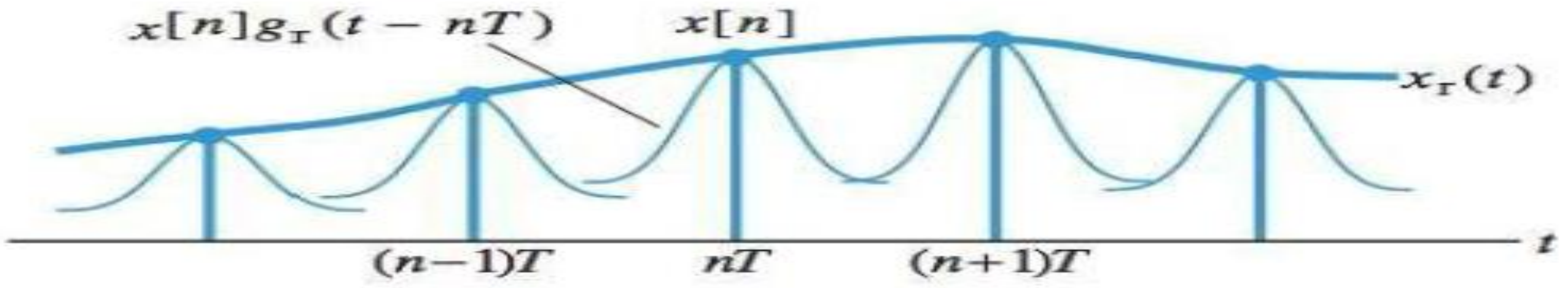
Reconstruction

- A general formula that describes a broad class of reconstruction processes is given by

$$x_r(t) = \sum_n x[n] g_r(t - nT)$$

where $g_r(t)$ is an interpolating reconstruction function.

- The process of fitting a continuous function to a set of samples is known as an *interpolation*.



- Thus, if the interpolation function has duration greater than or equal to T, the addition of the overlapping copies "fills the gaps" between samples.

Reconstruction

- In the Fourier domain, the interpolation formula becomes

$$X_r(j\Omega) = \sum_n x[n] G_r(j\Omega) e^{-j\Omega T} = G_r(j\Omega) \underbrace{\sum_n x[n] e^{-j\Omega T}}_{X(e^{j\Omega T})}$$

- Consequently, we obtain

$$X_r(j\Omega) = G_r(j\Omega)\, X(e^{j\Omega T})$$

- Specifically, if we choose $g_r(t)$ so that

$$G_r(j\Omega) \triangleq G_{BL}(j\Omega) = \begin{cases} T, & |\Omega| \leq \Omega_s/2 \\ 0, & |\Omega| > \Omega_s/2 \end{cases}$$

then $X_r(j\Omega) = X_c(j\Omega)$ and, therefore, $x_r(t) = x_c(t)$.

Reconstruction

- Evaluating the inverse Fourier transform of $G_{BL}(j\Omega)$, we obtain

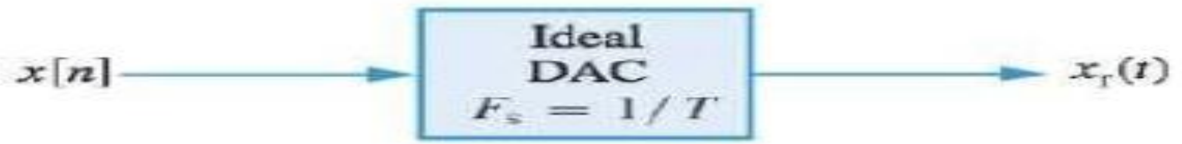
$$g_r(t) \triangleq g_{BL}(t) = \frac{\sin(\pi t/T)}{\pi t/T} = \mathrm{sinc}(t/T)$$

- In this case we obtain:

The ideal interpolation formula

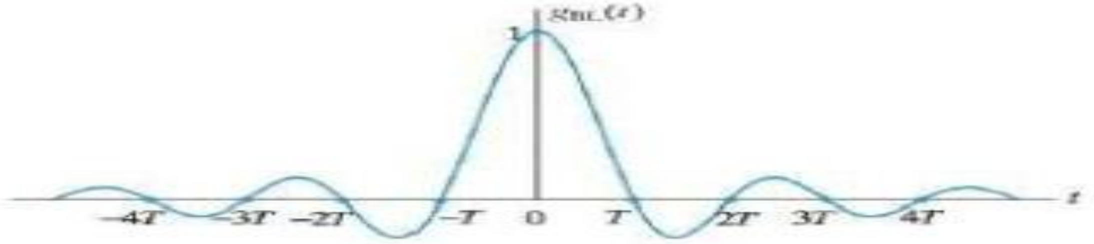
$$x_c(t) = \sum_n x[n] \frac{\sin[\pi(t - nT)/T]}{\pi(t - nT)/T}$$

