Module 3.a
RESONANT CIRCUITS

Objectives:

1. To study resonant circuits both in time and frequency domains.
2. Describe the conditions for electrical resonance.
3. Describe the mathematical strategy to develop the resonant frequency expression for a given resonant circuit.
4. Determine the resonant frequency of series, parallel, and series–parallel circuits.
5. Describe the quality factor.
6. Determine the quality factor of series, parallel, and series–parallel circuits.
7. Determine the three dB bandwidth from the resonant frequency and quality factor.
8. Decide whether a resonant circuit has a low Q or a high Q in order to select the 3 dB determination approach.

6.1 Introduction

A.C Circuits made up of resistors, inductors and capacitors are said to be resonant circuits when the current drawn from the supply is in phase with the impressed sinusoidal voltage. Then

1. the resultant reactance or susceptance is zero.
2. the circuit behaves as a resistive circuit.
3. the power factor is unity.

A second order series resonant circuit consists of R, L and C in series. At resonance, voltages across C and L are equal and opposite and these voltages are many times greater than the applied voltage. They may present a dangerous shock hazard.

A second order parallel resonant circuit consists of R, L and C in parallel. At resonance, currents in L and C are circulating currents and they are considerably larger than the input current. Unless proper consideration is given to the magnitude of these currents, they may become very large enough to destroy the circuit elements.

Resonance is the phenomenon which finds its applications in communication circuits: The ability of a radio or Television receiver (1) to select a particular frequency or a narrow band of frequencies transmitted by broad casting stations or (2) to suppress a band of frequencies from other broad casting stations, is based on resonance.

Thus resonance is desired in tuned circuits, design of filters, signal processing and control engineering. But it is to be avoided in other circuits. It is to be noted that if $R = 0$ in a series RLC circuit, the circuits acts as a short circuit at resonance and if $R = \infty$ in parallel RLC circuit, the circuit acts as an open circuit at resonance.

6.2 Transfer Functions

As ω is varied to achieve resonance, electrical quantities are expressed as functions of ω, normally denoted by $F(j\omega)$ and are called transfer functions. Accordingly the following notations are used.

$$Z(j\omega) = \frac{V(j\omega)}{I(j\omega)} = \text{Impedance function}$$
$$Y(j\omega) = \frac{I(j\omega)}{V(j\omega)} = \text{Admittance function}$$
$$G(j\omega) = \frac{V_2(j\omega)}{V_1(j\omega)} = \text{Voltage ratio transfer function}$$
$$\alpha(j\omega) = \frac{I_2(j\omega)}{I_1(j\omega)} = \text{Current ratio transfer function}$$

If we put $j\omega = s$ then the above quantities will be $Z(s), Y(s), G(s), \alpha(s)$ respectively. These are treated later in this book.

6.3 Series Resonance

Fig. 6.1 represents a series resonant circuit.

Resonance can be achieved by

1. varying frequency ω
2. varying the inductance L
3. varying the capacitance C

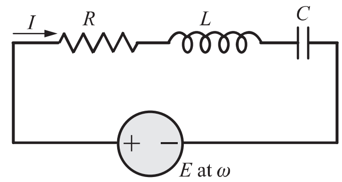

Figure 6.1 Series Resonant Circuit

The current in the circuit is

$$I = \frac{E}{R + j(X_L - X_C)} = \frac{E}{R + jX}$$

At resonance, X is zero. If ω_0 is the frequency at which resonance occurs, then $\omega_0 L = \dfrac{1}{\omega_0 C}$ or $\omega_0 = \dfrac{1}{\sqrt{LC}}$ = resonant frequency.

The current at resonance is $I_m = \dfrac{V}{R}$ = maximum current.

The phasor diagram for this condition is shown in Fig. 6.2.

The variation of current with frequency is shown in Fig. 6.3.

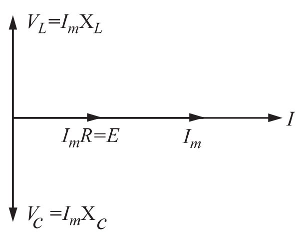

Figure 6.2

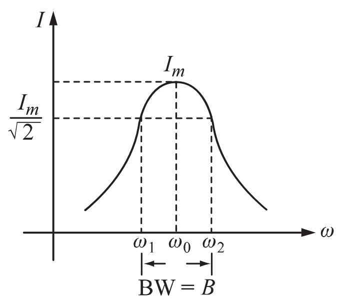

Figure 6.3

It is observed that there are two frequencies, one above and the other below the resonant frequency, ω_0 at which current is same.

Fig. 6.4 represents the variations of $X_L = \omega L; X_C = \dfrac{1}{\omega C}$ and $|Z|$ with ω.

From the equation $\omega_0 = \dfrac{1}{\sqrt{LC}}$ we see that any constant product of L and C give a particular resonant frequency even if the ratio $\dfrac{L}{C}$ is different. The frequency of a constant frequency source can also be a resonant frequency for a number of L and C combinations. Fig. 6.5 shows how the sharpness of tuning is affected by different $\dfrac{L}{C}$ ratios, but the product LC remaining constant.

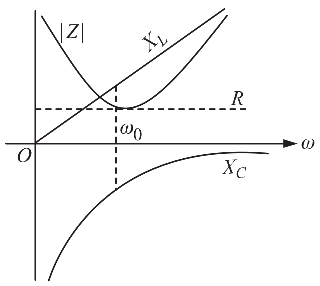

Figure 6.4

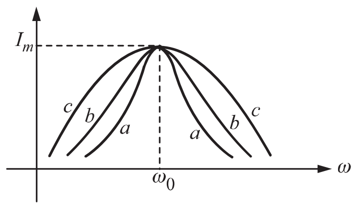

Figure 6.5

For larger $\dfrac{L}{C}$ ratio, current varies more abruptly in the region of ω_0. Many applications call for narrow band that pass the signal at one frequency and tend to reject signals at other frequencies.

6.4 Bandwidth, Quality Factor and Half Power Frequencies

At resonance $I = I_m$ and the power dissipated is

$$P_m = I_m^2 R \text{ watts.}$$

When the current is $I = \dfrac{I_m}{\sqrt{2}}$ power dissipated is

$$\frac{P_m}{2} = \frac{I_m^2 R}{2} \text{ watts.}$$

From $\omega - I$ characteristic shown in Fig. 6.3, it is observed that there are two frequencies ω_1 and ω_2 at which the current is $I = \dfrac{I_m}{\sqrt{2}}$. As at these frequencies the power is only one half of that at ω_0, these are called half power frequencies or cut off frequencies.

The ratio, $$\frac{\text{current at half power frequencies}}{\text{Maximum current}} = \frac{I_m}{\sqrt{2} I_m} = \frac{1}{\sqrt{2}}$$

When expressed in dB it is $20 \log \dfrac{1}{\sqrt{2}} = -3\text{dB}$.

Therefore ω_1 and ω_2 are also called -3 dB frequencies.

As $\frac{I_m}{\sqrt{2}} = \frac{E}{\sqrt{2}R}$, the magnitude of the impedance at half-power frequencies is $\sqrt{2}R = |R + j(X_L - X_C)|$

Therefore, the resultant reactance, $X = X_L - X_C = R$.

The frequency range between half - power frequencies is $\omega_2 - \omega_1$, and it is referred to as passband or band width.

$$\text{BW} = \omega_2 - \omega_1 = B.$$

The sharpness of tuning depends on the ratio $\frac{R}{L}$, a small ratio indicating a high degree of selectivity. The quality factor of a circuit can be expressed in terms of R and L of the inductor.

$$\text{Quality factor} = Q = \frac{\omega_0 L}{R}$$

Writing $\omega_0 = 2\pi f_0$ and multiplying numerator and denominator by $\frac{1}{2}I_m^2$, we get,

$$Q = 2\pi f_0 \frac{\frac{1}{2}L{I_m}^2}{\frac{1}{2}{I_m}^2 R} = 2\pi \times \frac{\frac{1}{2}L{I_m}^2}{\frac{1}{2}{I_m}^2 RT}$$
$$= 2\pi \times \frac{\text{Maximum energy stored}}{\text{total energy lost in a period}}$$

Selectivity is the reciprocal of Q.

As $\quad Q = \frac{\omega_0 L}{R}$ and $\omega_0 L = \frac{1}{\omega_0 C}$,

$$Q = \frac{1}{\omega_0 CR}$$

and since $\omega_0 = \frac{1}{\sqrt{LC}}$, we have

$$Q = \frac{1}{R}\sqrt{\frac{L}{C}}$$

6.5 Expressions for ω_1 and ω_2, and Bandwidth

At half power frequencies ω_1 and ω_2,

$$I = \frac{E}{\sqrt{2}R} = \frac{E}{\{R^2 + (X_L - X_C)^2\}^{\frac{1}{2}}}$$

$$\therefore \qquad |X_L - X_C| = R \quad \text{i.e.,} \quad \left|\omega L - \frac{1}{\omega C}\right| = R$$

At $\omega = \omega_2$, $\qquad R = \omega_2 L - \frac{1}{\omega_2 C}$

Simplifying, $\quad \omega_2^2 LC - \omega_2 CR - 1 = 0$

Solving, we get

$$\omega_2 = \frac{RC + \sqrt{R^2C^2 + 4LC}}{2LC} = \frac{R}{2L} + \sqrt{\left(\frac{R}{2L}\right)^2 + \frac{1}{LC}} \tag{6.1}$$

Note that only $+$ sign is taken before the square root. This is done to ensure that ω_2 is always positive.

At $\omega = \omega_1$,

$$R = \frac{1}{\omega_1 C} - \omega_1 L$$

$$\Rightarrow \quad \omega_1^2 LC + \omega_1 CR - 1 = 0$$

Solving,

$$\omega_1 = \frac{-RC + \sqrt{R^2C^2 + 4LC}}{2LC}$$

$$= \frac{-R}{2L} + \sqrt{\left(\frac{R}{2L}\right)^2 + \frac{1}{LC}} \tag{6.2}$$

While determining ω_1, only positive value is considered.

Subtracting equation(6.1) from equation (6.2), we get

$$\omega_2 - \omega_1 = \frac{R}{L} = \text{Band width.}$$

Since $Q = \dfrac{\omega_0 L}{R}$, Band width is expressed as

$$B = \omega_2 - \omega_1 = \frac{R}{L} = \frac{\omega_0}{Q}.$$

and therefore

$$Q = \frac{\omega_0}{\omega_2 - \omega_1} = \frac{\omega_0}{B}$$

Multiplying equations (6.1) and (6.2), we get

$$\omega_1\omega_2 = \frac{R^2}{4L^2} + \frac{1}{LC} - \frac{R^2}{4L^2} = \frac{1}{LC} = \omega_0^2$$

or

$$\omega_0 = \sqrt{\omega_1\omega_2}$$

The resonance frequency is the geometric mean of half power frequencies.

Normally $\dfrac{R}{2L} << \dfrac{1}{\sqrt{LC}}$, in which case $Q \geq 5$

Then,

$$\omega_1 \simeq -\frac{R}{2L} + \sqrt{\frac{1}{LC}} \quad \text{and} \quad \omega_2 \simeq \frac{R}{2L} + \frac{1}{\sqrt{LC}}$$

$$= \frac{R}{2L} + \omega_0 \quad \text{and} \quad \omega_2 = \frac{R}{2L} + \omega_0$$

$$\therefore \quad \omega_0 = \frac{\omega_1 + \omega_2}{2} = \text{Arithmetic mean of } \omega_1 \text{ and } \omega_2$$

Since $\dfrac{R}{L} = \dfrac{\omega_0}{Q}$, Equations for ω_1 and ω_2 as given by equations (6.1) and (6.2) can be expressed in terms of Q as

$$\omega_2 = \frac{\omega_0}{2Q} + \sqrt{\left(\frac{\omega_0}{2Q}\right)^2 + \omega_0^2}$$

$$= \omega_0 \left[\frac{1}{2Q} + \sqrt{1 + \left(\frac{1}{2Q}\right)^2} \right]$$

Similarly
$$\omega_1 = \omega_0 \left[-\frac{1}{2Q} + \sqrt{1 + \left(\frac{1}{2Q}\right)^2} \right]$$

Normally, $\frac{R}{2L} << \frac{1}{\sqrt{LC}}$ and then $Q > 5$.

Consequently ω_1 and ω_2 can be approximated as

$$\omega_1 \simeq -\frac{R}{2L} + \sqrt{\frac{1}{LC}} = -\frac{R}{2L} + \omega_0 = -\frac{B}{2} + \omega_0$$
$$\omega_2 \simeq \frac{R}{2L} + \sqrt{\frac{1}{LC}} = +\frac{R}{2L} + \omega_0 = \frac{B}{2} + \omega_0$$

so that
$$\omega_0 = \frac{\omega_1 + \omega_2}{2}.$$

6.6 Frequency Response of Voltage across L and C

As frequency is varied, both the voltages across L and C increase with frequency upto ω_0 and they are equal at ω_0. But their maximum values do not occur at ω_0. V_c reaches its maximum at $\omega < \omega_0$ and V_L reaches its maximum at $\omega > \omega_0$. This can be verified by calculating the frequency at which each occurs.

6.7 Expression for ω at which V_L is Maximum

Current in the circuit shown in Figure 6.1 is

$$I = \frac{E}{\sqrt{R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2}}$$

Voltage across L is

$$V_L = \omega L I = \frac{E \omega L}{\sqrt{R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2}}$$

Squaring

$$V_L^2 = \frac{E^2 \omega^2 L^2}{R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2}$$

This is maximum when $\frac{dV_L{}^2}{d\omega} = 0$

That is,

$$E^2L^2\left[\left\{R^2+\left(\omega C-\frac{1}{\omega C}\right)^2\right\}2\omega-\omega^2\left\{2\left(\omega L-\frac{1}{\omega C}\right)\left(L+\frac{1}{\omega^2 C}\right)\right\}\right]=0$$

$$R^2+\left(\omega L-\frac{1}{\omega C}\right)^2=\left(\omega L-\frac{1}{\omega C}\right)\left(\omega L+\frac{1}{\omega C}\right)$$

$$R^2+\omega^2\cancel{L^2}+\frac{1}{\omega^2C^2}-2\frac{L}{C}=\omega^2\cancel{L^2}-\frac{1}{\omega^2C^2}$$

$$R^2\omega^2C^2+1-2\omega^2LC=-1$$

or

$$\omega^2(2LC-R^2C^2)=2$$

$$\omega^2=\frac{2}{2LC-R^2C^2}$$

$$=\frac{1}{LC\left(1-\frac{R^2C}{2L}\right)}$$

Let this frequency be ω_L.

Then,

$$\omega_L^2=\omega_0^2\frac{1}{1-\frac{1}{2Q^2}}$$

$$\omega_L=\omega_0\sqrt{\frac{1}{1-\frac{1}{2Q^2}}}.$$

That is, $\omega_L>\omega_0$.

6.8 Expression for ω at which V_C is Maximum

Now

$$V_C=\frac{E}{\omega C\sqrt{R^2+\left(\omega^2L-\frac{1}{\omega C}\right)^2}}$$

$$V_C^2=\frac{E^2}{\omega^2C^2\left(R^2+\left(\omega L-\frac{1}{\omega C}\right)^2\right)}$$

This is maximum when $\dfrac{d}{d\omega}(V_C^2)=0$.
That is,

$$-\frac{E^2}{C^2}\left[\omega^2\left\{2\left(\omega L-\frac{1}{\omega C}\right)\left(L-\frac{1}{\omega^2C}\right)+2\omega\left\{R^2+\left(\omega L-\frac{1}{\omega C}\right)^2\right\}\right]=0$$

$$R^2+\left(\omega L-\frac{1}{\omega C}\right)^2=-\left(\omega L-\frac{1}{\omega C}\right)\left(\omega L+\frac{1}{\omega C}\right)$$

$$R^2 + \omega^2 L^2 + \cancel{\frac{1}{\omega^2 C^2}} - 2\frac{L}{C} = \cancel{\frac{1}{\omega^2 C^2}} - \omega^2 L^2$$

$$2\omega^2 L^2 + R^2 = 2\frac{L}{C}$$

$$\omega^2 = \frac{2\frac{L}{C} - R^2}{2L^2} = \frac{1}{LC} - \frac{R^2}{2L^2}$$

$$= \frac{1}{LC}\left(1 - \frac{R^2}{2}\frac{C}{L}\right) = \omega_0^2\left(1 - \frac{1}{2Q^2}\right)$$

Let this frequency be ω_C

$$\omega_C = \omega_0\sqrt{1 - \frac{1}{2Q^2}}$$

i.e., $\omega_C < \omega_0$

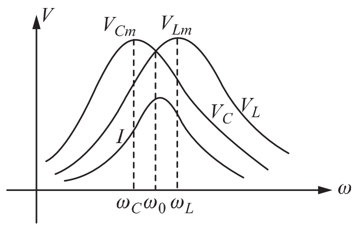

Figure 6.6

Variations of V_C and V_L as functions of ω are shown in Fig. 6.6.

We know that $$V_C = \frac{E}{\sqrt{\omega^2 C^2\left\{R^2 + \frac{(\omega^2 LC - 1)^2}{\omega^2 C^2}\right\}}} = \frac{E}{\sqrt{\{R^2\omega^2 C^2 + (\omega^2 LC - 1)^2\}}} \qquad (6.3)$$

Consider $\omega^2 C^2 R^2 + (\omega^2 LC - 1)^2$ and at $\omega = \omega_C$. Then equation(6.3) becomes

$$\omega_C^2 C^2 R^2 + (\omega_C^2 LC - 1)^2 = \omega_0^2\left(1 - \frac{1}{2Q^2}\right)C^2R^2 + \left\{\omega_0^2\left(1 - \frac{1}{2Q^2}\right)LC - 1\right\}^2$$

$$= \frac{1}{Q^2}\left(1 - \frac{1}{2Q^2}\right) + \left\{\cancel{\omega_0^2}\left(1 - \frac{1}{2Q^2}\right)\frac{1}{\cancel{\omega_0^2}} - 1\right\}^2$$

$$= \frac{1}{Q^2}\left(1 - \frac{1}{2Q^2}\right) + \left(\frac{1}{4Q^4}\right) = \frac{1}{Q^2} - \frac{1}{2Q^4} + \frac{1}{4Q^4} = \frac{1}{Q^2}\left[1 - \frac{1}{4Q^2}\right]$$

since $\frac{1}{LC} = \omega_0^2$ and $\omega_0 CR = \frac{1}{Q}$

Substituting the above expression in the denominator of equation (6.3), we get

$$V_{cm} = \frac{EQ}{\sqrt{1 - \frac{1}{4Q^2}}}$$

6.9 Selectivity with Variable L

In a series resonant circuit connected to a constant voltage, with a constant frequency, when L is varied to achieve resonance, the following conditions prevail:

1. X_C is constant and $I = \dfrac{E}{\sqrt{R^2 + X_C^2}}$ when $L = 0$.
2. With increase in L, X_L increases and $I_m = \dfrac{V}{R}$ at $X_L = X_C$
3. With further increase in L, I proceeds to fall.

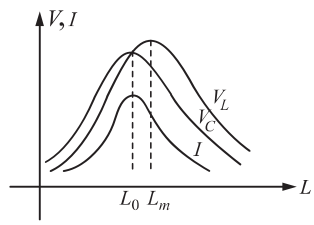

Figure 6.7

All these conditions are depicted in Fig. 6.7 $V_{C\,\max}$ occurs at ω_0 but $V_{L\,\max}$ occurs at a point beyond ω_0.
L at which V_L becomes a maximum is obtained in terms of other constants.

$$V_L = \frac{EX_L}{\{R^2 + (X_L - X_C)^2\}^{\frac{1}{2}}}$$

$$V_L^2 = \frac{E^2 X_L^2}{R^2 + (X_L - X_C)^2}$$

This is maximum when $\dfrac{dV_L^2}{dX_L} = 0$.

$$\text{Therefore,} \qquad \left\{R^2 + (X_L - X_C)^2\right\} 2X_L = X_L^2 \left\{2(X_L - X_C)\right\}$$

$$R^2 + X_L^2 + X_C^2 - 2X_L X_C = X_L^2 - X_L X_C$$

$$\text{Therefore,} \qquad X_L = \frac{R^2 + X_C^2}{X_C}$$

Let the corresponding value of L is L_m.

$$\text{Then,} \qquad L_m = C(R^2 + X_C^2)$$

and L_0 = value of L at ω_0 such that

$$\omega_0 L = \frac{1}{\omega_0 C}.$$

6.10 Selectivity with Variable C

In a series resonant circuit connected to a constant voltage, constant frequency supply, if C is varied to achive resonance, the following conditions prevail:

1. X_L is constant.
2. X_C varies as inversely as C

   when $C = 0$, $\qquad I = 0$.

   when $\omega C = \dfrac{1}{\omega L}$, $\quad I = I_m = \dfrac{V}{R}$.
3. with further increase in C, I starts decreasing as shown in Fig. 6.8, where C_m is the value of capacitance at maximum voltage across C and C_0 is the value of the capacitance at ω_0.

C at which V_C becomes maximum can be determined in terms of other circuit constants as follows.

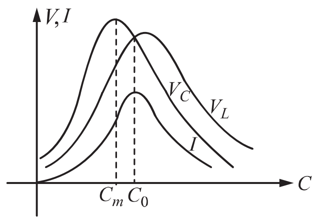

Figure 6.8

$$V_C = \frac{EX_C}{\sqrt{R^2 + (X_L - X_C)^2}}$$

$$V_C^2 = \frac{E^2 X_C^2}{R^2 + (X_L - X_C)^2}$$

For maximim V_C, $$\frac{dV_C^2}{dX_C} = 0$$

Then, $$\left\{R^2 + (X_L - X_C)^2\right\} 2X_C - X_C^2 \left\{2(X_L - X_C)(-1)\right\} = 0$$

$$R^2 + X_L^2 + \cancel{X_C^2} - 2X_L X_C = -X_L X_C + \cancel{X_C^2}$$

$$X_C = \frac{R^2 + X_L^2}{X_L}$$

Let the correrponding value of C be C_m.

Then, $$C_m = \frac{L}{R^2 + X_L^2}.$$

6.11 Transfer Functions

6.11.1 Voltage ratio transfer function of a series resonant circuit and frequency response

For the circuit shown in Fig. 6.9, we can write

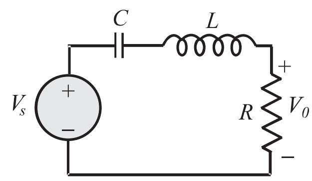

Figure 6.9

$$H(j\omega) = \frac{V_0(j\omega)}{V_s(j\omega)} = \frac{R}{R + j\left(\omega L - \frac{1}{\omega C}\right)}$$

$$= \frac{1}{1 + j\left\{\frac{\omega L}{R} - \frac{1}{\omega CR}\right\}}$$

$$= \frac{1}{1 + j\left\{\frac{\omega_0 L}{\omega_0 R}\omega - \frac{\omega_0}{\omega\omega_0 CR}\right\}}$$

$$= \frac{1}{1 + jQ\left[\frac{\omega}{\omega_0} - \frac{\omega_0}{\omega}\right]}$$

$$= \frac{1}{\left[1 + Q^2\left[\frac{\omega}{\omega_0} - \frac{\omega_0}{\omega}\right]^2\right]^{\frac{1}{2}}} \angle \tan^{-1}\left[Q\left(\frac{\omega}{\omega_0} - \frac{\omega_0}{\omega}\right)\right]$$

Let δ be a measure of the deviation in ω from ω_0. It is defined as

$$\delta = \frac{\omega - \omega_0}{\omega_0} = \frac{\omega}{\omega_0} - 1$$

Then $$\frac{\omega}{\omega_0} - \frac{\omega_0}{\omega} = (\delta + 1) - \frac{1}{\delta + 1} = \frac{(\delta + 1)^2 - 1}{\delta + 1} = \frac{\delta^2 + 2\delta}{\delta + 1}$$

For small deviations from $\omega_0, \delta << 1$. Then,

$$\frac{\omega}{\omega_0} - \frac{\omega_0}{\omega} \simeq 2\delta$$

Then, $$H(j\omega) = \frac{1}{1 + j2Q\delta} = \frac{1}{\sqrt{1 + 4Q^2\delta^2}} \underline{/ -\tan^{-1} 2Q\delta}$$

The amplitfude and phase response curves are as shown in Fig. 6.10.