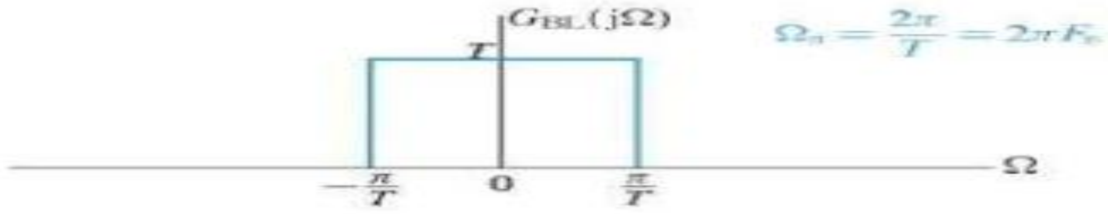
- The system used to implement the ideal interpolation is known as an *ideal DAC*.



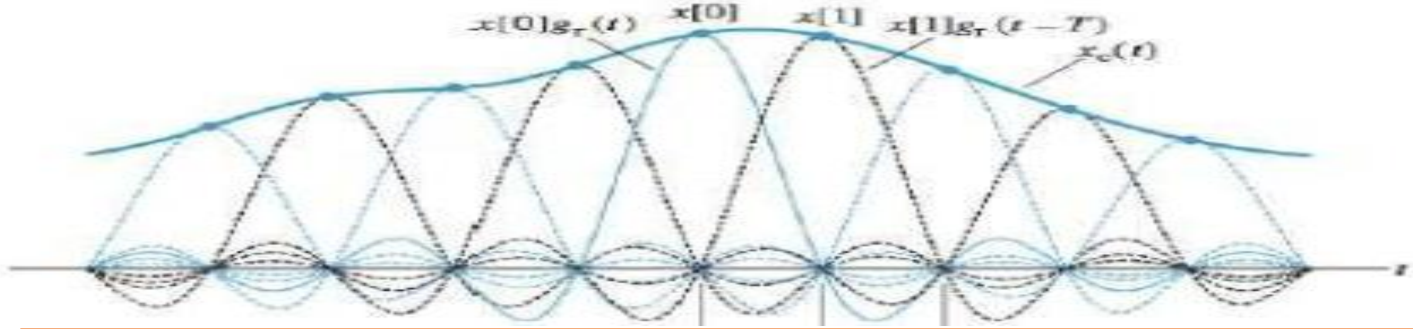
Reconstruction

- To understand the meaning and implications of the ideal interpolation we look more closely at the sinc function $g_{BL}(t)$.





- We note that $g_{BL}(t) = 0$ at all $t = nT$, except at $t = 0$ where $g_{BL}(t) = 1$. Thus, it is always true that $x_r(nT) = x_c(nT)$ regardless of whether aliasing occurred during sampling.



Signals

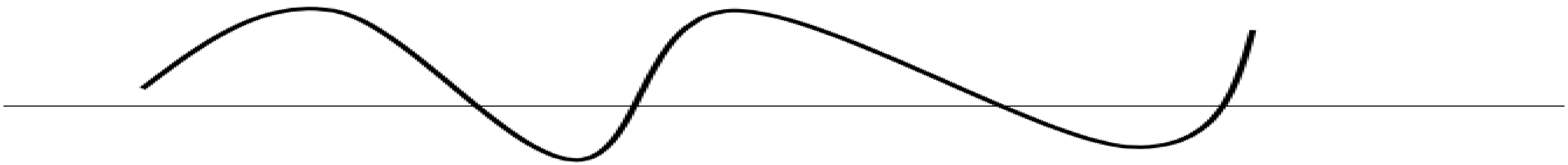
- *Continuous-time* signals are functions of a real argument
  $x(t)$ where t can take any real value
  $x(t)$ may be 0 for a given range of values of t
- *Discrete-time signals* are functions of an argument that takes values from a discrete set
  $x[n]$ where $n \in \{...-3,-2,-1,0,1,2,3...\}$
  Integer index n instead of time t for discrete-time systems
- $\boldsymbol{x}$ may be an array of values (multi channel signal)
- Values for $\boldsymbol{x}$ may be real or complex