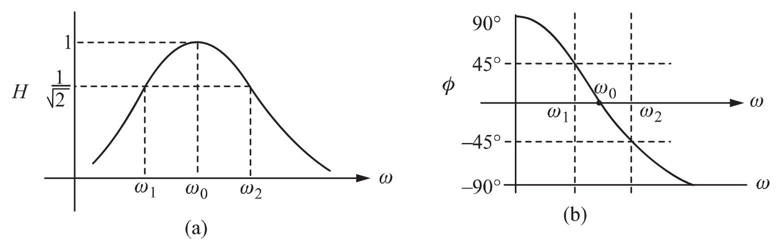

Figure 6.10 (a) and (b): Amplitude and Phase response of a series resonance circuit

6.11.2 Impedance function

The Impedance as a function of $j\omega$ is given by

$$\begin{aligned}
Z(j\omega) &= R + j\left(\omega L - \frac{1}{\omega C}\right) \\
&= R\left[1 + j\left(\frac{\omega L}{R} - \frac{1}{\omega C R}\right)\right] \\
&= R\left[1 + jQ\left(\frac{\omega}{\omega_0} - \frac{\omega_0}{\omega}\right)\right] \\
&= R\sqrt{1 + Q^2\left(\frac{\omega}{\omega_0} - \frac{\omega_0}{\omega}\right)^2} \underline{/ \tan^{-1} Q\left(\frac{\omega}{\omega_0} - \frac{\omega_0}{\omega}\right)}
\end{aligned}$$

For small deviations from ω_0, we can write

$$Z(j\omega) \simeq R[1 + j2Q\delta] = R\sqrt{1 + 4Q^2\delta^2} \underline{/ \tan^{-1} 2Q\delta}$$

6.12 Parallel Resonance

The dual of a series resonant circuit is often considered as a parallel resonant circuit and it is as shown in Fig. 6.11.

The phasor diagram for resonance is shown in Fig. 6.12.

The admittance as seen by the current source is

$$\begin{aligned} Y(j\omega) &= Y_R + Y_L + Y_C \\ &= \frac{1}{R} + j\left(\omega C - \frac{1}{\omega L}\right) = G + jB \end{aligned}$$

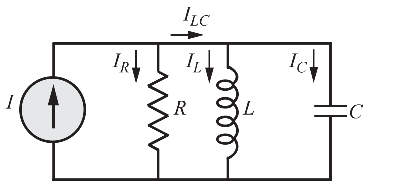

Figure 6.11 Parallel Resonance Circuit

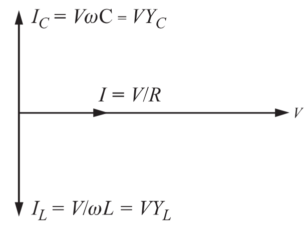

Figure 6.12 Phasor Diagram

If the resonance occurs at ω_0, then the susceptance B is zero. That is,

$$\omega_0 C = \frac{1}{\omega_0 L}$$

or

$$\omega_0 = \frac{1}{\sqrt{LC}} \text{ rad/ sec}.$$

At resonance,

$$I_{C0} = -I_{L0} = j\omega_0 CRI$$

and

$$I_{LC} = I_{C0} + I_{L0} = 0$$

The quality factor, as in the case of series resonant circuit is defined as

$$\begin{aligned} Q &= 2\pi \frac{\text{Maximum energy stored}}{\text{Energy dissipated in a period}} \\ &= 2\pi \frac{\frac{1}{2} C V_m^2}{\frac{1}{2}\frac{V_m^2}{R} T} \\ &= 2\pi f_0 CR = \omega_0 CR. \end{aligned}$$

Since $$\omega_0 C = \frac{1}{\omega_0 L},$$

$$Q = \frac{R}{\omega_0 L}.$$

On either side of ω_0 there are two frequencies at which the voltage is same. At resonance, the voltage is maximum and is given by $V_m = IR$ and is evident from the response curve as shown in Fig. 6.13. At this frequency, $p = p_m = \frac{V_m^2}{R}$ watts. The frequencies at which the voltage is $\frac{1}{\sqrt{2}}$ times the maximum voltage are called half power frequencies or cut off frequencies, since at these frequencies,

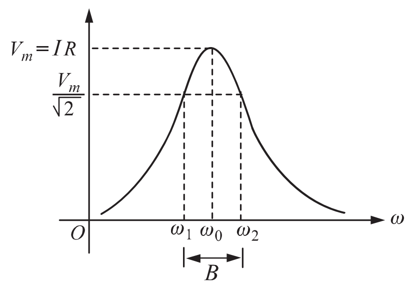

Figure 6.13

$$p = \frac{\left(\frac{V_m}{\sqrt{2}}\right)^2}{R} = \frac{V_m^2}{2R}$$ = half of the maximum power.

At any ω,

$$Y = \frac{1}{R} + j\left(\omega C - \frac{1}{\omega L}\right)$$

At ω_1 and ω_2,

$$|Y| = \frac{1}{\sqrt{2}R} = \sqrt{\left(\frac{1}{R}\right)^2 + \left(\omega C - \frac{1}{\omega L}\right)^2}$$

Squaring,

$$\frac{1}{2R^2} = \frac{1}{R^2} + \left(\omega C - \frac{1}{\omega L}\right)^2$$

Therefore, $$\left(\omega C - \frac{1}{\omega L}\right) = \frac{1}{R}$$

At $\omega = \omega_2$,

$$\omega_2 C - \frac{1}{\omega_2 L} = \frac{1}{R}$$

$$\omega_2^2 LC - 1 = \frac{\omega_2 L}{R}$$

$$\omega_2^2 LCR - R - \omega_2 L = 0$$

Hence, $$\omega_2 = \frac{L + \sqrt{L^2 + 4LCR^2}}{2LCR}$$

Note that only positive sign is used before the square root to ensure that ω_2 is positive.

Thus, $$\omega_2 = \frac{1}{2RC} + \sqrt{\left(\frac{1}{2RC}\right)^2 + \frac{1}{LC}}$$

Similarly, $$\omega_1 = -\frac{1}{2RC} + \sqrt{\left(\frac{1}{2RC}\right)^2 + \frac{1}{LC}}$$

So that, bandwidth

$$B = \omega_2 - \omega_1 = \frac{1}{RC}$$

and $$\omega_1\omega_2 = \left(\frac{1}{2RC}\right)^2 + \frac{1}{LC} - \left(\frac{1}{2RC}\right)^2$$
$$= \frac{1}{LC} = \omega_0^2$$

Thus, $$\omega_0 = \sqrt{\omega_1\omega_2}$$

As $\omega_0 = \dfrac{1}{\sqrt{LC}}$ and $Q = \omega_0 RC = \dfrac{R}{\omega_0 L}$

$$Q = \frac{R}{L}\sqrt{LC} = R\sqrt{\frac{C}{L}}$$

Since $\dfrac{1}{2RC} = \dfrac{B}{2}$

$$\omega_2 = \frac{B}{2} + \sqrt{\left(\frac{B}{2}\right)^2 + \omega_0^2}$$

and $$\omega_1 = -\frac{B}{2} + \sqrt{\left(\frac{B}{2}\right)^2 + \omega_0^2}$$

Using $B = \dfrac{\omega_0}{Q}$,

$$\omega_2 = \omega_0\left[\frac{1}{2Q} + \sqrt{1 + \left(\frac{1}{2Q}\right)^2}\right]$$

and $$\omega_1 = \omega_0\left[-\frac{1}{2Q} + \sqrt{1 + \left(\frac{1}{2Q}\right)^2}\right]$$

6.13 Transfer Function and Frequency Response

The transfer function for a parallel RLC circuit shown in Fig. 6.14. is $H(j\omega)$, the current ratio transfer function.

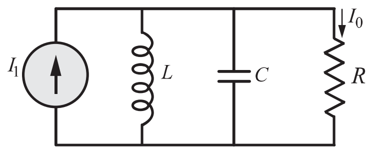

Figure 6.14 Parallel RLC Circuit

$$H(j\omega) = \frac{I_0(j\omega)}{I_1(j\omega)} = \frac{1}{RY(j\omega)}$$
$$= \frac{1}{R}\frac{1}{\frac{1}{R} + j\left(\omega C - \frac{1}{\omega L}\right)} = \frac{1}{1 + jR\left(\omega C - \frac{1}{\omega L}\right)}$$
$$= \frac{1}{1 + j\left(\frac{\omega\omega_0 CR}{\omega_0} - \frac{\omega_0 R}{\omega_0\omega_L}\right)} = \frac{1}{1 + jQ\left(\frac{\omega}{\omega_0} - \frac{\omega_0}{\omega}\right)}$$

As in the case of series resonance, here also let

$$\delta = \frac{\omega - \omega_0}{\omega_0} = \frac{\omega}{\omega_0} - 1$$

then,

$$\frac{\omega}{\omega_0} - \frac{\omega_0}{\omega} = \frac{\delta^2 + 2\delta}{\delta + 1}$$

For $\delta << 1$, for small deviations from ω_0

$$\frac{\omega}{\omega_0} - \frac{\omega_0}{\omega} \simeq 2\delta$$

Therefore,

$$H(j\omega) = \frac{1}{1 + j2Q\delta}$$

6.14 Resonance in a Two Branch $RL - RC$ Parallel Circuit

Consider the two branch parallel circuit shown in Fig. 6.15. Let E be the voltage across each of the parallel circuit shown in the figure. The vector diagram at resonance is shown in Figure 6.1.

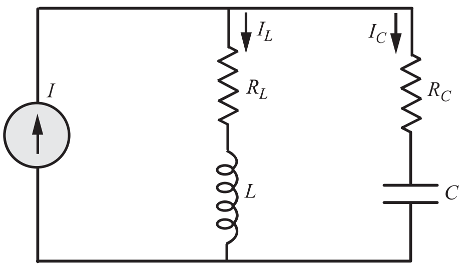

Figure 6.15 Two branch Parallel Circuit

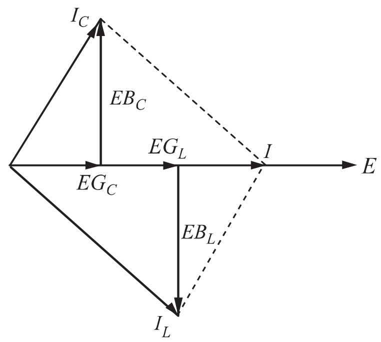

Figure 6.16

The admittance of the circuit is $Y(j\omega) = G_L - jB_L + G_C + jB_C$
For resonance,

$$B_L = B_C$$

If this occurs at $\omega = \omega_0$,

then
$$\frac{\omega_0 L}{RL^2 + \omega_0^2\, L^2} = \frac{\frac{1}{\omega_0 C}}{R_C^2 + \frac{1}{\omega_0^2\, C^2}}$$

$$= \frac{\omega_0 C}{R_C^2 \omega_0^2\, C^2 + 1}$$

$$L(1 + \omega_0^2 C^2\, R_C^2) = C(R_L^2 + \omega_0^2\, L^2)$$

$$\omega_0^2(LC^2\, R_C^2 - L^2 C) = R_L^2\, C - L$$

$$\omega_0^2 = \frac{R_L^2\,C - 1}{LC^2\,R_C^2 - L^2C}$$

$$= \frac{1}{LC}\frac{R_L^2\,C - L}{(R_C^2\,C - L)} = \frac{1}{LC}\frac{R_L^2 - \frac{L}{C}}{(R_C^2 - \frac{L}{C})}$$

Therefore, $$\omega_0 = \frac{1}{\sqrt{LC}}\sqrt{\frac{R_L^2 - \frac{L}{C}}{R_C^2 - \frac{L}{C}}}$$

This is the expression for resonant frequency. It is to be noted that

1. resonance is not possible for certain combination of circuit elements unlike in a series circuit where resonance is always possible.
2. resonance is also possible by varying of R_L or R_C.

Consider the case where

$$R_C^2 < \frac{L}{C} < R_L^2$$

or

$$R_L^2 < \frac{L}{C} < R_C^2$$

In both these cases, the quantity under radical is negative and therefore resonance is not possible.

The admittance at resonance of the above parallel circuit is

$$Y_0 = \left(\frac{R_L}{R_L^2 + X_{L_0}^2} + \frac{RC}{R_C^2 + X_{C_0}^2}\right)\text{S}$$

where X_{L_0} and X_{C_0} are the inductive and capacitive reactances respectively at resonance.

If $$R_L = R_C \neq \sqrt{\frac{L}{C}}$$

then $$\omega_0 = \frac{1}{\sqrt{LC}}$$

as in R, L, C series circuit.

If $$R_L = R_C = \sqrt{\frac{L}{C}}$$

which means

$$R_L^2 = R_C^2 = R^2 = \frac{L}{C} = X_L X_C.$$

Then, $$B_L - B_C = \frac{X_L}{R_L^2 + X_L^2} - \frac{X_C}{R_C^2 + X_C^2}$$

$$= \frac{1}{X_L + X_C} - \frac{1}{X_L + X_C} = 0$$

In this case, the circuit acts as a pure resistive circuit irrespective of frequency. That is, the circuit is resonant for all frequencies.

In this case the circuit admittance is

$$Y = \frac{R_L}{R_L^2 + X_L^2} + \frac{R_C}{R_C^2 + X_C^2}$$

$$= R\left[\frac{R^2 + X_L^2 + R^2 + X_C^2}{R^4 + R^2(X_L^2 + X_C^2) + X_L^2 X_C^2}\right]$$

$$= R\frac{2R^2 + X_L^2 + X_C^2}{2R^4 + R^2(X_L^2 + X_C^2)}$$

$$= \frac{R}{R^2}\left[\frac{2R^2 + X_L^2 + X_C^2}{2R^2 + X_L^2 + X_C^2}\right]$$

$$= \frac{1}{R} = \sqrt{\frac{C}{L}}$$

or

$$Z = R = \sqrt{\frac{L}{C}}$$

6.14.1 Resonance by varying inductance

If resonance is achieved by varying only L in the circuit shown in Figure 6.15 but with constant current constant frequency source, then the condition for resonance is

$$B_L = B_C$$

$$\Rightarrow \quad \frac{X_L}{R_L^2 + X_L^2} = \frac{X_C}{R_C^2 + X_C^2} = \frac{X_C}{Z_C^2} \quad \text{where} \quad Z_C^2 = R_C^2 + X_C^2$$

$$\text{Then,} \qquad X_L^2 X_C - X_L Z_C^2 + X_C R_L^2 = 0$$

$$\text{Solving, for} X_L \text{we get} \quad X_L = \frac{Z_C^2 \pm \sqrt{Z_C^4 - 4X_C^2 R_L^2}}{2X_C}$$

$$\text{Therefore,} \qquad L = \frac{C}{2}\left[Z_C^2 \pm \sqrt{Z_C^4 - 4X_C^2 R_L^2}\right] \quad \left(\text{since } X_L X_C = \frac{L}{C}\right)$$

The following conditions arise:

1. If $Z_C^4 > 4X_C^2\, R_L^2$, L has two values for the circuit to resonate.
2. For $Z_C^4 = 4X_C^2\, R_L^2$, $L = \frac{1}{2}CZ_C^2$ for reasonance.
3. For $Z_C^4 < 4X_C^2\, R_L^2$, No value of L makes the circuit to resonate.

6.14.2 Resonance by varying capacitance

As in the previous case, we have at resonance,,

$$B_L = B_C$$

$$\Rightarrow \qquad \frac{X_C}{R_C^2 + X_C^2} = \frac{X_L}{Z_L^2}, \text{ where } Z_L^2 = R_L^2 + X_L^2$$

Simplifying we get,

$$X_C^2 X_L - X_C Z_L^2 + R_C^2 X_L = 0$$

$$X_C = \frac{Z_L^2 \pm \sqrt{Z_L^4 - 4X_L^2 R_C^2}}{2X_L}$$

$$\text{Therefore,} \qquad C = \frac{2L}{Z_L^2 \pm \sqrt{Z_L^4 - 4X_L^2 R_C^2}}$$

The following conditions arise:

1. For $Z_L^4 > 4X_L^2\, R_C^2$, there are two values for C to resonante.
2. For $Z_4^4 = 4X_L^2\, R_C^2$, resonance occurs at $C = \dfrac{2L}{Z_L^2}$.
3. For $Z_L^4 < 4X_L^2\, R_C^2$, no value of C makes the circuit to resonate.

6.14.3 Resonance by varying R_L or R_C

It is often possible to adjust a two branch parallel combination to resonate by varying either R_L or R_C. This is because, when the supply is of constant current and, constant frequency, these resistors control inphase and quadratare components of the currents in the two parallel paths.

From the condition $B_L = B_C$, we get

$$\frac{X_L}{R_L^2 + X_L^2} = \frac{X_C}{R_C^2 + X_C^2}$$

$$R_L^2 = \frac{X_L}{X_C} R_C^2 + X_L X_C - X_L^2$$

$$R_L = \sqrt{\frac{X_L}{X_C} R_C^2 + X_L X_C - X_L^2} \tag{6.4}$$

This equation gives the value of R_L for resonance when all other quantities are constant and the term under radical is positive.

Similarly if only R_C is variable, keeping all other quantities constant, the value of R_C for resonanace is given by

$$R_C = \sqrt{\frac{X_C}{X_L} R_L^2 + X_L X_C - X_C^2}$$

provided the term under radical is positive.

6.15 Practical Parallel and Series Resonant Circuits

A practical resonant parallel circuit contains an inductive coil of resistance R and inductance L in parallel with a capacitor C as shown in Fig. 6.17. It is called a tank circuit because it stores energy in the magnetic field of the coil and in the electric field of the capacitor. Note that resistance R_C of the capacitor is negligibly small.

Condition for parallel resonance is shown by the phasor diagram of Fig. 6.18.

$$I_C = I_L \sin\phi$$

That is, $$B_C = B_L$$

$$\Rightarrow \quad \frac{\omega L}{R^2 + \omega^2 L^2} = \omega C.$$

Let the value of ω which satisfy this condition be ω_0.

Then, $$R^2 + \omega_0^2 L^2 = \frac{L}{C} \tag{6.5}$$

$$\omega_0^2 = \left(\frac{L}{C} - R^2\right)\frac{1}{L^2} = \frac{1}{LC}\left(1 - \frac{R^2 C}{L}\right)$$

$$\omega_0 = \frac{1}{\sqrt{LC}}\sqrt{1 - \frac{R^2 C}{L}} \tag{6.6}$$

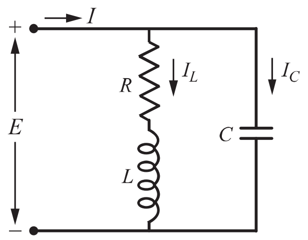

Figure 6.17

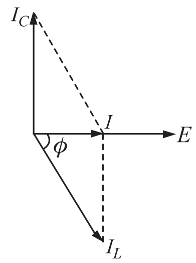

Figure 6.18

Admittance of the circuit shown in figure 6.17 is

$$Y(j\omega) = \frac{1}{R + j\omega L} + j\omega C$$

$$= \frac{R}{R^2 + \omega^2 L^2} - \frac{j\omega L}{R^2 + \omega^2 L^2} + j\omega C$$

At $\omega = \omega_0$, $Y(j\omega)$ is purely real.

Hence, $$Y(j\omega_0) = \frac{R}{R^2 + \omega_0^2 L^2} \tag{6.7}$$

Substituting for ω_0 in equation (6.7),

$$Y(j\omega_0) = \frac{R}{R^2 + \omega_0^2 L^2} = \frac{R}{\frac{L}{C}} = \frac{RC}{L} \tag{6.8}$$

and the circuit is a pure resistive with $R_0 = \dfrac{L}{CR}$, which is called the dynamic resistance of the circuit. This is greater than R if there is resonance. However, note that if $\dfrac{R^2 C}{L} > 1$, there is no resonance.

Fig. 6.19 shows a practical series resonant circuit. The input impedance as a function of ω is

$$Z(j\omega) = j\omega L + \frac{G}{G^2 + \omega^2 C^2} - j\frac{\omega C}{G^2 + \omega^2 C^2}$$

Condition for resonance is

$$\begin{aligned}
\omega L &= \frac{\omega C}{G^2 + \omega^2 C^2} \\
\omega^2 &= \left(\frac{C}{L} - G^2\right)\frac{1}{C^2} = \frac{1}{LC} - \frac{1}{C^2 R^2} \\
&= \frac{1}{LC}\left(1 - \frac{L}{CR^2}\right) \\
\omega &= \frac{1}{\sqrt{LC}}\left(1 - \frac{L}{CR^2}\right)
\end{aligned}$$

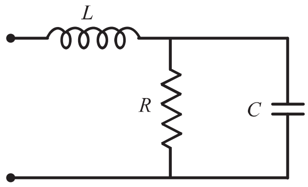

Figure 6.19

Impedance at resonance is

$$Z_0 = \frac{G}{G^2 + \omega C^2} = \frac{G}{\frac{C}{L}} = \frac{L}{CR}$$

The circuit at resonance is a purely resistive, and $Z_0 = R_0 = \dfrac{L}{CR}$. However, note that here also resonance is not possible for $\dfrac{L}{CR^2} > 1$.

In both the circuits, shown in Figs 6.18 and 6.19, resonance is achieved by varying either C or L until the input impedance or admittance is real and this process is called tuning. For this reason these circuits are called tuned circuits.

Series circuits

EXAMPLE 6.1

Two coils, one of $R_1 = 0.51\ \Omega$, $L_1 = 32$ mH, the other of $R_2 = 1.3\ \Omega$ and $L_2 = 15$ mH and two capacitors of 25 μF and 62 μF are all in series with a resistance of $0.24\ \Omega$. Determine the following for this circuit

(i) Resonance frequency

(ii) Q of each coil

(iii) Q of the circuit

(iv) Cut off frequencies

(v) Power dissipated at resonance if $E = 10$ V.

SOLUTION

From the given values, we find that

$$R_s = 0.51 + 1.3 + 0.24 = 2.05\Omega$$
$$L_s = 32 + 15 = 47 \text{ mH}$$
$$C_s = \frac{25 \times 62}{87}\ \mu\text{F} = 17.816\ \mu\text{F}$$

(i) Resonant frequency:

$$\omega_0 = \frac{1}{\sqrt{L_s C_s}} = \frac{1}{\sqrt{47 \times 10^{-3} \times 17.816 \times 10^{-6}}} = 1092.8 \text{ rad/ sec}$$

(ii) Q of coils:

For Coil 1, $$Q_1 = \frac{\omega_0 L_1}{R_1} = \frac{1092.8 \times 32 \times 10^{-3}}{0.51} = 68.57$$

For Coil 2, $$Q_2 = \frac{\omega_0 L_2}{R_2} = \frac{1092.8 \times 15 \times 10^{-3}}{1.3} = 12.6$$

(iii) Q of the circuit:

$$Q = \frac{\omega_0 L_s}{R_s} = \frac{1092.8 \times 47 \times 10^{-3}}{2.05} = 25$$

(iv) Cut off frequencies: Band width is,

$$B = \frac{\omega_0}{Q} = \frac{1092.8}{25} = 43.72$$

Considering $Q > 5$, the cut off frequencies,

$$\omega_{2,1} = \omega_0 \pm \frac{B}{2} = 1092.8 \pm 21.856$$

Therefore, $$\omega_2 = 1115 \text{ rad/ sec} \quad \text{and } \omega_1 = 1071 \text{ rad/ sec}.$$

(v) Power dissipated at resonance:

Given $E = 10$ V

We know that at resonance, only the resistance portion will come in to effect. Therefore

$$P = \frac{E^2}{R} = \frac{10^2}{2.05} = 48.78 \text{ W}$$

EXAMPLE 6.2

For the circuit shown in Fig. 6.20, find the out put voltages at

(i) $\omega = \omega_0$

(ii) $\omega = \omega_1$

(iii) $\omega = \omega_2$
when $v_s(t) = 800 \cos \omega t$ mV.

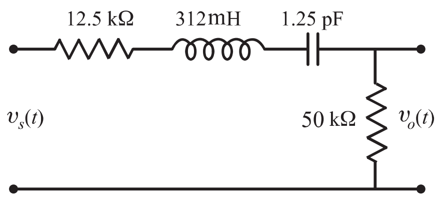

Figure 6.20

SOLUTION

For the circuit, using the values given, we can find that resonant frequency

$$\omega_0 = \frac{1}{\sqrt{LC}}$$
$$= \frac{1}{\sqrt{312 \times 10^{-3} \times 1.25 \times 10^{-12}}} = 1.6 \times 10^6 \text{ rad/ sec}$$

Quality factor:

$$Q = \frac{\omega_0 L}{R}$$
$$= \frac{1.6 \times 10^6 \times 312 \times 10^{-3}}{62.5 \times 10^3} = 8$$

Band width:

$$B = \frac{\omega_0}{Q}$$
$$= \frac{1.6 \times 10^6}{8} = 0.2 \times 10^6 \text{ rad/ sec}$$

As $Q > 5$,

$$\omega_{2,1} = \omega_0 \pm \frac{B}{2}$$
$$= (1.6 \pm 0.1)10^6 \text{ rad/ sec}$$

Hence, $\omega_2 = 1.7 \times 10^6$ rad/ sec

and $\omega_1 = 1.5 \times 10^6$ rad/ sec

(i) Output voltage at ω_0:

Using the relationship of transfer function, we get

$$H(j\omega)|_{\omega=\omega_0} = \frac{50I_m}{62.5I_m}$$
$$= 0.8 \angle 0^\circ$$

Since the current is maximum at resonance and is same in both resistors,

$$v_o(t) = 0.8 \times 800 \cos(1.6 \times 10^6 t) \text{ mV}$$
$$= 640 \cos(1.6 \times 10^6 t) \text{ mV}$$

At ω_1 and ω_2, $Z_{\text{in}} = \sqrt{2}R_s \angle \pm 45^\circ$. Therefore,

$$H(j\omega)|_{\omega=\omega_1} = \frac{R_{\text{out}}}{Z_{\text{in}}} = \frac{50}{\sqrt{2} \times 62.5} \angle 45^\circ$$
$$= 0.5657 \angle 45^\circ$$

and $$H(j\omega)|_{\omega=\omega_2} = 0.5657 \angle 45^\circ$$

(ii) Out put voltage at $\omega = \omega_1$

$$v_o(t) = 0.5657 \times 800 \cos(1.6 \times 10^6 t + 45^\circ) \text{ mv}$$
$$= 452.55 \cos(1.6 \times 10^6 t + 45^\circ) \text{ mV}$$

(iii) Out put voltage at $\omega = \omega_2$

$$v_o(t) = 452.55 \cos(1.6 \times 10^6 t - 45^\circ) \text{ mV}$$

EXAMPLE 6.3

In a series circuit $R = 6\ \Omega$, $\omega_0 = 4.1 \times 10^6$ rad/sec, band width = 10^5 rad/sec. Compute L, C, half power frequencies and Q.

SOLUTION

We know that Quality factor,

$$Q = \frac{\omega_0}{B} = \frac{4.1 \times 10^6}{10^5} = 41$$

Also, $$Q = \frac{\omega_0 L}{R}$$

Therefore, $$L = \frac{QR}{\omega_0} = \frac{41 \times 6}{4.1 \times 10^6} = 60\ \mu\text{H}$$

and $$Q = \frac{1}{\omega_0 CR}$$

Hence, $$C = \frac{1}{\omega_0 QR}$$
$$= \frac{1}{4.1 \times 10^6 \times 41 \times 6} = 991.5 \text{ pF}$$

As $Q > 5$,

$$\omega_{2,1} = \omega_0 \pm \frac{B}{2} = 4.1 \times 10^6 \pm \frac{10^5}{2}$$

That is, $\omega_2 = 4.15 \times 10^6 \text{ rad/sec}$

and $\omega_1 = 4.05 \times 10^6 \text{ rad/sec}$

EXAMPLE 6.4

In a series resonant circuit, the current is maximum when $C = 500$ pF and frequency is 1 MHz. If C is changed to 600 pF, the current decreases by 50%. Find the resistance, inductance and quality factor.

SOLUTION

■ Case 1

Given,

$$C = 500 \text{ pF}$$
$$I = I_m$$
$$f = 1 \times 10^6 \text{ Hz}$$
$$\Rightarrow \qquad \omega_0 = 2\pi \times 10^6 \text{ rad/sec}$$

We know that

$$\omega_0 = \frac{1}{\sqrt{LC}}$$

Therefore, Inductance,

$$L = \frac{1}{\omega_0^2 C} = \frac{10^{12}}{(2\pi \times 10^6)^2 \times 500}$$
$$= 0.0507 \text{ mH}$$

■ Case 2

When C = 600 pF,

$$I = \frac{I_m}{2} = \frac{E}{2R} \quad \Rightarrow \quad |Z| = 2R$$
$$\sqrt{R^2 + X^2} = 2R \quad \Rightarrow \quad X = \sqrt{3}R$$
$$X = X_L - X_C$$
$$= 2\pi \times 10^6 \times 0.0507 \times 10^{-3} - \frac{10^{12}}{2\pi \times 10^6 \times 600}$$
$$= 318.56 - 265.26$$
$$= 53.3\ \Omega = \sqrt{3}R$$

Therefore resistance,

$$R = \frac{53.3}{\sqrt{3}} = 30.77\Omega$$

Quality factor,

$$Q = \frac{\omega_0 L}{R} = \frac{318.56}{30.77} = 10.35$$

EXAMPLE 6.5

In a series circuit with $R = 50\ \Omega$, L = 0.05 H and $C = 20\ \mu$F, frequency is varied till the voltage across C is maximum. If the applied voltage is 100 V, find the maximum voltage across the capacitor and the frequency at which it occurs. Repeat the problem for $R = 10\ \Omega$.

SOLUTION

■ Case 1

Given $R = 50\ \Omega$, L = 0.05 H, $C = 20\ \mu$F

We know that

$$\omega_0 = \frac{1}{\sqrt{LC}} = \frac{10^3}{\sqrt{0.05 \times 20}} = 10^3 \text{ rad/sec}$$

$$Q = \frac{\omega_0 L}{R} = \frac{10^3 \times 0.05}{20} = 1$$

Using the given value of E = 100 V in the relationship

$$V_{Cm} = \frac{QE}{\sqrt{1 - \frac{1}{4Q^2}}}$$

we get

$$V_{Cm} = \frac{100}{\sqrt{1 - \frac{1}{4}}} = 115.5 \text{ V}$$

and the corresponding frequency at this voltage is

$$\omega_C = \omega_0 \sqrt{1 - \frac{1}{2Q^2}}$$

$$= 10^3 \sqrt{\frac{1}{2}} = 707 \text{ rad/sec}$$

■ Case 2

When $R = 10\ \Omega$,

$$Q = \frac{10^3 \times 0.05}{10} = 5$$

$$V_{Cm} = \frac{5 \times 100}{\sqrt{1 - \frac{1}{4 \times 25}}} = 502.5 \text{ V}$$

$$\omega_C = 10^3 \sqrt{1 - \frac{1}{50}} = 990 \text{ rad/sec}$$

EXAMPLE 6.6

(i) A series resonant circuit is tuned to 1 MHz. The quality factor of the coil is 100. What is the ratio of current at a frequency 20 kHz below resonance to the maximum current?
(ii) Find the frequency above resonance when the current is reduced to 90% of the maximum current.

SOLUTION

(i) Let ω_a be the frequency 20 kHz below the resonance, I_a be the current and Z_a be the impedance at this frequency.

Then $$\omega_a = 10^6 - 20 \times 10^3 = 980 \text{ kHz}$$

$$\frac{\omega_a}{\omega_0} - \frac{\omega_0}{\omega_a} = \frac{980}{10^3} - \frac{10^3}{980}$$
$$= -40.408 \times 10^{-3} = 2\delta$$

Now the ratio of current,

$$\frac{I_a}{I_m} = \frac{R}{Z_a} = \frac{1}{1 + j(2\delta)Q}$$
$$= \frac{1}{1 - j100(40.408 \times 10^{-3})}$$
$$= \frac{1}{1 - j4.0408}$$
$$= 0.2402 \underline{/76^\circ}$$

(ii) Let ω_b be the frequency at which $I_b = 0.9 I_m$

Then $$\left| \frac{I_b}{I_m} \right| = \frac{1}{1 + j(2\delta)Q} = 0.9$$

or $$\sqrt{1 + x^2} = \frac{1}{0.9}$$

where $$x = (2\delta)100$$

Then, $$1 + x^2 = \frac{1}{0.81} = 1.2346$$

or $$x^2 = 0.2346$$

and $$x = 0.4843$$

We know that

$$\delta = \frac{\omega_b}{\omega_0} - 1 = \frac{0.4843}{200}$$

Hence $$\omega_b = \left(1 + \frac{0.4843}{200}\right) \omega_0$$
$$= 1.00242 \text{ MHz}$$

EXAMPLE 6.7

For the circuit shown in Fig. 6.21, obtain the values of ω_0 and v_C at ω_0.

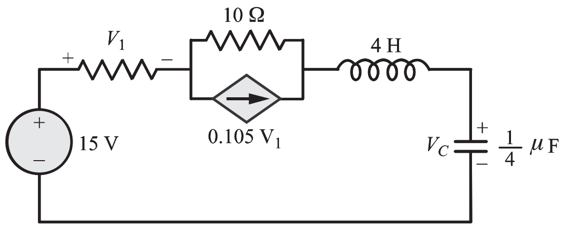

Figure 6.21

SOLUTION

For the series circuit,

$$\omega_0 = \frac{1}{\sqrt{LC}}$$
$$= \frac{1}{\sqrt{4 \times \frac{1}{4} \times 10^{-6}}} = 10^3 \text{ rad/sec}$$

At this $\omega_0, I = I_m$. Therefore,

$$V_1 = 125 I_m$$

and the circuit equation is

$$1.5 = V_1 + (I_m - 0 \cdot 105 V_1)10 + jV_L - jV_C$$

Since $V_L = V_C$, the above equation can be modified as

$$1.5 = 125 I_m + 10 I_m - 1.05 \times 125 I_m$$

Hence,
$$I_m = \frac{1.5}{3.75} \text{ A}$$

and
$$V_c = \frac{1.5}{3.75} \times \frac{4 \times 10^6}{10^3}$$
$$= 1600 \text{ V}$$

EXAMPLE 6.8

For the circuit shown in Fig. 6.22(a), obtain Z_{in} and then find ω_0 and Q.

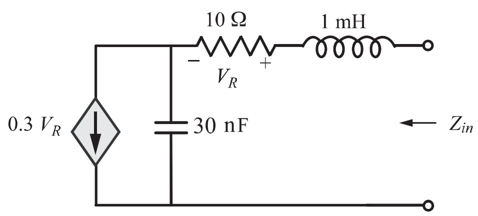

Figure 6.22(a)

SOLUTION

Taking I as the input current, we get

$$V_R = 10I$$

and the controlled current source,

$$0.3V_R = 0.3 \times 10I$$
$$= 3I$$

The input impedance can be obtained using the standard formula

$$Z_{in}(j\omega) = \frac{\text{Applied voltage}}{\text{Input current}} = \frac{V}{I} \tag{6.6}$$

For futher analysis, the circuit is redrawn as shown in Fig. 6.22(b). It may be noted that the controlled current source is transformed to its equivalent voltage source.

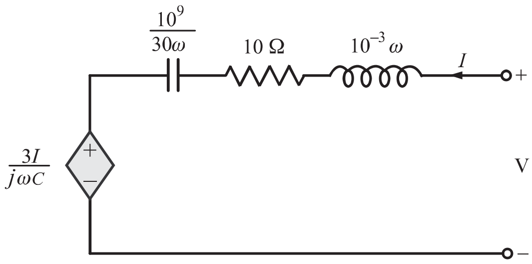

Figure 6.22(b)

Referring Fig. 6.22(b), the circuit equation may be obtained as

$$V = \left(10 + j10^{-3}\omega - \frac{j10^9}{30\omega} - \frac{j3}{30\omega \times 10^{-9}}\right) I \tag{6.7}$$

Substituting equation (6.7) in equation (6.6), we get

$$Z_{\text{in}} = 10 + j\left(10^{-3}\omega - \frac{4 \times 10^9}{30\omega}\right)\Omega$$

For resonance, Z_{in} should be purely real. This gives

$$10^{-3}\omega = \frac{4 \times 10^9}{30\omega}$$

Rearranging,

$$\omega^2 = \frac{4 \times 10^9}{30 \times 10^3}$$
$$= 0.133 \times 10^{12}$$

Solving we get

$$\omega = \omega_0 = \sqrt{0.133 \times 10^{12}}$$
$$= 365 \times 10^3 \text{ rad/ sec}$$

Quality factor

$$Q = \frac{\omega_0 L}{R}$$
$$= \frac{365 \times 10^3 \times 10^{-3}}{10}$$
$$= 36.5$$

Parallel circuits

EXAMPLE 6.9

For the circuit shown in Fig. 6.23(a), find ω_0, Q, BW and half power frequencies and the out put voltage V at ω_0.

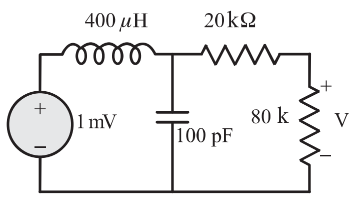

Figure 6.23(a)

SOLUTION

Transforming the voltage source into current source, the circuit in Fig. 6.28(a) can be redrawn as in Fig. 6.23(b).

Then, $$\omega_0 = \frac{1}{\sqrt{LC}}$$
$$= \frac{109}{\sqrt{400 \times 100}}$$
$$= 5 \times 10^6 \text{ rad/ sec}$$
$$Q = \omega_0 CR$$
$$= 5 \times 10^6 \times 100 \times 10^{-12} \times 100 \times 10^3 = 50$$
$$B = \frac{\omega_0}{Q} = \frac{5 \times 10^6}{50} = 10^5 \text{ rad/ sec}$$

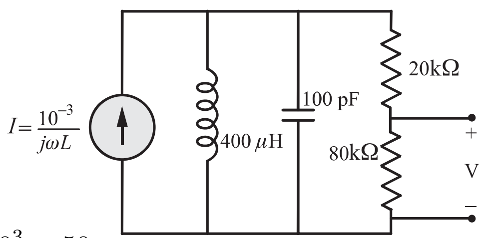

Figure 6.23(b)

As $Q > 10$,

$$\omega_{2,1} = \pm\frac{B}{2} + \omega_0$$
$$= 5 \times 10^6 \pm \frac{10^5}{2}$$

Hence, $\omega_2 = 5.05 \text{ M rad/ sec}$ and $\omega_1 = 4.95 \text{ M rad/ sec}$

Output voltage,

$$V = I \times 80 \text{ k}\Omega$$
$$= \frac{10^{-3} \times 80 \times 10^3}{j5 \times 10^6 \times 400 \times 10^{-6}}$$
$$= 0.04 \angle -90^\circ \text{ V}$$

EXAMPLE 6.10

In a parallel RLC circuit, $C = 50\ \mu$F. Determine BW, Q, R and L for the following cases.

(i) $\omega_0 = 100, \omega_2 = 120$
(ii) $\omega_0 = 100, \omega_1 = 80$

SOLUTION

(i) $\omega_0 = 100, \omega_2 = 120$

We know that

$$\omega_0 = \sqrt{\omega_1\omega_2}$$

Rearraging we get

$$\omega_1 = \frac{\omega_0^2}{\omega_2}$$
$$= \frac{100^2}{120} = 83.33 \text{ rad/sec}$$

Band width

$$B = \omega_2 - \omega_1$$
$$= 120 - 83.33 = 36.67 \text{ rad/ sec}$$

Quality factor,

$$Q = \frac{\omega_0}{B}$$
$$= \frac{100}{36.67} = 2.73$$

We know that

$$Q = \frac{R}{\omega_0 L} = \omega_0 RC \tag{6.8}$$

Rearraging equation (6.8),

$$R = \frac{Q}{\omega_0 C}$$

$$= \frac{2.73 \times 10^6}{100 \times 50} = 546\ \Omega$$

Similarly

$$L = \frac{1}{\omega_0^2 C}$$

$$= \frac{10^6}{100^2 \times 50} = 2\ \text{H}$$

(ii) $\omega_0 = 100, \omega_1 = 80$: Solving the same way as in case (i), we get

$$\omega_2 = \frac{100^2}{80} = 125$$

$$\text{BW} = B = 125 - 80 = 45\ \text{rad/sec}$$

$$Q = \frac{100}{45} = 2.22$$

EXAMPLE 6.11

In the circuit shown in Fig. 6.24(a), $v_s(t) = 100\cos\omega t$ volts. Find resonance frequency, quality factor and obtain i_1, i_2, i_3. What is the average power loss in 10 kΩ. What is the maximum stored energy in the inductors?

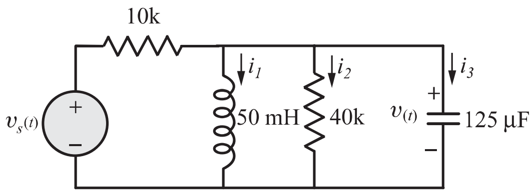

Figure 6.24(a)

SOLUTION

The circuit in Fig. 6.24(a) is redrawn by replacing its voltage source by equivalent current source as shown in Fig. 6.24(b).

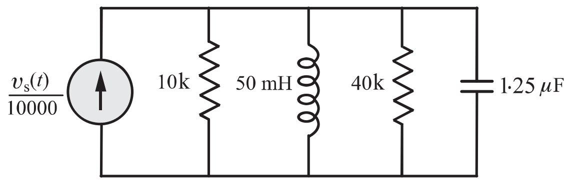

Figure 6.24(b)

Resonance frequency,

$$\omega_0 = \frac{1}{\sqrt{LC}}$$

$$= \frac{1}{\sqrt{50 \times 10^{-3} \times 1.25 \times 10^{-6}}}$$

$$= 4000\ \text{rad/sec}$$

Quality factor,

$$Q = \omega_0 C\ R_{\text{eq}}$$

$$= 4000 \times 1.25 \times 10^{-6} \times 8 \times 10^3$$

$$= 40$$

At resonance, the current source will branch into resistors only. Hence,

$$
\begin{aligned}
v(t) &= (10\text{k}\Omega||40\text{k}\Omega) \times \frac{v_s(t)}{10000} \\
&= 80\cos 4000t \text{ volts}
\end{aligned}
$$

$i_1(t)$ lags $v(t)$ by 90°. Therefore,

$$
\begin{aligned}
i_1(t) &= \frac{80}{50\times 10^{-3}\times 4000}\sin 40000t \\
&= 400\sin 4000t \text{ mA} \\
i_2(t) &= \frac{80}{40\times 1000}\cos 4000t \\
&= 2\cos 4000t \text{ mA} \\
i_3(t) &= -i_1(t) \\
&= -400\sin 4000t \text{ mA}
\end{aligned}
$$

Average power in 10 kΩ:

$$
\begin{aligned}
P_{\text{av}} &= \frac{\frac{80^2}{\sqrt{2}}}{10\times 10^3} \\
&= 0.32 \text{ W}
\end{aligned}
$$

Maximum stored energy in the inductance:

$$
\begin{aligned}
E &= \frac{1}{2}LI_m^2 \\
&= \frac{1}{2}\times 50\times 10^{-3}\times(400\times 10^{-3})^2 \\
&= 4 \text{ mJ}
\end{aligned}
$$

EXAMPLE 6.12

For the network shown in Fig. 6.25(a), obtain Y_{in} and then use it to determine the resonance frequency and quality factor.

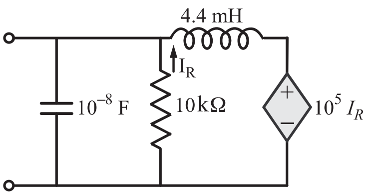

Figure 6.25(a)

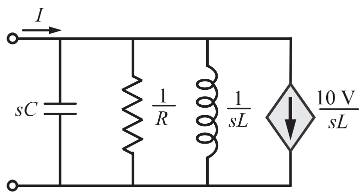

Figure 6.25(b)

SOLUTION

Considering V as the input voltage and I as the input current, it can be found that

$$10\text{k}\Omega \times I_R = -V \Rightarrow 10^4\ I_R = -V$$

The circuit in Fig. 6.25(a) is redrawn by replacing the controlled voltage source in to its equivalent current source by taking $s = j\omega$ and is shown in Fig. 6.25(b). Referring Fig. 6.25(b),

$$I - \frac{10V}{sL} = V\left(sC + \frac{1}{R} + \frac{1}{sL}\right)$$
$$\Rightarrow \qquad I = V\left(sC + \frac{1}{R} + \frac{11}{sL}\right)$$

Input admittance, with s is being replaced by $j\omega$ is

$$Y_{\text{in}} = \frac{I}{V} = \frac{1}{10^4} + j\omega 1 \times 10^{-8} - \frac{j11 \times 10^3}{\omega \times 4.4}$$
$$= 10^{-4} + j\omega \times 10^{-8} - \frac{j2500}{\omega}$$

At resonance, Y_{in} should be purely real. This enforces that

$$10^{-8}\omega = \frac{2500}{\omega}$$
$$\text{Therefore,} \qquad \omega_0 = \sqrt{10^8 \times 2500}$$
$$= 500 \text{ K rad/sec}$$

Quality factor:

$$Q = \omega_0 RC$$
$$= 500 \times 10^3 \times 10^4 \times 10^{-8}$$
$$= 50$$

EXAMPLE 6.13

In a parallel RLC circuit, cut off frequencies are 103 and 118 rad/sec. $|Z|$ at $\omega = 105$ rad/sec is 10 Ω. Find R, L and C.