Discrete-time Signals and Systems

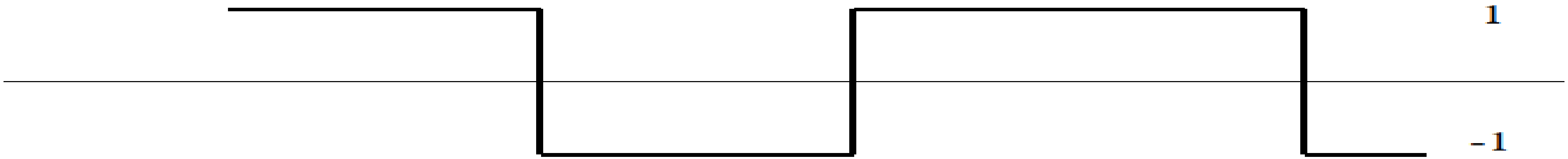
- *Continuous-time signals* are defined over a continuum of times and thus are represented by a continuous independent variable.
- *Discrete-time signals* are defined at discrete times and thus the independent variable has discrete values.
- *Analog signals* are those for which both time and amplitude are continuous.
- *Digital signals* are those for which both time and amplitude are discrete.

Analog vs. Digital

- The amplitude of an analog signal can take any real or complex value at each time/sample

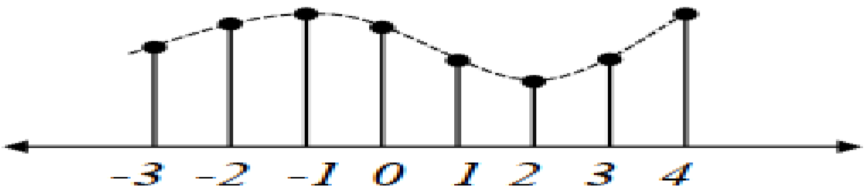


- The amplitude of a digital signal takes values from a discrete set



Periodic (Uniform) Sampling

- Sampling is a continuous to discrete-time conversion



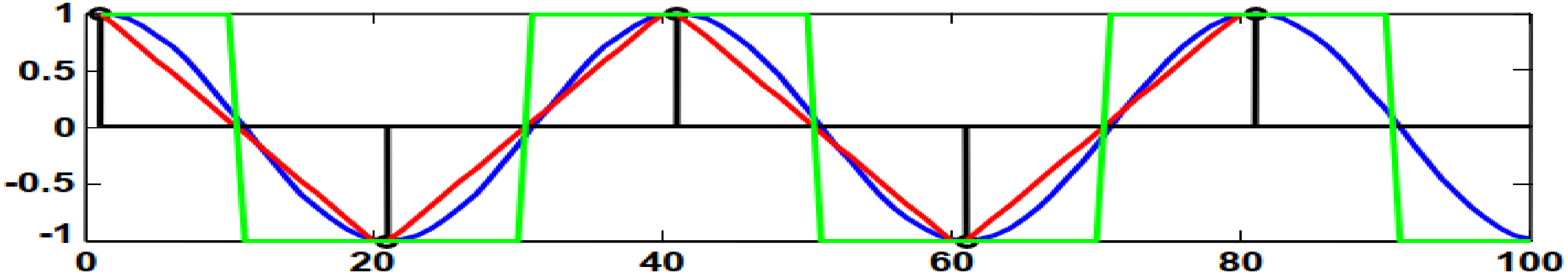
- Most common sampling is periodic

$$x[n] = x_c(nT) \qquad -\infty < n < \infty$$

- T is the sampling period in second
- $f_s = 1/T$ is the sampling frequency in Hz
- Sampling frequency in radian-per-second $\Omega_s = 2\pi f_s$ rad/sec
- Use [.] for discrete-time and (.) for continuous time signals
- This is the ideal case not the practical but close enough
  - In practice it is implement with an analog-to-digital converters
  - We get digital signals that are quantized in amplitude and time

Periodic Sampling

- Sampling is, in general, not reversible
- Given a sampled signal one could fit infinite continuous signals through the samples