SOLUTION

Given

$$\omega_1 = 103 \text{ rad/sec}$$
$$\omega_2 = 118 \text{ rad/sec}$$

Therefore

$$B = 118 - 103 = 15 \text{ rad/sec}$$

Resonant frequency,

$$\omega_0 = \sqrt{\omega_1 \omega_2}$$
$$= \sqrt{118 \times 103} = 110.245 \text{ rad/sec}$$

Quality factor

$$
\begin{aligned}
Q &= \frac{\omega_0}{B} \\
&= \frac{110.245}{15} = 7.35
\end{aligned}
$$

Admittance,

$$
\begin{aligned}
Y &= \frac{1}{R} + j\left(\omega C - \frac{1}{\omega L}\right) \\
&= \frac{1}{R}\left[1 + j\left(\omega CR - \frac{R}{\omega L}\right)\right] \\
&= \frac{1}{R}\left[1 + j\left(\frac{\omega_0 \omega CR}{\omega_0} - \frac{R\omega_0}{\omega\omega_0 L}\right)\right]
\end{aligned}
$$

Since $$Q = \omega_0 RC = \frac{R}{\omega_0 L},$$

we get $$Y = \frac{1}{R}\left[1 + jQ\left(\frac{\omega}{\omega_0} - \frac{\omega_0}{\omega}\right)\right]$$

Note that, $$\frac{\omega}{\omega_0} - \frac{\omega_0}{\omega} = \frac{105}{110.245} - \frac{110.245}{105} = -0.0975$$

Therefore,

$$
\begin{aligned}
Y &= \frac{1}{R}\left(1 + j7.35(-0.0975)\right) \\
&= \frac{1}{R}(1 - j0.7168) \qquad (6.12) \\
\Rightarrow |Y| &= \frac{1}{R}\sqrt{1 + (0.7168)^2} = \frac{1.23}{R}
\end{aligned}
$$

It is given that $|Z| = 10$ and therefore $|Y| = \dfrac{1}{10}$. Putting this value of Y in equation (6.9), we get

$$\frac{1}{10} = 1.23\frac{1}{R} \quad \Rightarrow \quad R = 12.3\Omega$$

From the relationship $Q = \omega_0 CR$, we get

$$\omega_0 CR = 7.35$$

Therefore,

$$
\begin{aligned}
C &= \frac{7.35}{12.3} \times \frac{1}{110.245} \\
&= 5.42\ \mu\text{F}
\end{aligned}
$$

Inductance,

$$L = \frac{1}{\omega_0^2 C}$$
$$= \frac{1}{110.245^2 \times 5.42 \times 10^{-3}}$$
$$= 15.18 \text{ mH}$$

EXAMPLE 6.14

For the circuit shown in Fig. 6.26(a), find ω_0, V_1 at ω_0, and V_1 at a frequency 15 k rad/sec above ω_0.

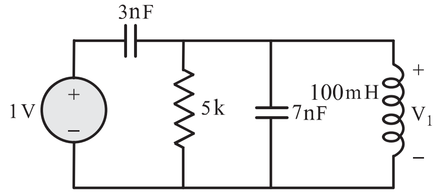

Figure 6.26(a)

SOLUTION

Changing voltage source of Fig. 6.26(a) into its equivalent current source, the circuit is redrawn as shown in Fig. 6.26(b).

Referring Fig. 6.26(b),

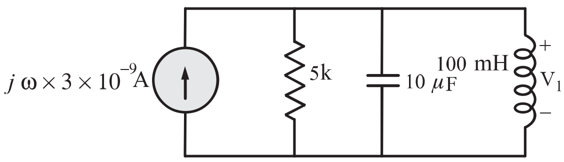

Figure 6.26(b)

$$\omega_0 = \frac{1}{\sqrt{LC}}$$
$$= \frac{1}{\sqrt{100 \times 10^{-6} \times 10 \times 10^{-9}}}$$
$$= 10^6 \text{ rad/sec}$$

Voltage across the inductor at ω_0 is,

$$V_1 = j10^6 \times 3 \times 10^{-9} \times 5 \times 10^3$$
$$= j15 \text{ V}$$

Quality factor,

$$Q = \omega_0 CR$$
$$= 10^6 \times 10 \times 10^{-9} \times 5 \times 10^3$$
$$= 50$$

Given

$$\omega_a = \omega_0 + 15 \text{ k rad/sec}$$
$$= 15 \times 10^3 + 10^6$$
$$= 1.015 \times 10^6 \text{ rad/sec}$$

Now,

$$\frac{\omega_a}{\omega_0} - \frac{\omega_0}{\omega_a} = 1.015 - \frac{1}{1.015} = 0.03$$

Using this relation in the equation,

$$Y = \frac{1}{R}\left[1 + jQ\left(\frac{\omega_a}{\omega_0} - \frac{\omega_0}{\omega_a}\right)\right]$$

we get

$$Y = \frac{1}{5000}(1 + j50 \times 0.03)$$
$$= 3.6 \times 10^{-4} \angle 56.31^\circ$$

The corresponding value of V_1 is

$$V_1 = I\,Y^{-1}$$
$$= j\omega_a \times 3 \times 10^{-9} \times Y^{-1}$$
$$= \frac{j\,1.015 \times 10^6 \times 3 \times 10^{-9}}{3.6 \times 10^{-4} \angle 56.31^\circ}$$
$$= 8.444 \angle 33.69^\circ \text{ V}$$

EXAMPLE 6.15

A parallel RLC circuit has a quality factor of 100 at unity power factor and operates at 1 kHz and dissipates 1 Watt when driven by 1 A at 1 kHz. Find Bandwidth and the numerical values of R, L and C.

SOLUTION

Given $f = 1$ kHz, $P = 1$ W, $I = 1$ A, $Q = 100$, $\cos\phi = 1$

$$B = \frac{\omega_0}{Q} = \frac{10^3 \times 2\pi}{100} = 20\pi \text{ rad/sec}$$
$$P = I^2 R$$

Therefore

$$R = 1\,\Omega$$
$$L = \frac{R}{\omega_0 Q}$$
$$= \frac{1}{20\pi \times 100}$$
$$= 159\,\mu\text{H}$$
$$C = \frac{1}{\omega_0^2 L}$$
$$= \frac{10}{(20\pi)^2 159}$$
$$= 16.9\,\mu\text{F}$$

EXAMPLE 6.16

For the circuit shown in Fig. 6.27, determine resonance frequency and the input impedance.

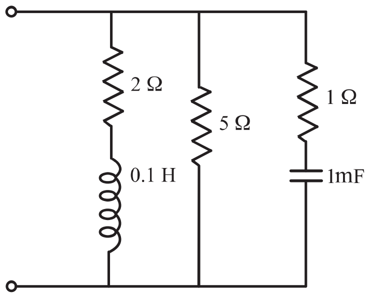

Figure 6.27

SOLUTION

Equation for resonance frequency is

$$\omega_L = \sqrt{\frac{1}{LC}\left(\frac{R_L^2 - \frac{L}{C}}{R_C^2 - \frac{L}{C}}\right)}$$

$$= \sqrt{\frac{1}{0.1 \times 10^{-3}}\left(\frac{2^2 - 100}{1 - 100}\right)}$$

$$= 98.47 \text{ rad/sec}$$

We know that

$$X_L = \omega_0 L$$

$$= 98.47 \times 0.1$$

$$= 9.847\ \Omega$$

and

$$X_C = \frac{1}{\omega_0 C}$$

$$= \frac{1}{98.47 \times 10^{-3}}$$

$$= 10.16\ \Omega$$

Admittance Y at resonance is purely real and is given by

$$Y = G_1 + G_2 + G_3$$
$$= \frac{2}{2 + (0.1\omega_0)^2} + \frac{1}{5} + \frac{1}{1 + \left(\frac{10^3}{\omega_0}\right)^2}$$
$$Y = \frac{2}{2^2 + 9.847^2} + \frac{1}{5} + \frac{1}{1 + 10.16^2}$$
$$= 0.23 \text{ S}$$

and the input impedance,

$$Z = \frac{1}{Y} = 4.35\ \Omega$$

EXAMPLE 6.17

The impedance of a parallel RLC circuit as a function of ω is depicted in the diagram shown in Fig. 6.28. Determine R, L and C of the circuit. What are the new values of ω_0 and bandwidth if C is increased by 4 times?

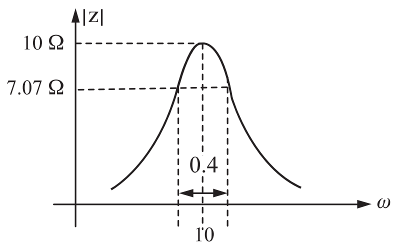

Figure 6.28

SOLUTION

It can be seen from the figure that

$$\omega_0 = 10 \text{ rad/sec}$$
$$B = 0.4 \text{ rad/sec}$$
$$R = 10\ \Omega$$

Then Quality factor

$$Q = \frac{\omega_0}{\text{BW}} = \frac{10}{0.4} = 25$$

We know that

$$L = \frac{R}{\omega_0 Q} = \frac{10}{10 \times 25} = 0.04 \text{ H}$$

As $Q = \omega_0 CR$,

$$C = \frac{25}{10 \times 10} = 0.25 \text{ F}$$

If C is increased by 4 times, the new value of C is 1 Farad. Therefore,

$$\omega_0 = \frac{1}{\sqrt{LC}} = \frac{1}{\sqrt{0.04}} = 5$$

and the corresponding bandwith

$$B = \frac{1}{RC} = 0.1$$

EXAMPLE 6.18

In a two branch $RL - RC$ parallel resonant circuit, $L = 0.4$ H and $C = 40\ \mu$F. Obtain resonant frequency for the following values of R_L and R_C.

(i) $R_L = 120$; $R_C = 80$

(ii) $R_L = R_C = 80$

(iii) $R_L = 80$; $R_C = 0$

(iv) $R_L = R_C = 100$

(v) $R_L = R_C = 120$

SOLUTION

As R_L and R_C are given separately, we can use the following formula to calculate the resonant frequency.

$$\omega_0 = \frac{1}{\sqrt{LC}} \sqrt{\left(\frac{R_L^2 - \frac{L}{C}}{R_C^2 - \frac{L}{C}}\right)} \tag{6.10}$$

Let us compute the following values

$$\begin{aligned} LC &= 0.4 \times 40 \times 10^{-6} \\ &= 16 \times 10^{6} \\ \frac{1}{\sqrt{LC}} &= 250 \\ \frac{L}{C} &= 10^4 \end{aligned}$$

(i) $R_L = 120$; $R_C = 80$
Using equation (6.10),

$$\omega_0 = 250\sqrt{\frac{120^2 - 10^4}{50^2 - 10^4}}$$

As the result is an imaginary number resonance is not possible in this case.

(ii) $R_L = R_C = 80$

$$\omega_0 = 250\sqrt{\frac{80^2 - 10^4}{80^2 - 10^4}}$$
$$= 250 \text{ rad/sec}$$

(iii) $R_L = 80$; $R_C = 0$

$$\omega_0 = 250\sqrt{\frac{80^2 - 10^4}{-10^4}}$$
$$= 150 \text{ rad/sec}$$

(iv) $R_L = R_C = 100$

$$\omega_0 = 250\sqrt{\frac{100^2 - 10^4}{100^2 - 10^4}}$$

As the result is indeterminate, the circuit resonates at all frequencies.

(v) $R_L = R_C = 120$

$$\omega_0 = 250\sqrt{\frac{120^2 - 10^4}{120^2 - 10^4}}$$
$$= 250 \text{ rad/sec}$$

EXAMPLE 6.19

The following information is given in connection with a two branch parallel circuit: $R_L = 10\ \Omega$, $R_C = 20\ \Omega$, $X_C = 40\ \Omega$, $E = 120$ V and frequency = 60 Hz. What are the values of L for resonance and what currents are drawn from the supply under this condition?

SOLUTION

As the frequency is constant, the condition for resonance is

$$\frac{X_L}{R_L^2 + X_L^2} = \frac{X_C}{R_C^2 + X_C^2}$$
$$\Rightarrow \qquad \frac{X_L}{10^2 + X_L^2} = \frac{40}{20^2 + 40^2} = \frac{1}{50}$$
$$\Rightarrow \qquad X_L^2 - 50X_L + 100 = 0$$

Solving we get

$$X_L = 47.913\ \Omega \quad \text{or} \quad 2.087\ \Omega$$

Then the corresponding values of inductances are

$$L = \frac{X_L}{\omega} = 0.127 \text{ H} \quad \text{or} \quad 5.536 \text{ mH}$$

The supply current is

$$I = EG = E\left(G_L + G_C\right)$$

Thus,
$$I = 120\left(\frac{10}{10^2 + 47.913^2} + 0.02\right) = 1.7\text{ A} \quad \text{for } X_L = 47.913\ \Omega$$

or
$$I = 120\left(\frac{10}{10^2 + 2.087^2} + 0.02\right) = 12.7\text{ A} \quad \text{for } X_L = 2.087\ \Omega$$

Exercise Problems

E.P 6.1

Refer the circuit shown in Fig. E.P. 6.1, where R_i is the source resistance

(a) Determine the transfer function of the circuit.
(b) Sketch the magnitude plot with $R_i \neq 0$ and $R_i = 0$.

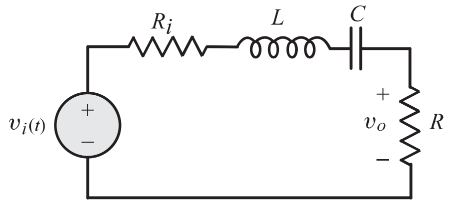

Figure E.P. 6.1

Ans: $\boldsymbol{H(s) = \dfrac{V_o(s)}{V_i(s)} = \dfrac{\frac{R}{L}s}{s^2 + \left(\frac{R+R_i}{L}\right)s + \frac{1}{LC}}}$

E.P 6.2

For the circuit shown in Fig. E.P. 6.2, calculate the following:

(a) $\mathbf{f_0}$, (b) **Q**, (c) $\mathbf{f_{c_1}}$, (d) $\mathbf{f_{c_2}}$ **and** (e) **B**

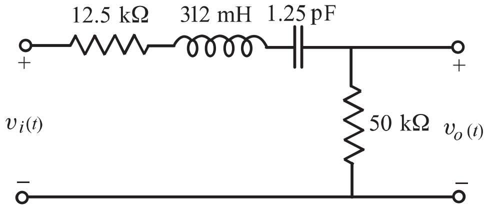

Figure E.P. 6.2

Ans: (a) **254.65 kHz** (b) **8** (c) **239.23 kHz** (d) **271.06 kHz** (e) **31.83 kHz**

E.P 6.3

Refer the circuit shown in Fig. E.P. 6.3, find the output voltage, when (a) $\omega = \omega_0$ (b) $\omega = \omega_1$, and (c) $\omega = \omega_{c2}$.

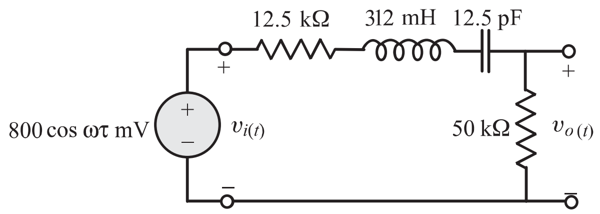

Figure E.P. 6.3

Ans: (a) $\mathbf{640\cos(1.6 \times 10^6\, t)mV}$
(b) $\mathbf{452.55\cos(1.5 \times 10^6\, t + 45°)mV}$
(c) $\mathbf{452.55\cos(1.7 \times 10^6\, t - 45°)mV}$

E.P 6.4

Refer the circuit shown in Fig. E.P. 6.4. Calculate $Z_i(s)$ and then find (a) ω_0 and (b) Q.

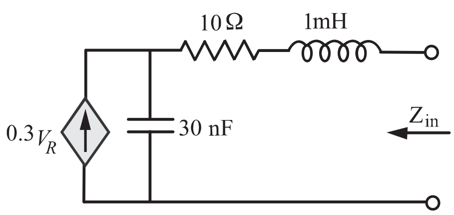

Figure E.P. 6.4

Ans: (a) **364.69 krad/sec**, (b) **36**

E.P 6.5

Refer the circuit shown in Fig. E.P. 6.5. Show that at resonance, $|V_o|_{\max} = \frac{Q|V_s|}{\sqrt{1-\frac{1}{4Q^2}}}$.

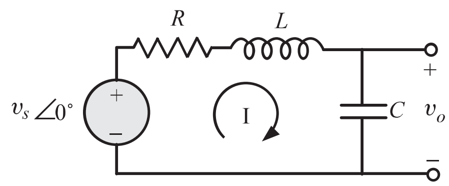

Figure E.P. 6.5

E.P 6.6

Refer the circuit given in Fig. E.P. 6.6, calculate ω_0, Q and $|V_o|_{\max}$

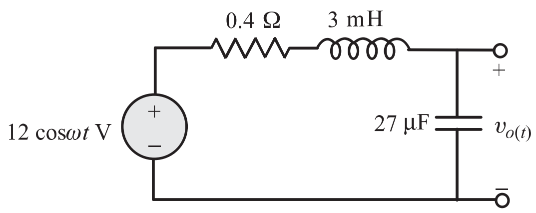

Figure E.P. 6.6

Ans: **3513.64 rad/sec**, **26.35**, **316 volts**.

E.P 6.7

A parallel network, which is driven by a variable frequency of 4 A current source has the following values: $R = 1$ kΩ , $L = 10$ mH, $C = 100$ μF. Find the band width of the network, the half power frequenies and the voltage across the network at half-power frequencies.

Ans: **10 rad/sec**, **995 rad/sec**, **10005 rad/sec**

E.P 6.8

For the circuit shown in Fig. E.P. 6.8, determine the expression for the magnitude response, $|Z_{in}|$ versus ω and Z_{in} at $\omega_0 = \frac{1}{\sqrt{LC}}$.

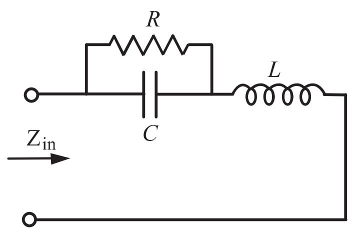

Figure E.P. 6.8

Ans: (a) $|\boldsymbol{Z}_{\mathbf{in}}| = \sqrt{\frac{(R-\omega^2 RLC)^2+(\omega L)^2}{1+(\omega RC)^2}}$, (b) $|\boldsymbol{Z}_{\mathbf{in}}| = \frac{1}{\sqrt{\frac{C}{L}\left(1+R^2\frac{C}{L}\right)}}$

E.P 6.9

A coil under test may be represented by the model of L in series with R. The coil is connected in series with a variable capacitor. A voltage source $v(t) = 10\cos 1000\, t$ volts is connected to the coil. The capacitor is varied and it is found that the current is maximum when $C = 10\mu$F. Also, when $C = 12.5\mu$F, the current is 0.707 of the maximum value. Find Q of the coil at $\omega = 1000$ rad/sec.

Ans: **5**

E.P 6.10

A fresher in the devices lab for sake of curiosity sets up a series RLC network as shown in Fig. E.P.6.10. The capacitor can withstand very high voltages. Is it safe to touch the capacitor at resonance? Find the voltage across the capacitor.

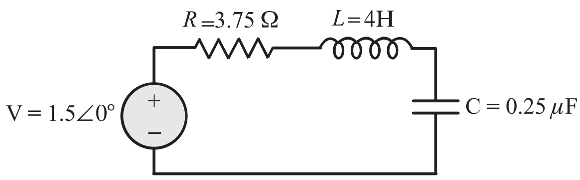

Figure E.P. 6.10

Ans: *Not safe*, $|\boldsymbol{V_c}|_{\mathbf{max}} = \mathbf{1600\,V}$

Outcomes:

1. Analyze resonant circuits both in time and frequency domains.
2. Determine the resonant frequency and bandwidth of a series or parallel circuit.
3. Sketch the impedance, current, and power in resonant circuits.
4. Derive the network function for an RLC resonant circuit
5. Analyze the frequency response of an RLC resonant circuit
6. Perform laboratory measurements to determine the frequency response of a circuit
7. Understand the impact of component values on the BW, Q, and □0.

Resources:

1. http://www.electronics-tutorials.ws/accircuits/series-resonance.html
2. http://en.wikipedia.org/wiki/RLC_circuit
3. http://www.electronics-tutorials.ws/accircuits/series-circuit.html
4. http://en.wikipedia.org/wiki/LC_circuit

UNIT - 4
TRANSIENT BEHAVIOUR AND INITIAL CONDITIONS

Objectives:
1. To Develop and solve mathematical representations for simple RLC circuits.
2. Compute initial conditions for current and voltage in first order R-L and R-C capacitor and inductor circuits.
3. Compute time response of current and voltage in first order R-L and R-C capacitor and inductor circuits.
4. Compute initial conditions for current and voltage in second order RLC Circuits.
5. Compute time response of current and voltage in second order RLC circuits.
6. Compute time response of current and voltage in second order RC circuits.

4.1 Introduction

There are many reasons for studying initial and final conditions. The most important reason is that the initial and final conditions evaluate the arbitrary constants that appear in the general solution of a differential equation.

In this chapter, we concentrate on finding the change in selected variables in a circuit when a switch is thrown open from closed position or vice versa. The time of throwing the switch is considered to be $t = 0$, and we want to determine the value of the variable at $t = 0^-$ and at $t = 0^+$, immediately before and after throwing the switch. Thus a switched circuit is an electrical circuit with one or more switches that open or close at time $t = 0$. We are very much interested in the change in currents and voltages of energy storing elements after the switch is thrown since these variables along with the sources will dictate the circuit behaviour for $t > 0$.

Initial conditions in a network depend on the past history of the circuit (before $t = 0^-$) and structure of the network at $t = 0^+$, (after switching). Past history will show up in the form of capacitor voltages and inductor currents. The computation of all voltages and currents and their derivatives at $t = 0^+$ is the main aim of this chapter.

4.2 Initial and final conditions in elements

4.2.1 The inductor

The switch is closed at $t = 0$. Hence $t = 0^-$ corresponds to the instant when the switch is just open and $t = 0^+$ corresponds to the instant when the switch is just closed.

The expression for current through the inductor is given by

$$i = \frac{1}{L}\int_{-\infty}^{t} v d\tau$$

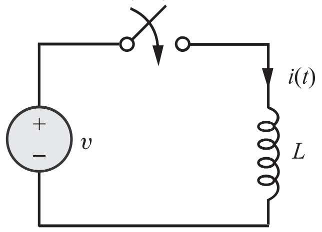

Figure 4.1 Circuit for explaining switching action of an inductor

$$\Rightarrow \qquad i = \frac{1}{L}\int_{-\infty}^{0^-} v d\tau + \frac{1}{L}\int_{0^-}^{t} v d\tau$$

$$\Rightarrow \qquad i(t) = i(0^-) + \frac{1}{L}\int_{0^-}^{t} v d\tau$$

Putting $t = 0^+$ on both sides, we get

$$i(0^+) = i(0^-) + \frac{1}{L}\int_{0^-}^{0^+} v d\tau$$

$$\Rightarrow \qquad i(0^+) = i(0^-)$$

The above equation means that the current in an inductor cannot change instantaneously. Consequently, if $i(0^-) = 0$, we get $i(0^+) = 0$. This means that at $t = 0^+$, inductor will act as an open circuit, irrespective of the voltage across the terminals. If $i(0^-) = I_o$, then $i(0^+) = I_o$. In this case at $t = 0^+$, the inductor can be thought of as a current source of I_o A. The equivalent circuits of an inductor at $t = 0^+$ is shown in Fig. 4.2.

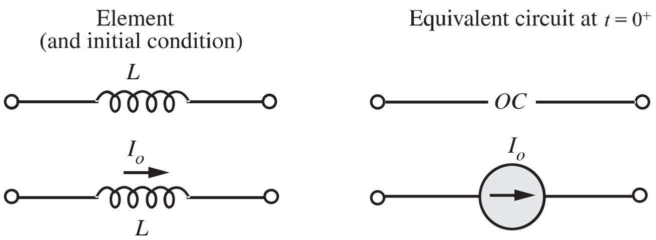

Figure 4.2 The initial-condition equivalent circuits of an inductor

The final-condition equivalent circuit of an inductor is derived from the basic relationship

$$v = L\frac{di}{dt}$$

Under steady condition, $\frac{di}{dt} = 0$. This means, $v = 0$ and hence L acts as short at $t = \infty$ (final or steady state). The final-condition equivalent circuits of an inductor is shown in Fig.4.3.

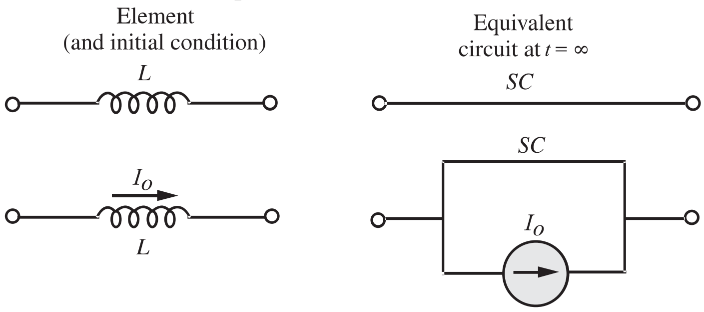

Figure 4.3 The final-condition equivalent circuit of an inductor

4.2.2 The capacitor

The switch is closed at $t = 0$. Hence, $t = 0^-$ corresponds to the instant when the switch is just open and $t = 0^+$ corresponds to the instant when the switch is just closed. The expression for voltage across the capacitor is given by

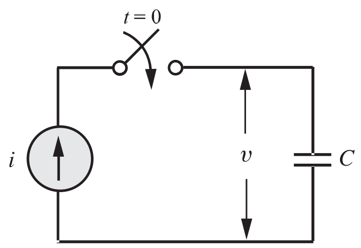

Figure 4.4 Circuit for explaining switching action of a Capacitor

$$v = \frac{1}{C}\int_{-\infty}^{t} i d\tau$$

$$\Rightarrow \qquad v(t) = \frac{1}{C}\int_{-\infty}^{0^-} i d\tau + \frac{1}{C}\int_{0^-}^{t} i d\tau$$

$$\Rightarrow \qquad v(t) = v(0^-) + \frac{1}{C}\int_{0^-}^{t} i d\tau$$

Evaluating the expression at $t = 0^+$, we get

$$v(0^+) = v(0^-) + \frac{1}{C}\int_{0^-}^{0^+} i d\tau \quad \Rightarrow \quad v(0^+) = v(0^-)$$

Thus the voltage across a capacitor cannot change instantaneously.

If $v(0^-) = 0$, then $v(0^+) = 0$. This means that at $t = 0^+$, capacitor C acts as short circuit. Conversely, if $v(0^-) = \frac{q_0}{C}$ then $v(0^+) = \frac{q_0}{C}$. These conclusions are summarized in Fig. 4.5.

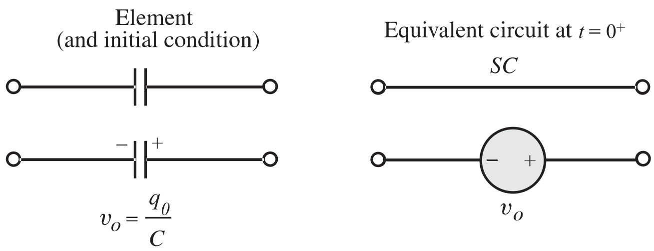

Figure 4.5 Initial-condition equivalent circuits of a capacitor

The final–condition equivalent network is derived from the basic relationship

$$i = C\frac{dv}{dt}$$

Under steady state condition, $\frac{dv}{dt} = 0$. This is, at $t = \infty$, $i = 0$. This means that $t = \infty$ or in steady state, capacitor C acts as an open circuit. The final condition equivalent circuits of a capacitor is shown in Fig. 4.6.

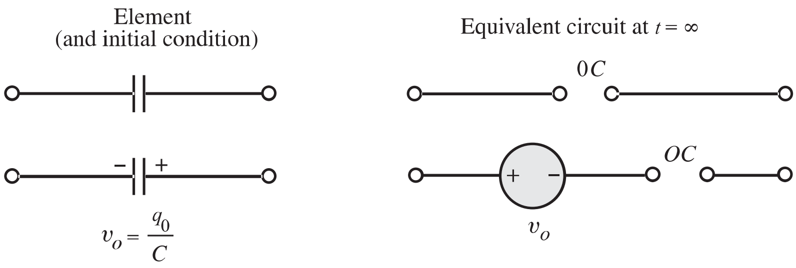

Figure 4.6 Final-condition equivalent circuits of a capacitor

4.2.3 The resistor

The cause–effect relation for an ideal resistor is given by $v = Ri$. From this equation, we find that the current through a resistor will change instantaneously if the voltage changes instantaneously. Similarly, voltage will change instantaneously if current changes instantaneously.

4.3 Procedure for evaluating initial conditions

There is no unique procedure that must be followed in solving for initial conditions. We usually solve for initial values of currents and voltages and then solve for the derivatives. For finding initial values of currents and voltages, an equivalent network of the original network at $t = 0^+$ is constructed according to the following rules:

(1) Replace all inductors with open circuit or with current sources having the value of current flowing at $t = 0^+$.

(2) Replace all capacitors with short circuits or with a voltage source of value $v_o = \frac{q_0}{C}$ if there is an initial charge.

(3) Resistors are left in the network without any changes.

EXAMPLE 4.1

Refer the circuit shown in Fig. 4.7(a). Find $i_1(0^+)$ and $i_L(0^+)$. The circuit is in steady state for $t < 0$.

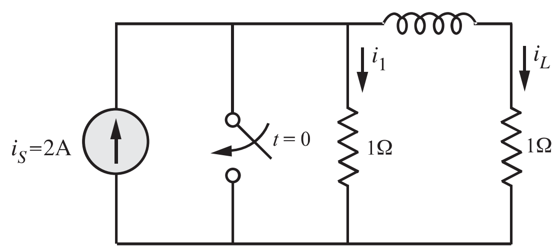

Figure 4.7(a)

SOLUTION

The symbol for the switch implies that it is open at $t = 0^-$ and then closed at $t = 0^+$. The circuit is in steady state with the switch open. This means that at $t = 0^-$, inductor L is short. Fig.4.7(b) shows the original circuit at $t = 0^-$.
Using the current division principle,

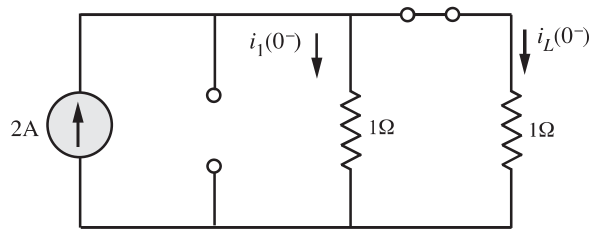

Figure 4.7(b)

$$i_L(0^-) = \frac{2 \times 1}{1+1} = 1\text{A}$$

Since the current in an inductor cannot change instantaneously, we have

$$i_L(0^+) = i_L(0^-) = 1\text{A}$$

At $t = 0^-$, $i_1(0^-) = 2 - 1 = 1\text{A}$. Please note that the current in a resistor can change instantaneously. Since at $t = 0^+$, the switch is just closed, the voltage across R_1 will be equal to zero because of the switch being short circuited and hence,

$$i_1(0^+) = 0\text{A}$$

Thus, the current in the resistor changes abruptly form 1A to 0A.

EXAMPLE 4.2

Refer the circuit shown in Fig. 4.8. Find $v_C(0^+)$. Assume that the switch was in closed state for a long time.

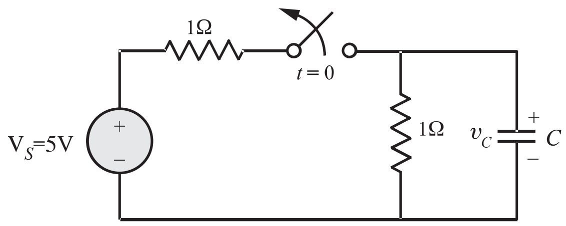

Figure 4.8

SOLUTION

The symbol for the switch implies that it is closed at $t = 0^-$ and then opens at $t = 0^+$. Since the circuit is in steady state with the switch closed, the capacitor is represented as an open circuit at $t = 0^-$. The equivalent circuit at $t = 0^-$ is as shown in Fig. 4.9.

$$v_C(0^-) = i(0^-)R_2$$

Using the principle of voltage divider,

$$v_C(0^-) = \frac{V_S}{R_1 + R_2} R_2 = \frac{5 \times 1}{1+1} = 2.5 \text{ V}$$

Since the voltage across a capacitor cannot change instaneously, we have

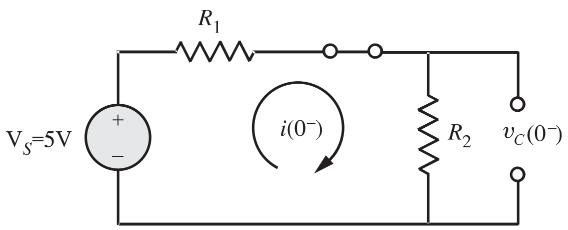

Figure 4.9

$$v_C(0^+) = v_C(0^-) = 2.5\text{V}$$

That is, when the switch is opened at $t = 0$, and if the source is removed from the circuit, still $v_C(0^+)$ remains at 2.5 V.

EXAMPLE 4.3

Refer the circuit shown in Fig 4.10. Find $i_L(0^+)$ and $v_C(0^+)$. The circuit is in steady state with the switch in closed condition.

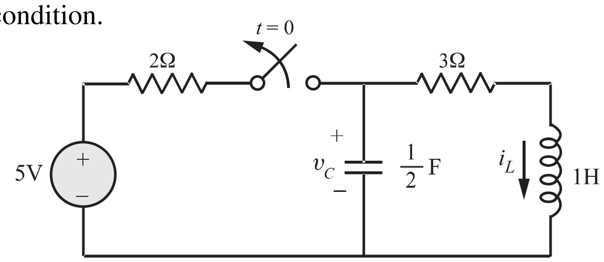

Figure 4.10

SOLUTION

The symbol for the switch implies, it is closed at $t = 0^-$ and then opens at $t = 0^+$. In order to find $v_C(0^-)$ and $i_L(0^-)$ we replace the capacitor by an open circuit and the inductor by a short circuit, as shown in Fig.4.11, because in the steady state L acts as a short circuit and C as an open circuit.

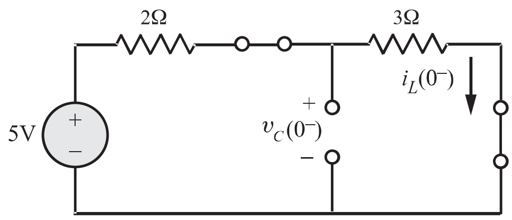

Figure 4.11

$$i_L(0^-) = \frac{5}{2+3} = 1 \text{ A}$$

Using the voltage divider principle, we note that

$$v_C(0^-) = \frac{5 \times 3}{3+2} = 3 \text{ V}$$

Then we note that:

$$v_C(0^+) = v_C(0^-) = 3 \text{ V}$$
$$i_L(0^+) = i_L(0^-) = 2 \text{ A}$$

EXAMPLE 4.4

In the given network, K is closed at $t = 0$ with zero current in the inductor. Find the values of $i, \frac{di}{dt}, \frac{d^2i}{dt^2}$ at $t = 0^+$ if $R = 8\Omega$ and $L = 0.2$H. Refer the Fig. 4.12(a).

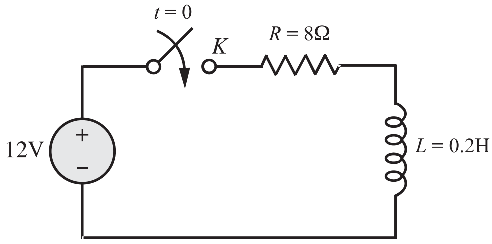

Figure 4.12(a)

SOLUTION

The symbol for the switch implies that it is open at $t = 0^-$ and then closes at $t = 0^+$. Since the current i through the inductor at $t = 0^-$ is zero, it implies that $i(0^+) = i(0^-) = 0$.

To find $\frac{di(0^+)}{dt}$ and $\frac{d^2i(0^+)}{dt^2}$:

Applying KVL clockwise to the circuit shown in Fig. 4.12(b), we get

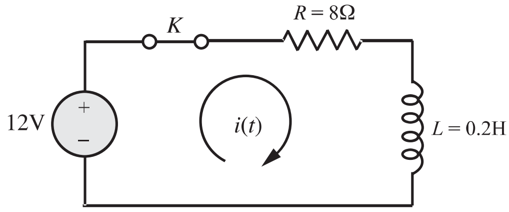

Figure 4.12(b)

$$Ri + L\frac{di}{dt} = 12$$

$$\Rightarrow \quad 8i + 0.2\frac{di}{dt} = 12 \tag{4.1}$$

At $t = 0^+$, the equation (4.1) becomes

$$8i(0^+) + 0.2\frac{di(0^+)}{dt} = 12$$

$$\Rightarrow \quad 8 \times 0 + 0.2\frac{di(0^+)}{dt} = 12$$

$$\Rightarrow \quad \frac{di(0^+)}{dt} = \frac{12}{0.2} = \mathbf{60\ A/sec}$$

Differentiating equation (4.1) with respect to t, we get

$$8\frac{di}{dt} + 0.2\frac{d^2i}{dt^2} = 0$$

At $t = 0^+$, the above equation becomes

$$8\frac{di(0^+)}{dt} + 0.2\frac{d^2i(0^+)}{dt^2} = 0$$

$$\Rightarrow \quad 8 \times 60 + 0.2\frac{d^2i(0^+)}{dt^2} = 0$$

Hence $$\frac{d^2i(0^+)}{dt^2} = \mathbf{-2400\ A/sec^2}$$

EXAMPLE 4.5

In the network shown in Fig. 4.13, the switch is closed at $t = 0$. Determine $i, \frac{di}{dt}, \frac{d^2i}{dt^2}$ at $t = 0^+$.

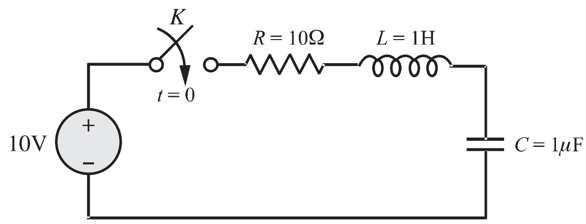

Figure 4.13

SOLUTION

The symbol for the switch implies that it is open at $t = 0^-$ and then closes at $t = 0^+$. Since there is no current through the inductor at $t = 0^-$, it implies that $i(0^+) = i(0^-) = 0$.

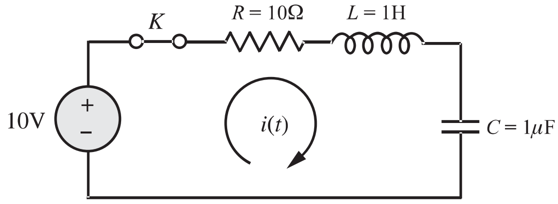

Figure 4.14

Writing *KVL clockwise* for the circuit shown in Fig. 4.14, we get

$$Ri + L\frac{di}{dt} + \frac{1}{C}\int_0^t i(\tau)d\tau = 10 \tag{4.2}$$

$$\Rightarrow \qquad Ri + L\frac{di}{dt} + v_C(t) = 10 \tag{4.2a}$$

Putting $t = 0^+$ in equation (4.2a), we get

$$Ri\left(0^+\right) + L\frac{di\left(0^+\right)}{dt} + v_C\left(0^+\right) = 10$$

$$\Rightarrow \qquad R \times 0 + L\frac{di\left(0^+\right)}{dt} + 0 = 10$$

$$\Rightarrow \qquad \frac{di\left(0^+\right)}{dt} = \frac{10}{L} = \mathbf{10\ A/sec}$$

Differentiating equation (4.2) with respect to t, we get

$$R\frac{di}{dt} + L\frac{d^2i}{dt^2} + \frac{i(t)}{C} = 0$$

At $t = 0^+$, the above equation becomes

$$R\frac{di\left(0^+\right)}{dt} + L\frac{d^2i\left(0^+\right)}{dt^2} + \frac{i\left(0^+\right)}{C} = 0$$

$$\Rightarrow \qquad R \times 10 + L\frac{d^2i\left(0^+\right)}{dt^2} + \frac{0}{C} = 0$$

$$\Rightarrow \qquad 100 + \frac{d^2i\left(0^+\right)}{dt^2} = 0$$

Hence at $t = 0^+$,
$$\frac{d^2i\left(0^+\right)}{dt^2} = \mathbf{-100\ A/sec^2}$$

EXAMPLE 4.6

Refer the circuit shown in Fig. 4.15. The switch K is changed from position 1 to position 2 at $t = 0$. Steady-state condition having been reached at position 1. Find the values of i, $\frac{di}{dt}$, and $\frac{d^2i}{dt^2}$ at $t = 0^+$.

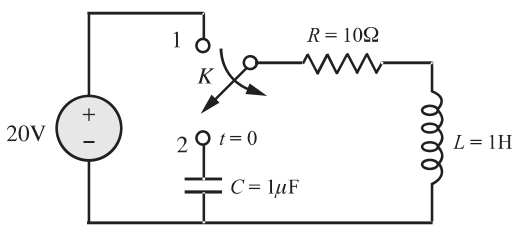

Figure 4.15

SOLUTION

The symbol for switch K implies that it is in position 1 at $t = 0^-$ and in position 2 at $t = 0^+$. Under steady-state condition, inductor acts as a short circuit. Hence at $t = 0^-$, the circuit diagram is as shown in Fig. 4.16.

$$i\left(0^-\right) = \frac{20}{10} = \mathbf{2A}$$

Since the current through an inductor cannot change instantaneously, $i\left(0^+\right) = i\left(0^-\right) = 2\text{A}$. Since there is no initial charge on the capacitor, $v_C\left(0^-\right) = 0$. Since the voltage across a capacitor cannot change instantaneously, $v_C\left(0^+\right) = v_C\left(0^-\right) = 0$. Hence at $t = 0^+$ the circuit diagram is as shown in Fig. 4.17(a).

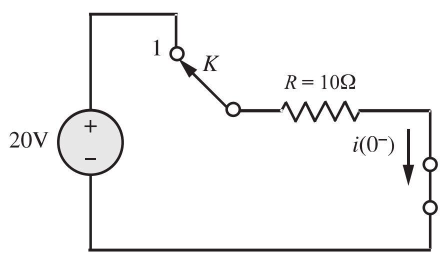

Figure 4.16

For $t \geq 0^+$, the circuit diagram is as shown in Fig. 4.17(b).

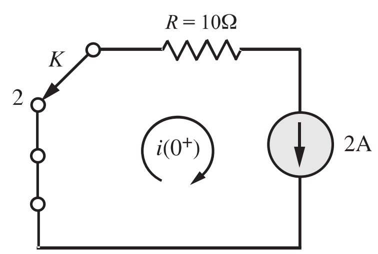

Figure 4.17(a)

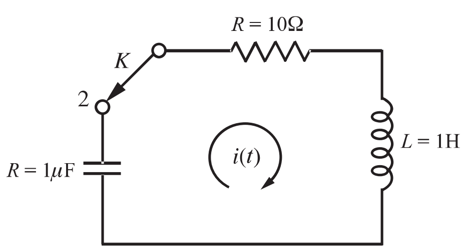

Figure 4.17(b)

Applying KVL clockwise to the circuit shown in Fig. 4.17(b), we get

$$Ri(t) + L\frac{di(t)}{dt} + \frac{1}{C}\int_{0^+}^{t} i(\tau)d\tau = 0 \tag{4.3}$$

$$\Rightarrow \qquad Ri(t) + L\frac{di(t)}{dt} + v_C(t) = 0 \tag{4.3a}$$

At $t = 0^+$ equation (4.3a) becomes

$$Ri\left(0^+\right) + L\frac{di\left(0^+\right)}{dt} + v_C\left(0^+\right) = 0$$

$$\Rightarrow \qquad R \times 2 + L\frac{di\left(0^+\right)}{dt} + 0 = 0$$

$$\Rightarrow \qquad 20 + \frac{di\left(0^+\right)}{dt} = 0$$

$$\Rightarrow \qquad \frac{di\left(0^+\right)}{dt} = \mathbf{-20\ A/sec}$$

Differentiating equation (4.3) with respect to t, we get

$$R\frac{di}{dt} + L\frac{d^2i}{dt^2} + \frac{i}{C} = 0$$

At $t = 0^+$, we get

$$R\frac{di\left(0^+\right)}{dt} + L\frac{d^2i\left(0^+\right)}{dt^2} + \frac{i\left(0^+\right)}{C} = 0$$

$$\Rightarrow \qquad R \times (-20) + L\frac{d^2i\left(0^+\right)}{dt^2} + \frac{2}{C} = 0$$

Hence, $$\frac{d^2i\left(0^+\right)}{dt^2} \approx \mathbf{-2 \times 10^6\ A/sec^2}$$

EXAMPLE 4.7

In the network shown in Fig. 4.18, the switch is moved from position 1 to position 2 at $t = 0$. The steady-state has been reached before switching. Calculate i, $\frac{di}{dt}$, and $\frac{d^2i}{dt^2}$ at $t = 0^+$.

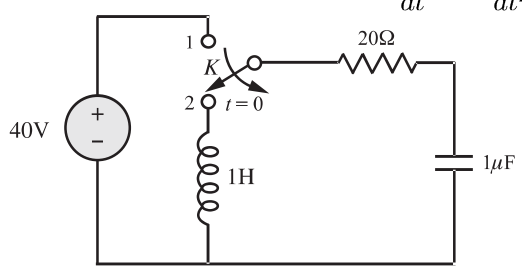

Figure 4.18

SOLUTION

The symbol for switch K implies that it is in position 1 at $t = 0^-$ and in position 2 at $t = 0^+$. Under steady-state condition, a capacitor acts as an open circuit. Hence at $t = 0^-$, the circuit diagram is as shown in Fig. 4.18(a).