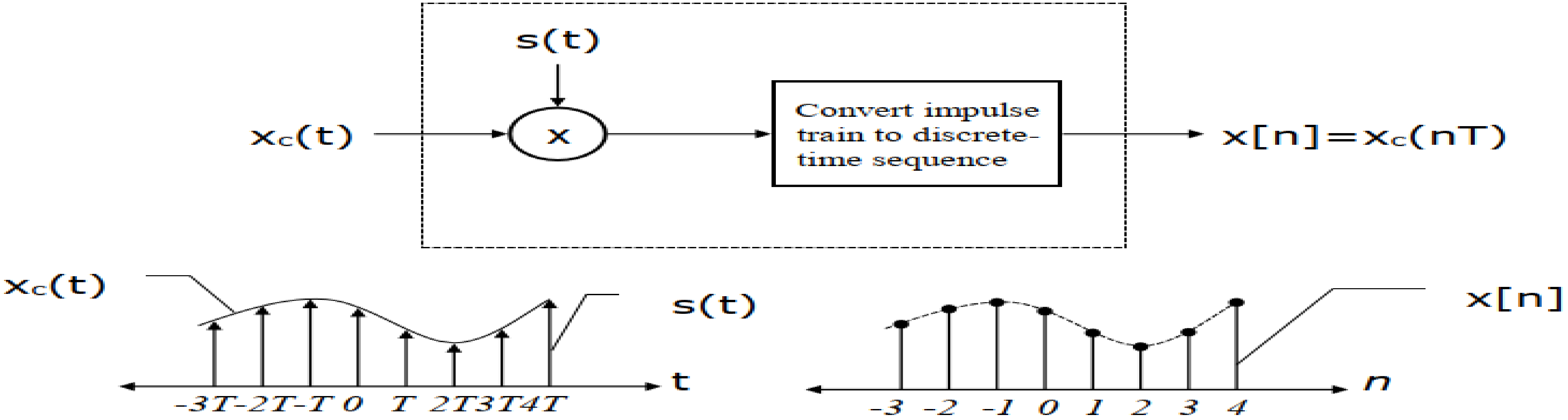


- Fundamental issue in digital signal processing
  - If we loose information during sampling we cannot recover it
- *Under certain conditions an analog signal can be sampled without loss so that it can be reconstructed perfectly*

Representation of Sampling

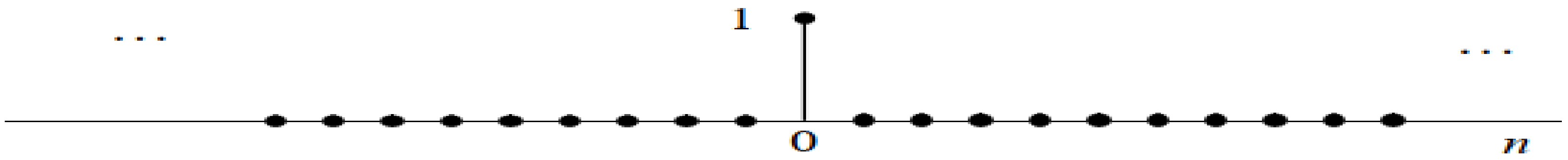
Mathematically convenient to represent in two stages

- Impulse train modulator
- Conversion of impulse train to a sequence



Unit Sample Sequence

$$\delta[n] = 0, \quad n \neq 0$$
$$= 1, \quad n = 0.$$



The unit sample sequence plays the same role for discrete-time sequences and systems that the unit impulse (Dirac delta function) does for continuous-time signals and systems.

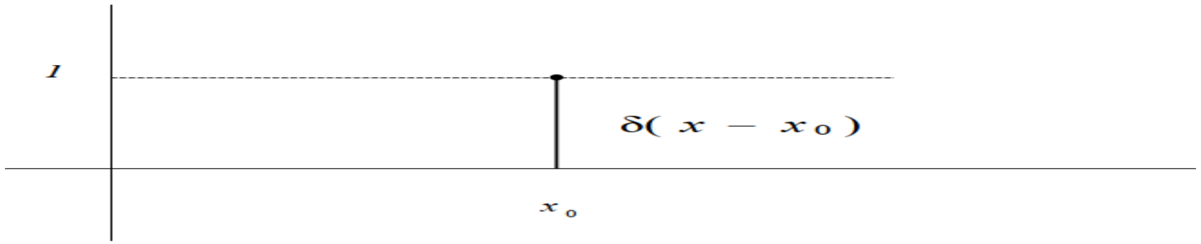
Impulse Function

The *impulse function*, also known as Dirac's delta function, is used to represented quantities that are highly localized in space. Examples include point optical sources and electrical charges.

The impulse function can be visualized as a narrow spike having infinite height and zero width, such that its area is equal to unity.

Graphical Representation

On graphs we will represent $\delta(x-x_0)$ as a spike of unit height located at the point x_0.



Definition of Impulse Function

The impulse function may be defined from its basic properties.

$$\delta(x - x_0) = 0, \quad x \neq x_0$$

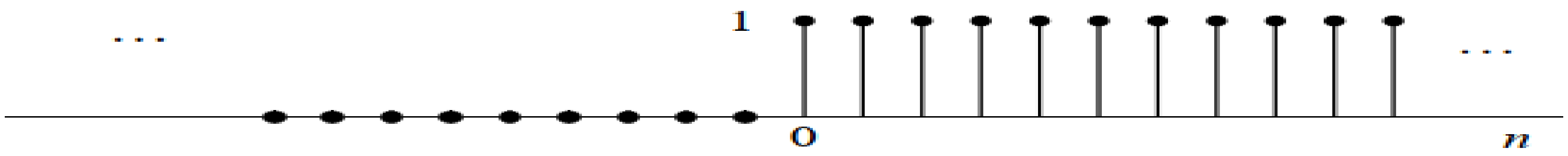
$$\int_{x_1}^{x_2} f(x)\,\delta(x - x_0)\,dx = f(x_0), \quad x_1 < x_0 < x_2$$

Where $f(x)$ is any complex-valued function of x. If $f(x)$ is discontinuous at the point x_0, the value of $f(x_0)$ is taken as the average of the limiting values as x approaches x_0 from above and below.

This property is called the *sifting* property.

Unit Step Sequence

$$u[n] = \begin{cases} 1, & n \geq 0 \\ 0, & n < 0. \end{cases}$$



$$u[n] = \delta[n] + \delta[n-1] + \delta[n-2] + \cdot$$

$$u[n] = \sum_{k=0}^{\infty} \delta[n-k]$$

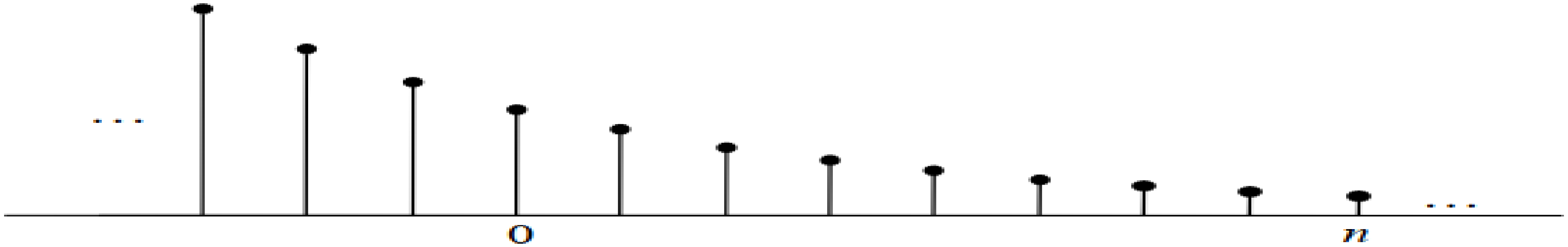
r $$u[n] = \sum_{k=-\infty}^{n} \delta[k]$$

Conversely, the impulse sequence can be expressed as the first backward difference of the unit step sequence:

$$\delta[n] = u[n] - u[n-1]$$

Exponential Sequence

$$x[n] = A\alpha^n$$



If we want an exponential sequence that is zero for $n < 0$, we can write this as:

$$x[n] = A\alpha^n u[n]$$

Geometric Series

A one-sided exponential sequence of the form

α^n, for $n \geq 0$ and α an arbitrary constant

is called a *geometric series*. The series converges for $|a| < 1$, and its sum converges to