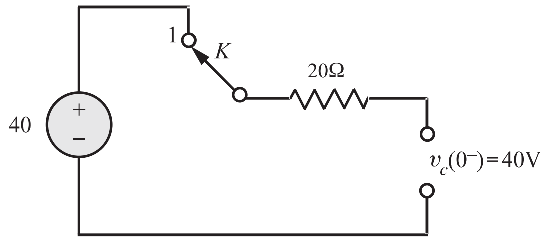

Figure 4.18(a)

We know that the voltage across a capacitor cannot change instantaneously. This means that $v_C\left(0^+\right) = v_C\left(0^-\right) = 40$ V.

At $t = 0^-$, inductor is not energized. This means that $i\left(0^-\right) = 0$. Since current in an inductor cannot change instantaneously, $i\left(0^+\right) = i\left(0^-\right) = \mathbf{0}$. Hence, the circuit diagram at $t = 0^+$ is as shown in Fig. 4.18(b).

The circuit diagram for $t \geq 0^+$ is as shown in Fig.4.18(c).

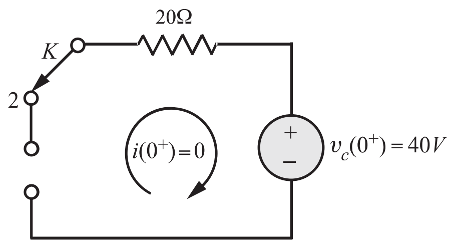

Figure 4.18(b)

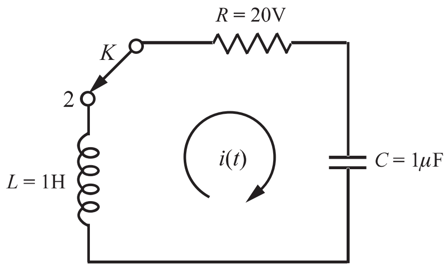

Figure 4.18(c)

Applying KVL clockwise, we get

$$Ri + L\frac{di}{dt} + \frac{1}{C}\int_{0^+}^{t} i(\tau)d\tau = 0 \tag{4.4}$$

$$\Rightarrow \qquad Ri + L\frac{di}{dt} + v_C(t) = 0$$

At $t = 0^+$, we get

$$Ri(0^+) + L\frac{di(0^+)}{dt} + v_C(0^+) = 0$$

$$\Rightarrow \qquad 20 \times 0 + 1\frac{di(0^+)}{dt} + 40 = 0$$

$$\Rightarrow \qquad \frac{di(0^+)}{dt} = \mathbf{-40A/\,sec}$$

Diferentiating equation (4.4) with respect to t, we get

$$R\frac{di}{dt} + L\frac{d^2i}{dt^2} + \frac{i}{C} = 0$$

Putting $t = 0^+$ in the above equation, we get

$$R\frac{di(0^+)}{dt} + L\frac{d^2i(0^+)}{dt^2} + \frac{i(0^+)}{C} = 0$$

$$\Rightarrow \qquad R \times (-40) + L\frac{d^2i(0^+)}{dt^2} + \frac{0}{C} = 0$$

Hence $$\frac{d^2i(0^+)}{dt^2} = \mathbf{800A/\,sec^2}$$

EXAMPLE 4.8

In the network shown in Fig. 4.19, $v_1(t) = e^{-t}$ for $t \geq 0$ and is zero for all $t < 0$. If the capacitor is initially uncharged, determine the value of $\frac{d^2v_2}{dt^2}$ and $\frac{d^3v_2}{dt^3}$ at $t = 0^+$.

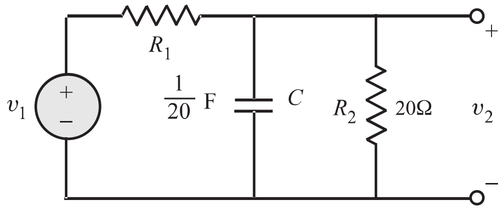

Figure 4.19

SOLUTION

Since the capacitor is initially uncharged, $v_2(0^+) = 0$

Referring to Fig. 4.19(a) and *applying KCL at node* $v_2(t)$:

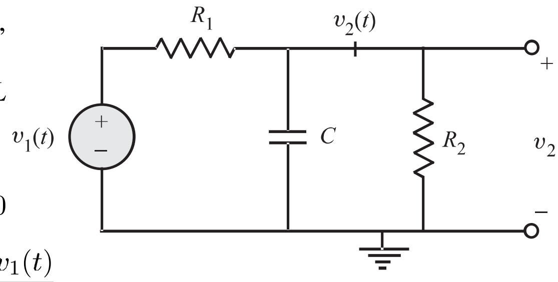

Figure 4.19(a)

$$\frac{v_2(t) - v_1(t)}{R_1} + C\frac{dv_2(t)}{dt} + \frac{v_2(t)}{R_2} = 0$$

$$\Rightarrow \qquad \left(\frac{1}{R_1} + \frac{1}{R_2}\right)v_2(t) + C\frac{dv_2(t)}{dt} = \frac{v_1(t)}{R_1}$$

$$\Rightarrow \qquad 0.15v_2 + 0.05\frac{dv_2}{dt} = 0.1e^{-t} \tag{4.5}$$

Putting $t = 0^+$, we get

$$0.15v_2(0^+) + 0.05\frac{dv_2(0^+)}{dt} = 0.1$$

$$\Rightarrow \qquad 0.15 \times 0 + 0.05\frac{dv_2(0^+)}{dt} = 0.1$$

$$\Rightarrow \qquad \frac{dv_2(0^+)}{dt} = \frac{0.1}{0.05} = 2\text{ Volts/ sec}$$

Differentiating equation (4.5) with respect to t, we get

$$0.15\frac{dv_2}{dt} + 0.05\frac{d^2v_2}{dt^2} = -0.1e^{-t} \tag{4.6}$$

Putting $t = 0^+$ in equation (4.6), we find that

$$\frac{d^2v_2(0^+)}{dt^2} = \frac{-0.1 - 0.3}{0.05} = \mathbf{-8\ Volts/\ sec^2}$$

Again differentiating equation (4.6) with respect to t, we get

$$0.15\frac{d^2v_2}{dt^2} + 0.05\frac{d^3v_2}{dt^3} = 0.1e^{-t} \tag{4.7}$$

Putting $t = 0^+$ in equation (4.7) and solving for $\frac{d^3v_2}{dt^3}(0^+)$, we find that

$$\frac{d^3v_2(0^+)}{dt^3} = \frac{0.1 + 1.2}{0.05} = \mathbf{26\ Volts/\ sec^3}$$

EXAMPLE 4.9

Refer the circuit shown in Fig. 4.20. The circuit is in steady state with switch K closed. At $t = 0$, the switch is opened. Determine the voltage across the switch, v_K and $\frac{dv_K}{dt}$ at $t = 0^+$.

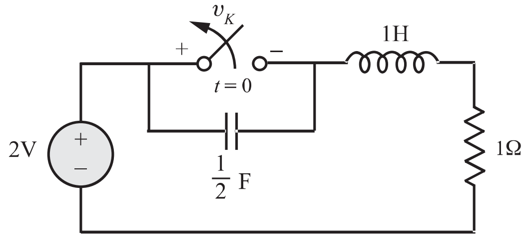

Figure 4.20

SOLUTION

The switch remains closed at $t = 0^-$ and open at $t = 0^+$. Under steady condition, inductor acts as a short circuit and hence the circuit diagram at $t = 0^-$ is as shown in Fig. 4.21(a).

$$\text{Therefore,} \quad v_K(0^+) = v_K(0^-)$$
$$= 0\text{ V}$$

For $t \geq 0^+$ the circuit diagram is as shown in Fig. 4.21(b).

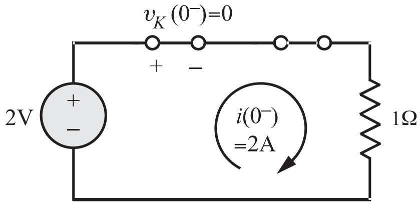

Figure 4.21(a)

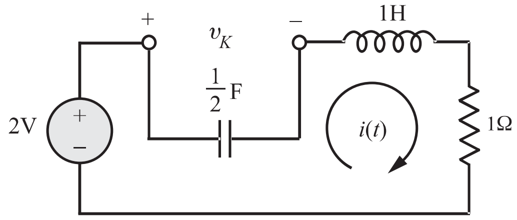

Figure 4.21(b)

$$i(t) = C\frac{dv_K}{dt}$$

At $(t) = 0^+$, we get

$$i(0^+) = C\frac{dv_K(0^+)}{dt}$$

Since the current through an inductor cannot change instantaneously, we get

$$i(0^+) = i(0^-) = 2\text{A}$$

Hence, $$2 = C\frac{dv_K(0^+)}{dt}$$

$$\frac{dv_K(0^+)}{dt} = \frac{2}{C} = \frac{2}{\frac{1}{2}} = \mathbf{4V/\,sec}$$

EXAMPLE 4.10

In the given network, the switch K is opened at $t = 0$. At $t = 0^+$, solve for the values of v, $\frac{dv}{dt}$ and $\frac{d^2v}{dt^2}$ if $I = 2$ A, $R = 200\ \Omega$ and $L = 1$ H

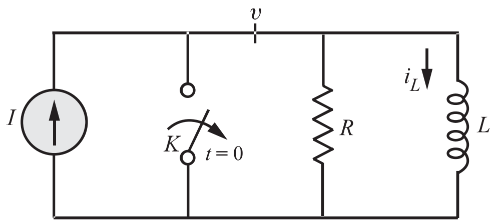

Figure 4.22

SOLUTION

The switch is opened at $t = 0$. This means that at $t = 0^-$, it is closed and at $t = 0^+$, it is open. Since $i_L(0^-) = 0$, we get $i_L(0^+) = 0$. The circuit at $t = 0^+$ is as shown in Fig. 4.23(a).

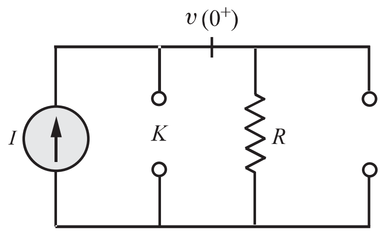

Figure 4.23(a)

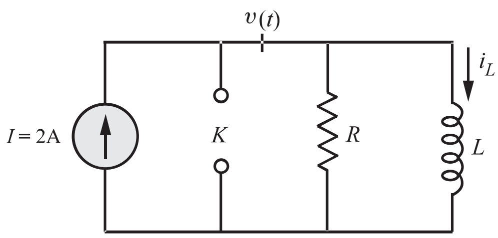

Figure 4.23(b)

$$v(0^+) = IR$$
$$= 2 \times 200$$
$$= \mathbf{400\ Volts}$$

Refer to the circuit shown in Fig. 4.23(b).

For $t \geq 0^+$, the *KCL* at node $v(t)$ gives

$$I = \frac{v(t)}{R} + \frac{1}{L}\int_{0^+}^{t} v(\tau)d\tau \tag{4.8}$$

Differentiating both sides of equation (4.8) with respect to t, we get

$$0 = \frac{1}{R}\frac{dv(t)}{dt} + \frac{1}{L}v(t) \tag{4.8a}$$

At $t = 0^+$, we get

$$\frac{1}{R}\frac{dv(0^+)}{dt} + \frac{1}{L}v(0^+) = 0$$

$$\Rightarrow \quad \frac{1}{200}\frac{dv(0^+)}{dt} + \frac{1}{1} \times 400 = 0$$

$$\Rightarrow \quad \frac{dv(0^+)}{dt} = \mathbf{-8 \times 10^4\ V/\sec}$$

Again differentiating equation (4.8a), we get

$$\frac{1}{R}\frac{d^2v(t)}{dt^2} + \frac{1}{L}\frac{dv(t)}{dt} = 0$$

At $t = 0^+$, we get

$$\frac{1}{200}\frac{d^2v(0^+)}{dt^2} + \frac{1}{1}\frac{dv(0^+)}{dt} = 0$$

$$\Rightarrow \quad \frac{d^2v(0^+)}{dt^2} = 200 \times 8 \times 10^4$$

$$= \mathbf{16 \times 10^6\ V/\sec^2}$$

EXAMPLE 4.11

In the circuit shown in Fig. 4.24, a steady state is reached with switch K open. At $t = 0$, the switch is closed. For element values given, determine the values of $v_a(0^-)$ and $v_a(0^+)$.

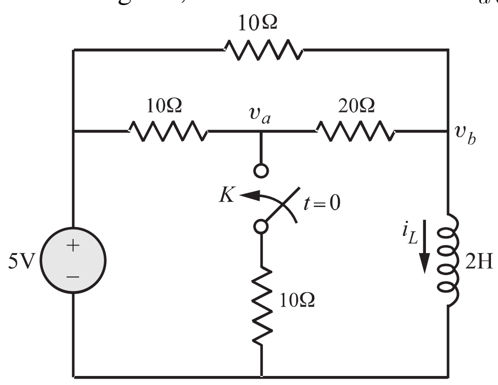

Figure 4.24

SOLUTION

At $t = 0^-$, the switch is open and at $t = 0^+$, the switch is closed. Under steady conditions, inductor L acts as a short circuit. Also the steady state is reached with switch K open. Hence, the circuit diagram at $t = 0^-$ is as shown in Fig.4.25(a).

$$i_L(0^-) = \frac{5}{30} + \frac{5}{10} = \frac{2}{3}\text{A}$$

Using the voltage divider principle:

$$v_a(0^-) = \frac{5 \times 20}{10 + 20} = \frac{\mathbf{10}}{\mathbf{3}}\ \mathbf{V}$$

Since the current in an inductor cannot change instantaneously,

$$i_L(0^+) = i_L(0^-) = \frac{2}{3}\ \text{A}.$$

At $t = 0^+$, the circuit diagram is as shown in Fig. 4.25(b).

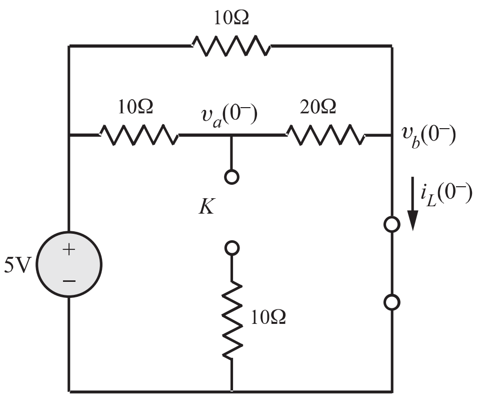

Figure 4.25(a)

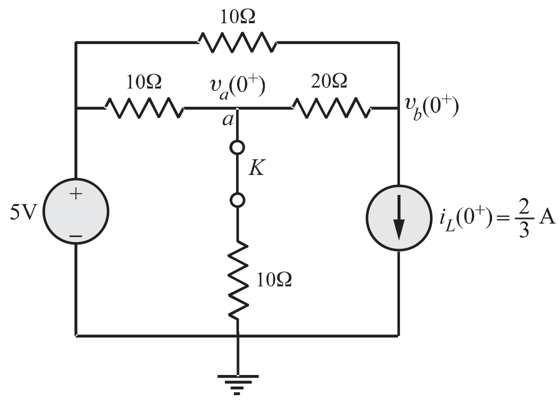

Figure 4.25(b)

Refer the circuit in Fig. 4.25(b).

KCL at node a:

$$\frac{v_a(0^+) - 5}{10} + \frac{v_a(0^+)}{10} + \frac{v_a(0^+) - v_b(0^+)}{20} = 0$$

$$\Rightarrow \quad v_a(0^+)\left[\frac{1}{10} + \frac{1}{10} + \frac{1}{20}\right] - v_b(0^+)\left[\frac{1}{20}\right] = \frac{5}{10}$$

KCL at node b:

$$\frac{v_b(0^+) - v_a(0^+)}{20} + \frac{v_b(0^+) - 5}{10} + \frac{2}{3} = 0$$

$$\Rightarrow \quad -v_a(0^+)\left[\frac{1}{20}\right] + v_b(0^+)\left[\frac{1}{20} + \frac{1}{10}\right] = \frac{5}{10} - \frac{2}{3}$$

Solving the above two nodal equations, we get,

$$v_a(0^+) = \frac{\mathbf{40}}{\mathbf{21}}\mathbf{V}$$

EXAMPLE 4.12

Find $i_L(0^+), v_C(0^+), \dfrac{dv_C(0^+)}{dt}$ and $\dfrac{di_L(0^+)}{dt}$ for the circuit shown in Fig. 4.26.

Assume that switch 1 has been opened and switch 2 has been closed for a long time and steady–state conditions prevail at $t = 0^-$.

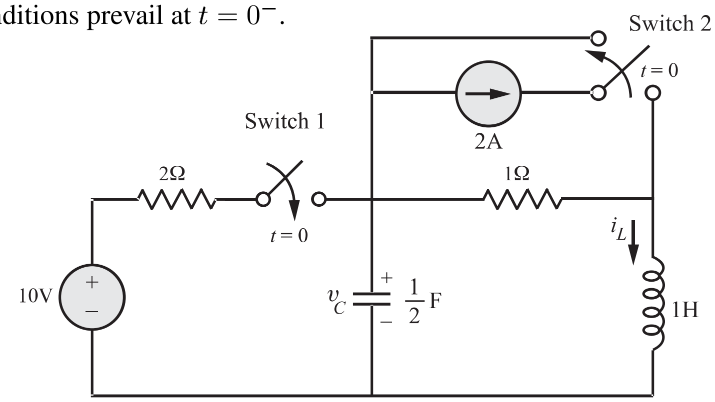

Figure 4.26

SOLUTION

At $t = 0^-$, switch 1 is open and switch 2 is closed, whereas at $t = 0^+$, switch 1 is closed and switch 2 is open.

First, let us redraw the circuit at $t = 0^-$ by replacing the inductor with a short circuit and the capacitor with an open circuit as shown in Fig. 4.27(a).

From Fig. 4.27(b), we find that $i_L(0^-) = 0$

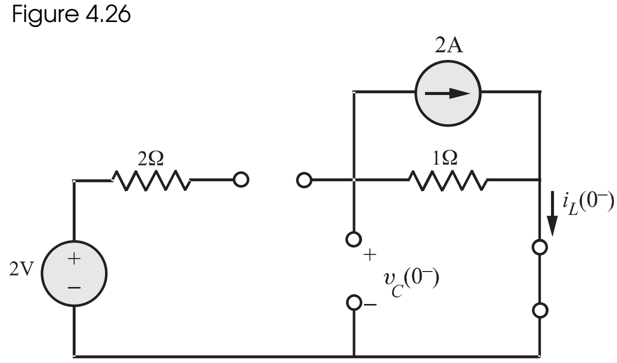

Figure 4.27(a)

$\Rightarrow$

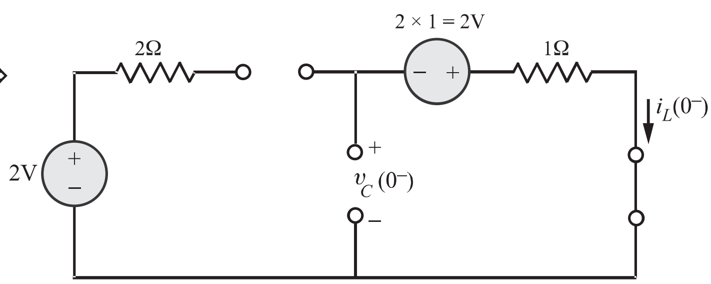

Figure 4.27(b)

Applying KVL clockwise to the loop on the right, we get

$$-v_C(0^-) - 2 + 1 \times 0 = 0$$
$$\Rightarrow \qquad v_C(0^-) = -2 \text{ V}$$

Hence, at $t = 0^+$: $i_L(0^+) = i_L(0^-) = \mathbf{0A}$
$$v_C(0^+) = v_C(0^-) = \mathbf{-2V}$$

The circuit diagram for $t \geq 0^+$ is shown in Fig. 4.27(c).

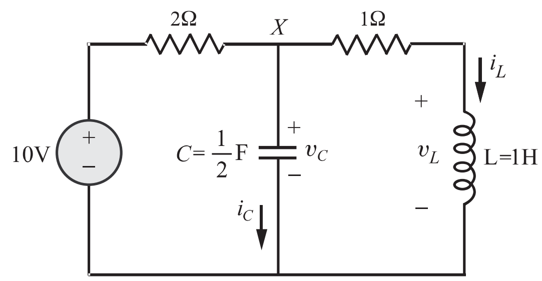

Figure 4.27(c)

Applying KVL for right–hand mesh, we get

$$v_L - v_C + i_L = 0$$

At $t = 0^+$, we get

$$v_L(0^+) = v_C(0^+) - i_L(0^+)$$
$$= -2 - 0 = -2 \text{ V}$$

We know that $$v_L = L\frac{di_L}{dt}$$

At $t = 0^+$, we get

$$\frac{di_L(0^+)}{dt} = \frac{v_L(0^+)}{L} = \frac{-2}{1} = \mathbf{-2A/\,sec}$$

Applying KCL at node X, $$\frac{v_C - 10}{2} + i_C + i_L = 0$$

Consequently, at $t = 0^+$

$$i_C(0^+) = \frac{10 - v_C(0^+)}{2} - i_L(0^+) = 6 - 0 = 6 \text{ A}$$

Since $$i_C = C\frac{dv_C}{dt}$$

We get, $$\frac{dv_C(0^+)}{dt} = \frac{i_C(0^+)}{C} = \frac{6}{\frac{1}{2}} = \mathbf{12V/\,sec}$$

EXAMPLE 4.13

For the circuit shown in Fig. 4.28, find:

(a) $i(0^+)$ and $v(0^+)$

(b) $\dfrac{di(0^+)}{dt}$ and $\dfrac{dv(0^+)}{dt}$

(c) $i(\infty)$ and $v(\infty)$

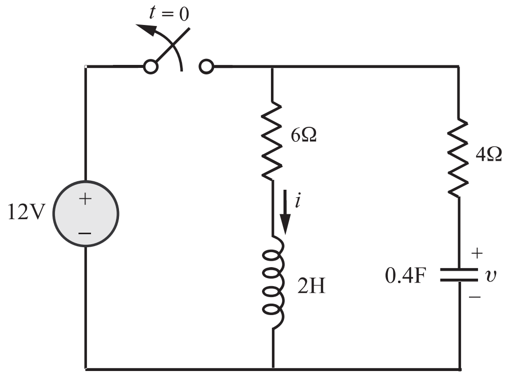

Figure 4.28

SOLUTION

(a) From the symbol of switch, we find that at $t = 0^-$, the switch is closed and $t = 0^+$, it is open. At $t = 0^-$, the circuit has reached steady state so that the equivalent circuit is as shown in Fig.4.29(a).

$$i(0^-) = \frac{12}{6} = 2\text{A}$$

$$v(0^-) = 12 \text{ V}$$

Therefore, we have

$$i(0^+) = i(0^-)$$

$$= \mathbf{2}\text{A}$$

$$v(0^+) = v(0^-) = \mathbf{12V}$$

(b) For $t \geq 0^+$, we have the equivalent circuit as shown in Fig.4.29(b).

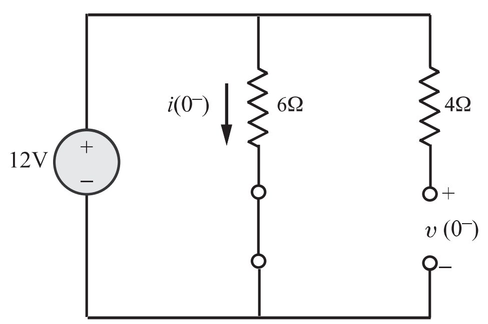

Figure 4.29(a)

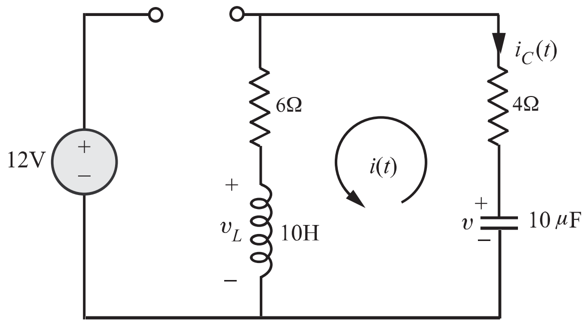

Figure 4.29(b)

Applying KVL anticlockwise to the mesh on the right, we get

$$v_L(t) - v(t) + 10i(t) = 0$$

Putting $t = 0^+$, we get

$$v_L(0^+) - v(0^+) + 10i(0^+) = 0$$

$$\Rightarrow \quad v_L(0^+) - 12 + 10 \times 2 = 0$$

$$\Rightarrow \quad v_L(0^+) = -8\text{V}$$

The voltage across the inductor is given by

$$v_L = L\frac{di}{dt}$$

$$\Rightarrow \quad v_L(0^+) = L\frac{di(0^+)}{dt}$$

$$\Rightarrow \quad \frac{di(0^+)}{dt} = \frac{1}{L}v_L(0^+)$$

$$= \frac{1}{10}(-8) = \mathbf{-0.8A/\,sec}$$

Similarly, the current through the capacitor is

$$i_C = C\frac{dv}{dt}$$

or $$\frac{dv(0^+)}{dt} = \frac{i_C(0^+)}{C} = \frac{-i(0^+)}{C}$$

$$= \frac{-2}{10\times10^{-6}} = \mathbf{-0.2\times10^6V/\,sec}$$

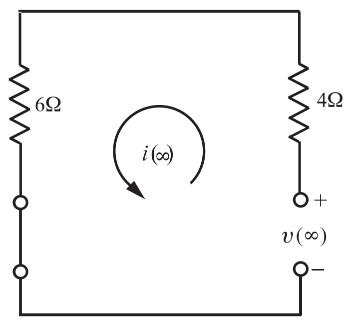

Figure 4.29(c)

(c) As t approaches infinity, the switch is open and the circuit has attained steady state. The equivalent circuit at $t = \infty$ is shown in Fig.4.29(c).

$$i(\infty) = \mathbf{0}$$

$$v(\infty) = \mathbf{0}$$

EXAMPLE 4.14

Refer the circuit shown in Fig.4.30. Find the following:

(a) $v(0^+)$ and $i(0^+)$

(b) $\dfrac{dv(0^+)}{dt}$ and $\dfrac{di(0^+)}{dt}$

(c) $v(\infty)$ and $i(\infty)$

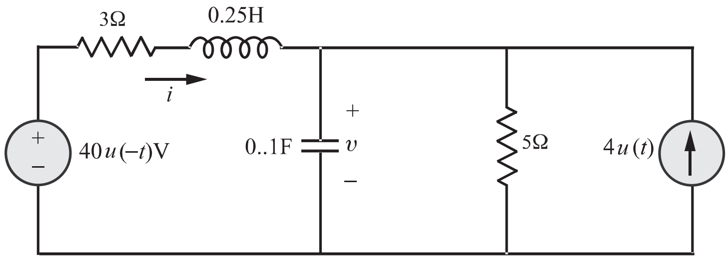

Figure 4.30

SOLUTION

From the definition of step function,

$$u(t) = \begin{cases} 1, & t > 0 \\ 0, & t < 0 \end{cases}$$

From Fig.4.31(a), $u(t) = 0$ at $t = 0^-$.

$$\text{Similarly,} \quad u(-t) = \begin{cases} 1, & -t > 0 \\ 0, & -t < 0 \end{cases}$$

$$\text{or} \qquad u(-t) = \begin{cases} 1, & t < 0 \\ 0, & t > 0 \end{cases}$$

From Fig.4.31(b), we find that $u(-t) = 1$, at $t = 0^-$.

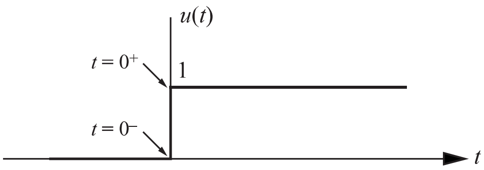

Figure 4.31(a)

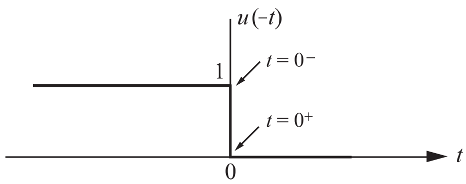

Figure 4.31(b)

Due to the presence of $u(-t)$ and $u(t)$ in the circuit of Fig.4.30, the circuit is an implicit switching circuit. We use the word implicit since there are no conventional switches in the circuit of Fig.4.30.

The equivalent circuit at $t = 0^-$ is shown in Fig.4.31(c). Please note that at $t = 0^-$, the independent current source is open because $u(t) = 0$ at $t = 0^-$ and the circuit is in steady state.

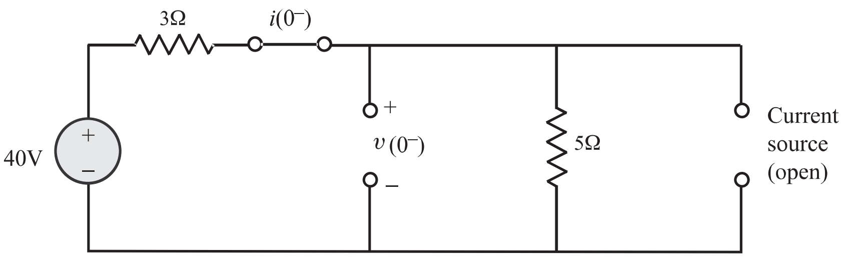

Figure 4.31(c)

$$\begin{aligned} i(0^-) &= \frac{40}{3+5} = 5\text{A} \\ v(0^-) &= 5i(0^-) = 25\text{V} \\ \text{Therefore} \quad i(0^+) &= i(0^-) = \mathbf{5A} \\ v(0^+) &= v(0^-) = \mathbf{25V} \end{aligned}$$

(b) For $t \geq 0^+$, $u(-t) = 0$. This implies that the independent voltage source is zero and hence is represented by a short circuit in the circuit shown in Fig.4.31(d).

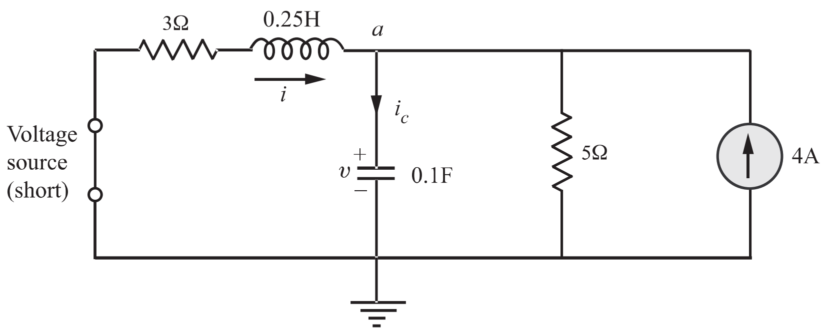

Figure 4.31(d)

Applying KVL at node a, we get

$$4 + i = C\frac{dv}{dt} + \frac{v}{5}$$

At $t = 0^+$, We get

$$4 + i(0^+) = C\frac{dv(0^+)}{dt} + \frac{v(0^+)}{5}$$

$$\Rightarrow \qquad 4 + 5 = 0.1\frac{dv(0^+)}{dt} + \frac{25}{5}$$

$$\Rightarrow \qquad \frac{dv(0^+)}{dt} = \mathbf{40V/\sec}$$

Applying KVL to the left–mesh, we get

$$3i + 0.25\frac{di}{dt} + v = 0$$

Evaluating at $t = 0^+$, we get

$$3i(0^+) + 0.25\frac{di(0^+)}{dt} + v(0^+) = 0$$

$$\Rightarrow \qquad 3 \times 5 + 0.25\frac{di(0^+)}{dt} + 25 = 0$$

$$\Rightarrow \qquad \frac{di(0^+)}{dt} = \frac{-40}{\frac{1}{4}} = \mathbf{-160A/\sec}$$

(c) As t approaches infinity, again the circuit is in steady state. The equivalent circuit at $t = \infty$ is shown in Fig.4.31(e).

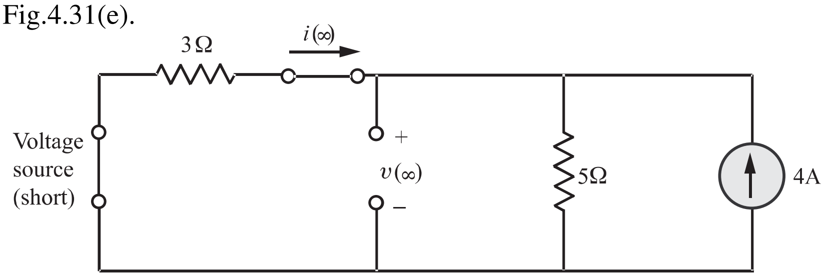

Figure 4.31(e)

Using the principle of current divider, we get

$$\begin{aligned} i(\infty) &= -\left(\frac{4\times 5}{3+5}\right) = \mathbf{-2.5A} \\ v(\infty) &= (i(\infty)+4)\,5 \\ &= (-2.5+4)5 \\ &= \mathbf{7.5V} \end{aligned}$$

EXAMPLE 4.15

Refer the circuit shown in Fig.4.32. Find the following:

(a) $i(0^+)$ and $v(0^+)$

(b) $\dfrac{di(0^+)}{dt}$ and $\dfrac{dv(0^+)}{dt}$

(c) $i(\infty)$ and $v(\infty)$

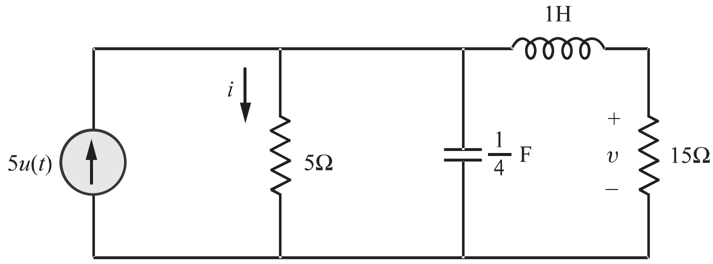

Figure 4.32

SOLUTION

Here the function $u(t)$ behaves like a switch. Mathematically,

$$u(t) = \begin{cases} 1, & t > 0 \\ 0, & t < 0 \end{cases}$$

The above expression means that the switch represented by $u(t)$ is open for $t < 0$ and remains closed for $t > 0$. Hence, the circuit diagram of Fig.4.32 may be redrawn as shown in Fig.4.33(a).

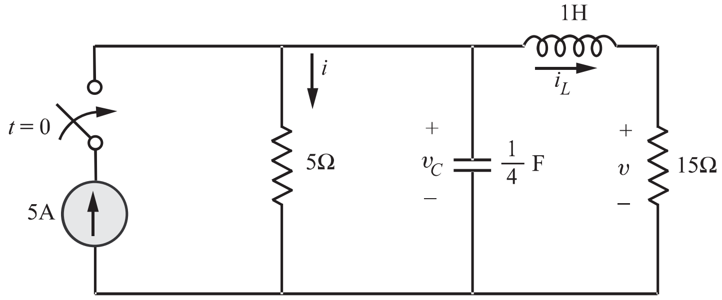

Figure 4.33(a)

For $t < 0$, the circuit is not active because switch is in open state, This implies that all the initial conditions are zero.

That is, $\quad i_L(0^-) = 0$ and $v_C(0^-) = 0$

for $t \geq 0^+$, the equivalent circuit is as shown in Fig.4.33(b).

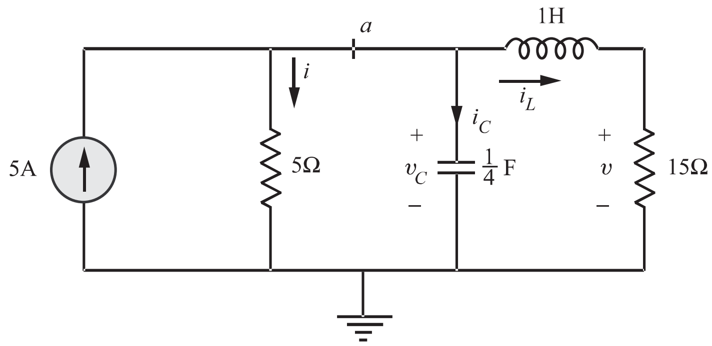

Figure 4.33(b)

From the circuit diagram of Fig.4.33(b), we find that

$$i = \frac{v_C}{5}$$

At $t = 0^+$, we get

$$i(0^+) = \frac{v_C(0^+)}{5} = \frac{v_C(0^-)}{5} = \frac{0}{5} = \mathbf{0A}$$

Also $$v = 15 i_L$$

Evaluating at $t = 0^+$, we get

$$v(0^+) = 15 i_L(0^+)$$
$$= 15 i_L(0^-) = 15 \times 0 = \mathbf{0V}$$

(b) The equivalent circuit at $t = 0^+$ is shown in Fig.4.33(c).

We find from Fig.4.33(c) that

$$i_C(0^+) = 5\text{A}$$

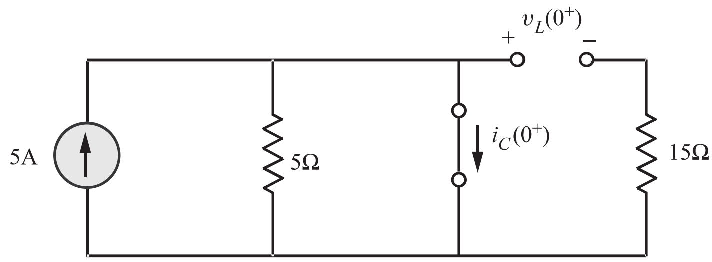

Figure 4.33(c)

From Fig.4.33(b), we can write

$$v_C = 5i$$

$$\Rightarrow \qquad \frac{dv_C}{dt} = 5\frac{di}{dt}$$

Multiplying both sides by C, we get

$$C\frac{dv_C}{dt} = 5C\frac{di}{dt}$$

$$\Rightarrow \qquad i_C = 5C\frac{di}{dt}$$

Putting $t = 0^+$, we get

$$\frac{di(0^+)}{dt} = \frac{1}{5C} i_C(0^+)$$
$$= \frac{1}{5\left(\frac{1}{4}\right)} \times 5$$
$$= \mathbf{4A/\,sec}$$

Also $\qquad v = 15 i_L$

$$\Rightarrow \qquad \frac{dv}{dt} = 15\frac{di_L}{dt}$$
$$\Rightarrow \qquad \frac{dv}{dt} = 15\left[1 \times \frac{di_L}{dt}\right]$$
$$\Rightarrow \qquad \frac{dv}{dt} = 15 v_L$$

At $t = 0^+$, we find that

$$\Rightarrow \qquad \frac{dv(0^+)}{dt} = 15 v_L(0^+)$$

From Fig.4.33(b), we find that $v_L(0^+) = 0$

Hence, $\qquad \dfrac{dv(0^+)}{dt} = 15 \times 0$

$$= \mathbf{0V/\,sec}$$

EXAMPLE 4.16

In the circuit shown in Fig. 4.34, steady state is reached with switch K open. The switch is closed at $t = 0$.

Determine: $i_1,\ i_2,\ \dfrac{di_1}{dt}$ and $\dfrac{di_2}{dt}$ at $t = 0^+$

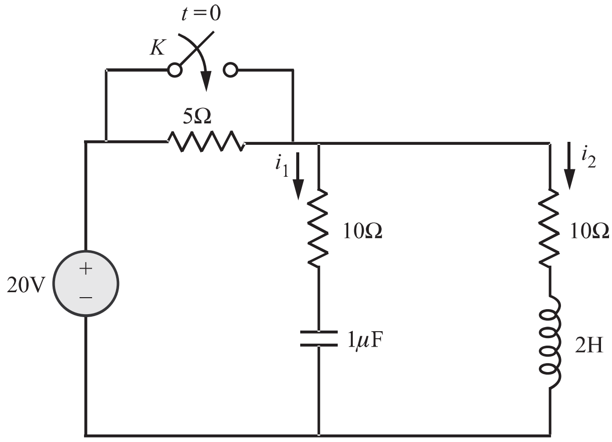

Figure 4.34

SOLUTION

At $t = 0^-$, switch K is open and at $t = 0^+$, it is closed. At $t = 0^-$, the circuit is in steady state and appears as shown in Fig.4.35(a).

$$i_2(0^-) = \frac{20}{10+5} = 1.33\text{A}$$

Hence, $$v_C(0^-) = 10i_2(0^-) = 10 \times 1.33 = 13.3\text{V}$$

Since current through an inductor cannot change instantaneously, $i_2(0^+) = i_2(0^-) = \mathbf{1.33\ A}$.
Also, $v_C(0^+) = v_C(0^-) = 13.3\text{V}$.
The equivalent circuit at $t = 0^+$ is as shown in Fig.4.35(b).

$$i_1(0^+) = \frac{20 - 13.3}{10} = \frac{6.7}{10} = \mathbf{0.67A}$$

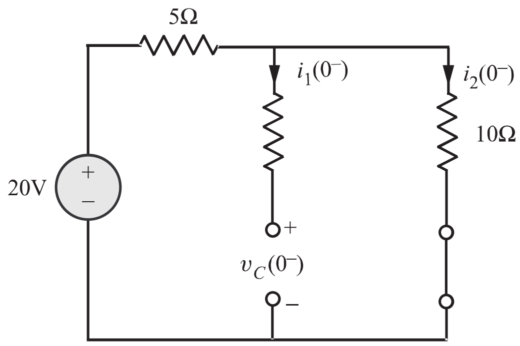

Figure 4.35(a)

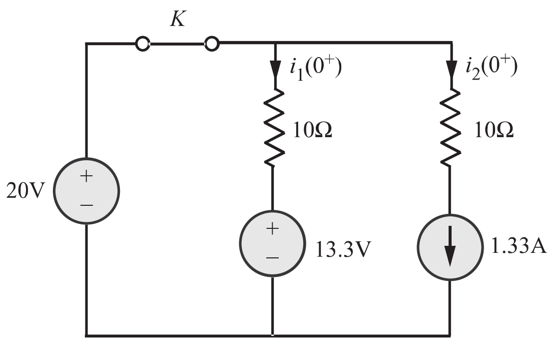

Figure 4.35(b)

For $t \geq 0^+$, the circuit is as shown in Fig.4.35(c).

Writing KVL clockwise for the left–mesh, we get

$$10i_1 + \frac{1}{C}\int_{0^+}^{t} i_1(\tau)d\tau = 20$$

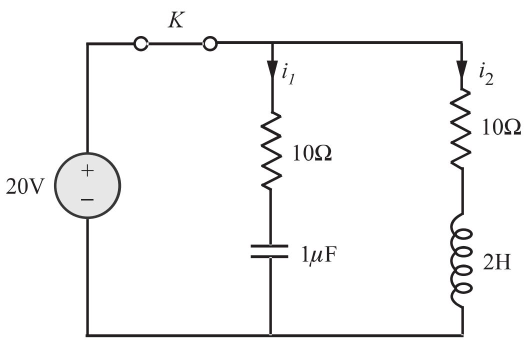

Figure 4.35(c)

Differentiating with respect to t, we get

$$10\frac{di_1}{dt} + \frac{1}{C}i_1 = 0$$

Putting $t = 0^+$, we get

$$10\frac{di_1(0^+)}{dt} + \frac{1}{C}i_1(0^+) = 0$$

$$\Rightarrow \quad \frac{di_1(0^+)}{dt} = \frac{-1}{10 \times 1 \times 10^{-6}}i_1(0^+) = \mathbf{-0.67 \times 10^5 A/\,sec}$$

Writing KVL equation to the path made of $20\text{V} \rightarrow K \rightarrow 10\Omega \rightarrow 2\text{H}$, we get

$$10i_2 + \frac{2di_2}{dt} = 20$$

At $t = 0^+$, the above equation becomes

$$10i_2(0^+) + \frac{2di_2(0^+)}{dt} = 20$$

$$\Rightarrow \qquad 10 \times 1.33 + \frac{2di_2(0^+)}{dt} = 20$$

$$\Rightarrow \qquad \frac{di_2(0^+)}{dt} = \mathbf{3.35A/\,sec}$$

EXAMPLE 4.17

Refer the citcuit shown in Fig.4.36. The switch K is closed at $t = 0$. Find:

(a) v_1 and v_2 at $t = 0^+$

(b) v_1 and v_2 at $t = \infty$

(c) $\dfrac{dv_1}{dt}$ and $\dfrac{dv_2}{dt}$ at $t = 0^+$

(d) $\dfrac{d^2v_1}{dt^2}$ at $t = 0^+$

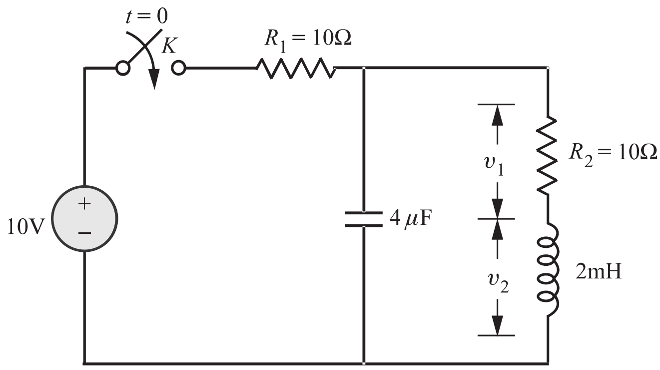

Figure 4.36

SOLUTION

(a) The circuit symbol for switch conveys that at $t = 0^-$, the switch is open and $t = 0^+$, it is closed. At $t = 0^-$, since the switch is open, the circuit is not activated. This implies that all initial conditions are zero. Hence, at $t = 0^+$, inductor is open and capactor is short. Fig 4.37(a) shows the equivalent circuit at $t = 0^+$.