$$\sum_{n=0}^{\infty} \alpha^n \rightarrow \frac{1}{1-\alpha}$$

The sum of a finite number N of terms is

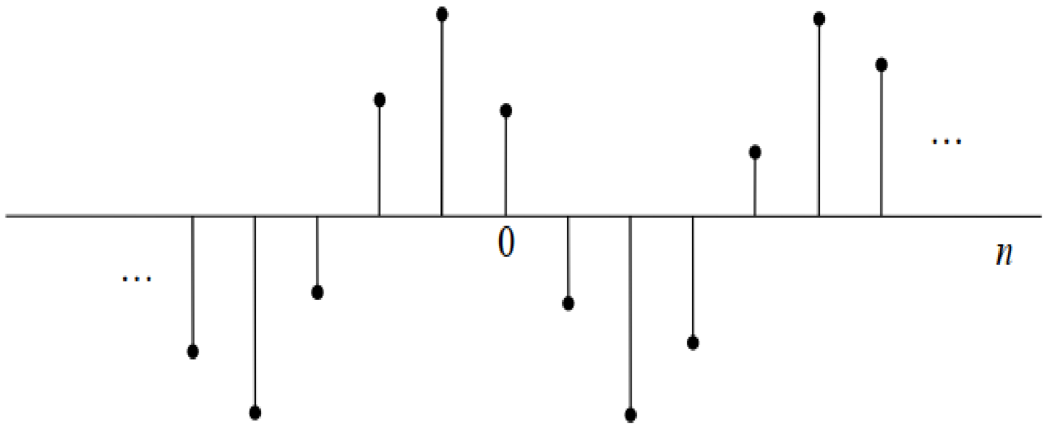
$$\sum_{n=0}^{N} \alpha^n \rightarrow \frac{1-\alpha^{N+1}}{1-\alpha}$$

A general form can also be written:

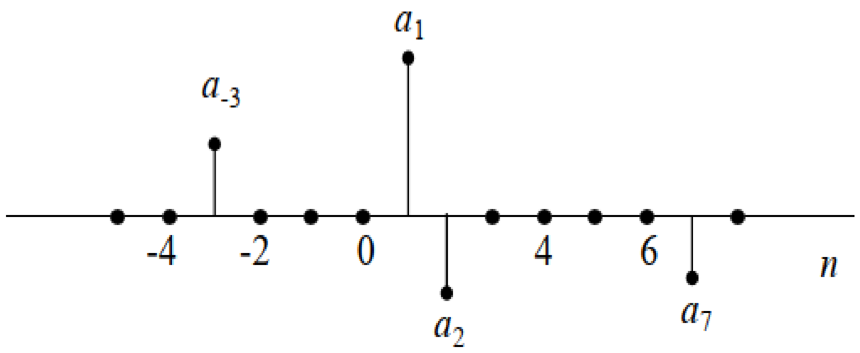
$$\sum_{n=N_1}^{N_2} \alpha^n \rightarrow \frac{\alpha^{N_1}-\alpha^{N_2+1}}{1-\alpha}$$

Sinusoidal Sequence

$$x[n] = A\cos(\omega_o n + \phi)$$



Sequence as a sum of scaled, delayed impulses



$$p[n] = a_{-3}\,\delta[n+3] + a_1\,\delta[n-1] - a_2\,\delta[n-2] - a_7\,\delta[n-7]$$

Sequence Operations

- The product and sum of two sequences are defined as the sample-by-sample product and sum, respectively.
- Multiplication of a sequence by a number is defined as multiplication of each sample value by this number.
- A sequence $y[n]$ is said to be a delayed or shifted version of a sequence $x[n]$ if

$$y[\mathrm{n}] = x[n - n_d]$$

where n_d is an integer.

- Combination of Basic Sequences

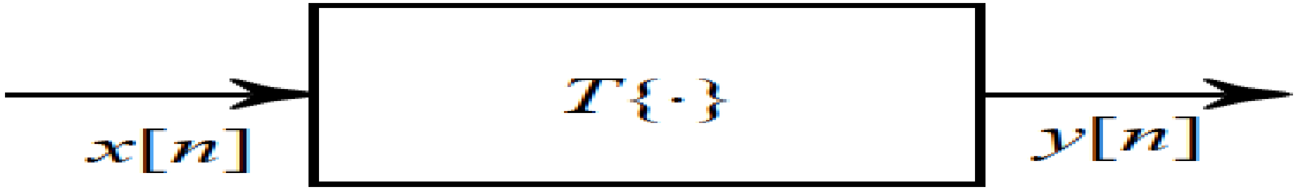
Ex 1

$$x[n] = K\alpha^n, \quad n \geq 0,$$
$$= 0, \quad n < 0,$$

or

$$x[n] = K\alpha^n u[n].$$

Systems



A discrete-time system is a transformation that maps an input sequence $x[n]$ into an output sequence $y[n]$.

System Characteristics

1. Linear vs. non-linear
2. Causal vs. non-causal
3. Time invariant

System Characteristics

$$x[n] \longrightarrow \boxed{T\{\bullet\}} \longrightarrow y[n]$$

1. Linear vs. non-linear
2. Time invariant vs. time variant
3. Causal vs. non-causal
4. Stable vs. unstable
5. Memoryless vs. state-sensitive
6. Invertible vs. non-invertible