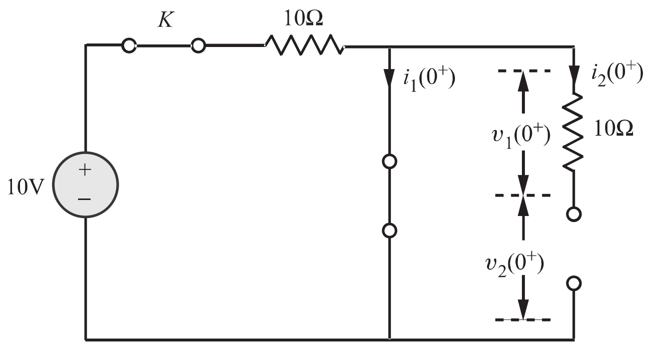

Figure 4.37(a)

$$i_1(0^+) = \frac{10}{10} = 1\text{A}$$
$$v_1(0^+) = 0, \quad i_2(0^+) = 0$$

Applying KVL to the path, 10V source $\to K \to 10\Omega \to 10\Omega \to 2\text{mH}$, we get

$$-10 + 10i_1(0^+) + v_1(0^+) + v_2(0^+) = 0$$
$$\Rightarrow \quad -10 + 10 + 0 + v_2(0^+) = 0$$
$$\Rightarrow \quad v_2(0^+) = \mathbf{0}$$

(b) At $t = \infty$, switch K remains closed and circuit is in steady state. Under steady state conditions, capacitor C is open and inductor L is short. Fig. 4.37(b) shows the equivalent circuit at $t = \infty$.

$$i_2(\infty) = \frac{10}{10+10} = \mathbf{0.5A}$$
$$i_1(\infty) = \mathbf{0}$$
$$v_1(\infty) = 0.5 \times 10 = \mathbf{5V}$$
$$v_2(\infty) = \mathbf{0}$$

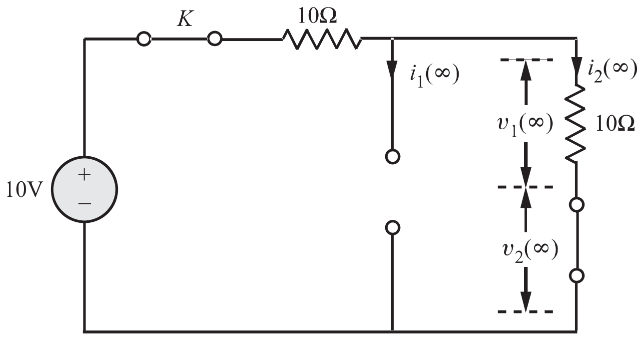

Figure 4.37(b)

(c) For $t \geq 0^+$, the circuit is as shown in Fig. 4.37(c).

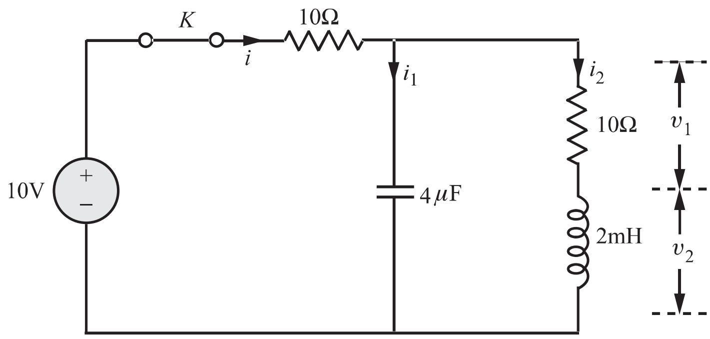

Figure 4.37(c)

$$i_2 = \frac{1}{L}\int\limits_{0^+}^{t} v_2(\tau)d\tau = \frac{v_1(t)}{R_2}$$

Differentiating with respect to t, we get

$$\frac{v_2}{L} = \frac{1}{R_2}\frac{dv_1}{dt}$$

Evaluating at $t = 0^+$ we get

$$\frac{dv_1(0^+)}{dt} = \frac{R_2}{L_2}v_2(0^+)$$

$$\Rightarrow \qquad \frac{dv_1(0^+)}{dt} = \mathbf{0V/\,sec}$$

Applying KVL clockwise to the path 10 V source $\rightarrow K \rightarrow 10\Omega \rightarrow 4\mu$F, we get

$$-10 + 10i + \frac{1}{C}\int\limits_{0^+}^{t} [i(\tau) - i_2(\tau)]d\tau = 0$$

Differentiating with respect to t, we get

$$10\frac{di}{dt} + \frac{1}{C}[i - i_2] = 0$$

Evaluating at $t = 0^+$, we get

$$\begin{aligned} \frac{di(0^+)}{dt} &= \frac{i_2(0^+) - i(0^+)}{C \times 10} \\ &= \frac{0 - 1}{10 \times 4 \times 10^{-6}} \qquad \left[\begin{aligned} \because\ i(0^+) &= i_1(0^+) + i_2(0^+) \\ &= 1 + 0 \\ &= 1\text{A} \end{aligned} \right] \\ &= \mathbf{-25000A/\,sec} \end{aligned}$$

Applying KVL clockwise to the path 10 V source $\rightarrow K \rightarrow 10\Omega \rightarrow 10\Omega \rightarrow 2$ mH, we get

$$-10 + 10i + 10i_2 + v_2 = 0$$

$$\Rightarrow \qquad 10i + v_1 + v_2 = 10$$

Differentiating with respect to t, we get

$$10\frac{di}{dt} + \frac{dv_1}{dt} + \frac{dv_2}{dt} = 0$$

At $t = 0^+$, we get

$$10\frac{di(0^+)}{dt} + \frac{dv_1(0^+)}{dt} + \frac{dv_2(0^+)}{dt} = 0$$

$$\Rightarrow \qquad 10(-25000) + 0 + \frac{dv_2(0^+)}{dt} = 0$$

$$\Rightarrow \qquad \frac{dv_2(0^+)}{dt} = \mathbf{25 \times 10^4 V/\,sec}$$

(d) From part (c), we have

$$\frac{1}{L}\int_{0^+}^{t} v_2(\tau)d\tau = \frac{v_1}{10}$$

Differentiating with respect to t twice, we get

$$\frac{1}{L}\frac{dv_2}{dt} = \frac{1}{10}\frac{d^2v_1}{dt^2}$$

At $t = 0^+$, we get

$$\frac{1}{L}\frac{dv_2(0^+)}{dt} = \frac{1}{10}\frac{d^2v_1(0^+)}{dt^2}$$

Hence,
$$\frac{d^2v_1(0^+)}{dt^2} = \mathbf{125 \times 10^7 V/\,sec^2}$$

EXAMPLE 4.18

Refer the network shown in Fig. 4.38. Switch K is changed from a to b at $t = 0$ (a steady state having been established at position a).

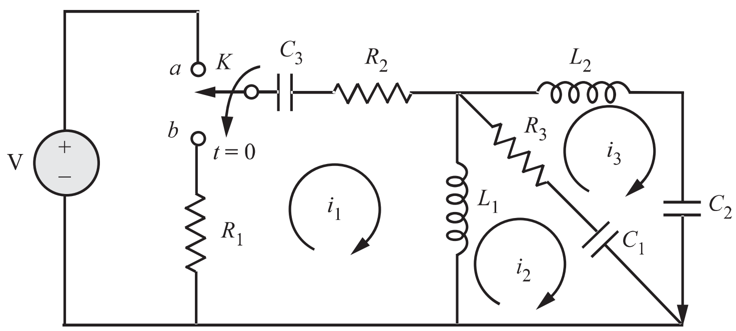

Figure 4.38

Show that at $t = 0^+$.

$$i_1 = i_2 = \frac{-V}{R_1 + R_2 + R_3}, \quad i_3 = 0$$

SOLUTION

The symbol for switch indicates that at $t = 0^-$, it is in position a and at $t = 0^+$, it is in position b. The circuit is in steady state at $t = 0^-$. Fig 4.39(a) refers to the equivalent circuit at $t = 0^-$. Please remember that at steady state C is open and L is short.

$$i_{L_1}(0^-) = 0, \quad i_{L_2}(0^-) = 0, \quad v_{C_2}(0^-) = 0, \quad v_{C_1}(0^-) = 0$$

Applying KVL clockwise to the left-mesh, we get

$$-V + v_{C_3}(0^-) + 0 \times R_2 + 0 = 0$$

$$\Rightarrow \qquad v_{C_3}(0^-) = \text{V volts.}$$

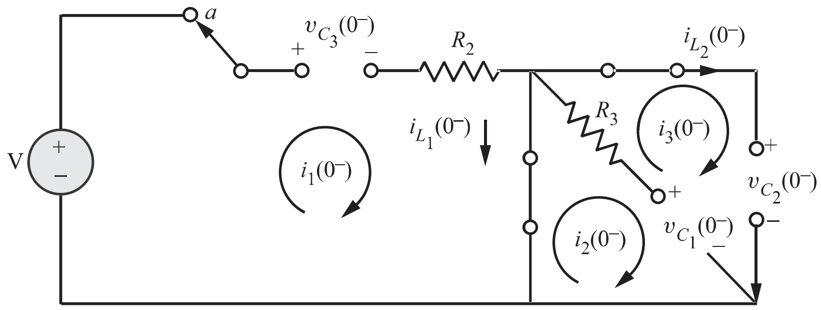

Figure 4.39(a)

Since current in an inductor and voltage across a capacitor cannot change instantaneously, the equivalent circuit at $t = 0^+$ is as shown in Fig. 4.39(b).

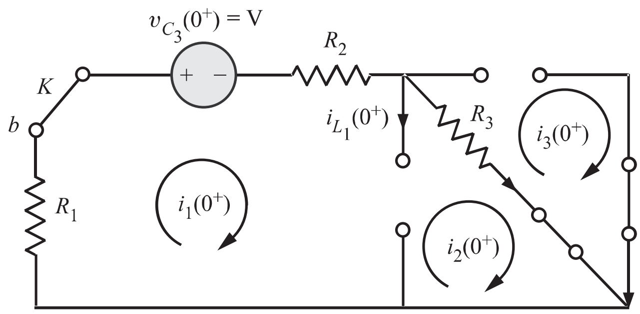

Figure 4.39(b)

$$i_1(0^+) = i_2(0^+) \text{ since } i_{L_1}(0^+) = 0$$
$$i_3(0^+) = 0 \text{ since } i_{L_2}(0^+) = 0$$

Applying KVL to the path $v_{C_3}(0^+) \to R_2 \to R_3 \to R_1 \to K$ we get,

$$V + R_2 i_1(0^+) + R_3 i_2(0^+) + R_1 i_1(0^+) = 0$$

Since $i_1(0^+) = i_2(0^+)$, the above equation becomes

$$-V = [R_1 + R_2 + R_3]\, i_1(0^+)$$

Hence, $$i_1(0^+) = i_2(0^+) = \frac{\boldsymbol{-V}}{\boldsymbol{R_1 + R_2 + R_3}}\,\mathbf{A}$$

EXAMPLE 4.19

Refer the circuit shown in Fig. 4.40. The switch K is closed at $t = 0$.

Find (a) $\dfrac{di_1(0^+)}{dt}$ and (b) $\dfrac{di_2(0^+)}{dt}$

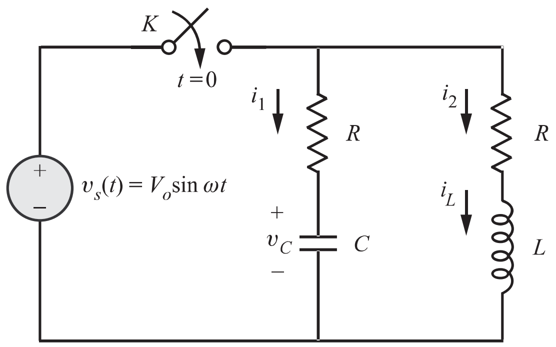

Figure 4.40

SOLUTION

The circuit symbol for the switch shows that at $t = 0^-$, it is open and at $t = 0^+$, it is closed. Hence, at $t = 0^-$, the circuit is not activated. This implies that all initial conditions are zero. That is, $v_C(0^-) = 0$ and $i_L(0^-) = i_2(0^-) = 0$. The equivalent circuit at $t = 0^+$ keeping in mind that $v_C(0^+) = v_C(0^-)$ and $i_L(0^+) = i_L(0^-)$ is as shown in Fig. 4.41 (a).

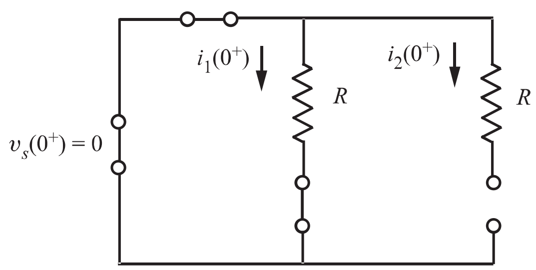

Figure 4.41(a)

$$i_1(0^+) = 0 \text{ and } i_2(0^+) = 0.$$

Figure. 4.41(b) shows the circuit diagram for $t \geq 0^+$.

$$V_o \sin \omega t = i_1 R + \frac{1}{C}\int_{0^+}^{t} i_1(\tau)d\tau$$

Differentiating with respect to t, we get

$$V_o \omega \cos \omega t = R\frac{di_1}{dt} + \frac{i_1}{C}$$

At $t = 0^+$, we get

$$V_o\omega = R\frac{di_1(0^+)}{dt} + \frac{i_1(0^+)}{C}$$

$$\Rightarrow \quad \frac{di_1(0^+)}{dt} = \frac{\boldsymbol{V_0\omega}}{\boldsymbol{R}}\,\mathbf{A/\,sec}$$

Also, $\quad V_o \sin\omega t = i_2R + L\dfrac{di_2}{dt}$

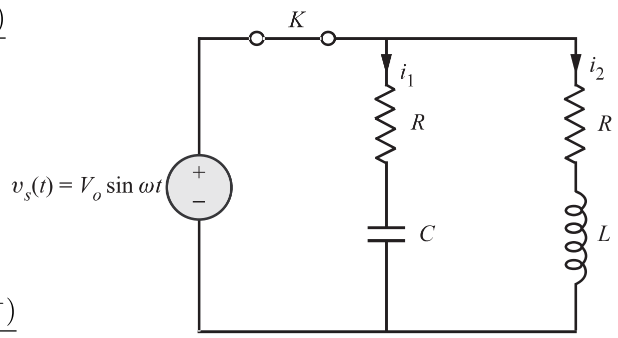

Figure 4.41(b)

Evaluating at $t = 0^+$, we get

$$0 = i_2(0^+)R + L\frac{di_2(0^+)}{dt}$$

$$\Rightarrow \quad \frac{di_2(0^+)}{dt} = \mathbf{0A/\,sec}$$

EXAMPLE 4.20

In the network of the Fig. 4.42, the switch K is opened at $t = 0$ after the network has attained steady state with the switch closed.

(a) Find the expression for v_K at $t = 0^+$.

(b) If the parameters are adjusted such that $i(0^+) = 1$, and $\dfrac{di(0^+)}{dt} = -1$, what is the value of the derivative of the voltage across the switch at $t = 0^+$, $\left(\dfrac{dv_K}{dt}(0^+)\right)$?

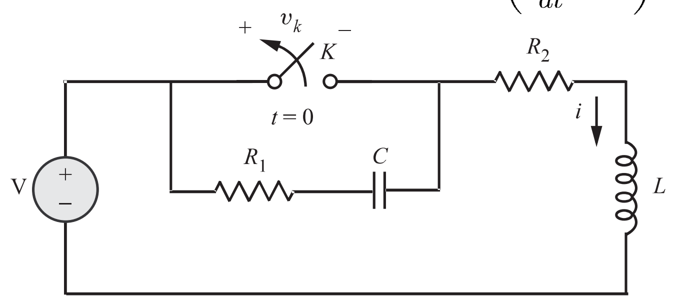

Figure 4.42

SOLUTION

At $t = 0^-$, switch is in the closed state and at $t = 0^+$, it is open. Also at $t = 0^-$, the circuit is in steady state. The equivalent circuit at $t = 0^-$ is as shown in Fig. 4.43(a).

$$i(0^-) = \frac{V}{R_2} \text{ and } v_C(0^-) = 0$$

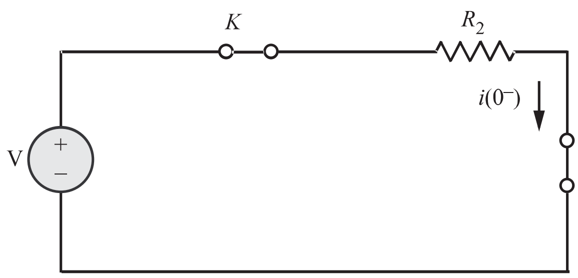

Figure 4.43(a)

For $t \geq 0^+$, the equivalent circuit is as shown in Fig. 4.43(b).
From Fig. 4.43 (b),

$$v_K = R_1 i + \frac{1}{C}\int_{0^+}^{t} i(\tau)d\tau$$

$$\Rightarrow \qquad v_K = R_1 i + v_C(t)$$

At $t = 0^+$, $v_K(0^+) = R_1 i(0^+) + v_C(0^+)$

$$\Rightarrow \qquad v_K(0^+) = R_1\frac{V}{R_2} + v_C(0^-)$$

$$= \boldsymbol{R_1\frac{V}{R_2}}\,\textbf{volts}$$

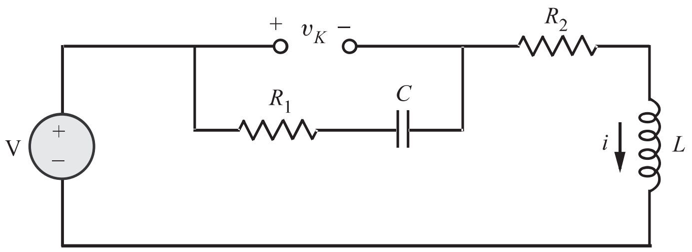

Figure 4.43(b)

(b)

$$v_K = R_1 i + \frac{1}{C}\int_{0^+}^{t} i(\tau)d\tau$$

$$\Rightarrow \qquad \frac{dv_K}{dt} = R_1\frac{di}{dt} + \frac{i}{C}$$

Evaluating at $t = 0^+$, we get

$$\frac{dv_K(0^+)}{dt} = R_1\frac{di(0^+)}{dt} + \frac{i(0^+)}{C}$$

$$= R_1 \times (-1) + \frac{1}{C}$$

$$= \boldsymbol{\frac{1}{C} - R_1}\,\textbf{volts/ sec}$$

Reinforcement Problems

R.P 4.1

Refer the circuit shown in Fig RP.4.1(a). If the switch is closed at $t = 0$, find the value of $\frac{d^2 i_L(0^+)}{dt^2}$ at $t = 0^+$.

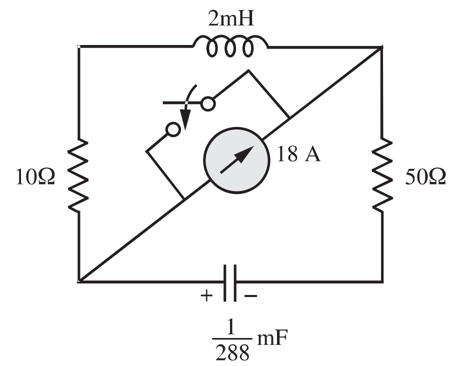

Figure R.P.4.1(a)

SOLUTION

The circuit at $t = 0^-$ is as shown in Fig RP 4.1(b).

Since current through an inductor and voltage across a capacitor cannot change instantaneously, it implies that $i_L(0^+) = 18$A and $v_C(0^+) = -180$ V.

The circuit for $t \geq 0^+$ is as shown in Fig. RP 4.1 (c).

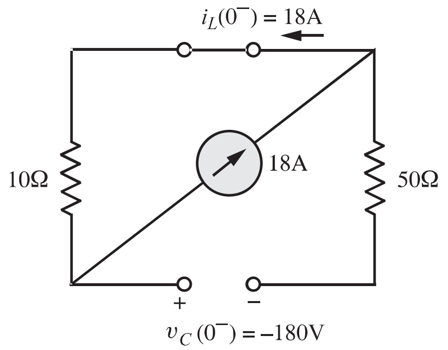

Figure R.P.4.1(b)

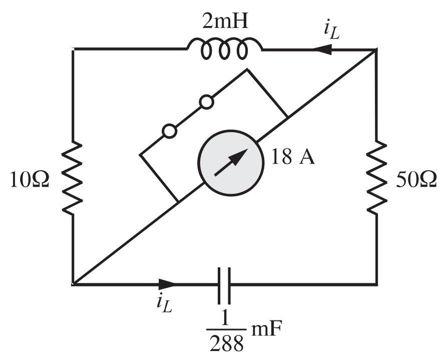

Figure R.P.4.1(c)

Referring Fig RP 4.1 (c), we can write

$$2 \times 10^{-3} \frac{di_L}{dt} + 60 i_L + 288 \times 10^3 \int_{0^+}^{t} i_L(t) dt = 0 \tag{4.9}$$

At $t = 0^+$, we get

$$\frac{di_L(0^+)}{dt} = \frac{-60 \times 18 + 180}{2 \times 10^{-3}}$$
$$= -450 \times 10^3 \text{ A/ sec}$$

Differentiating equation (4.9) with respect to t, we get

$$2 \times 10^{-3} \frac{d^2 i_L}{dt^2} + 60 \frac{di_L}{dt} + 288 \times 10^3 i_L = 0$$

At $t = 0^+$, we get

$$\frac{d^2 i_L(0^+)}{dt^2} = \frac{60(450)10^3 - 288 \times 10^3(18)}{2 \times 10^{-3}}$$
$$= 1.0908 \times 10^{10} \text{ A/ sec}^2$$

R.P 4.2

For the circuit shown in Fig. RP 4.2, determine $\frac{d^2 v_C(0^+)}{dt^2}$ and $\frac{d^3 v_C(0^+)}{dt^3}$.

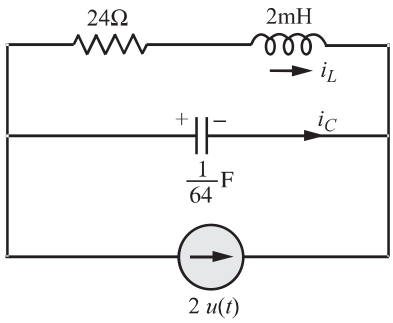

Figure R.P.4.2

SOLUTION

Given

$$i(t) = 2u(t) = \begin{cases} 2, & t \geq 0^+ \\ 0, & t \leq 0^- \end{cases}$$

Hence, at $t = 0^-$, $v_C(0^-) = 0$ and $i_L(0^-) = 0$.
For $t \geq 0^+$, the circuit equations are

$$\frac{1}{64} \frac{dv_C(t)}{dt} + \frac{1}{2} \int_{0^+}^{t} v_L(t) dt = -2 \qquad (4.10)$$

$$\Rightarrow \qquad \frac{1}{64} \frac{dv_C(t)}{dt} + i_L(t) = -2 \qquad (4.11)$$

[$\mathbf{Note}$: $i_C + i_L = -2$ because of the capacitor polarity]
At $t = 0^+$, equation (4.10) gives

$$\frac{1}{64}\frac{dv_C(0^+)}{dt} + i_L(0^+) = -2$$

Since, $i_L(0^+) = i_L(0^-) = 0$, we get

$$\frac{1}{64}\frac{dv_C(0^+)}{dt} + 0 = -2$$

$$\Rightarrow \qquad \frac{dv_C(0^+)}{dt} = -128 \text{ volts/ sec}$$

Differentiating equation (4.10) with respect to t we get

$$\frac{1}{64}\frac{d^2v_C(t)}{dt} + \frac{1}{2}v_L(t) = 0 \tag{4.12}$$

Also, $$\frac{v_C - v_L}{24} = \frac{1}{2}\int_{0^+}^{t} v_L dt = i_L \tag{4.13}$$

At $t = 0^+$, we get

$$\frac{v_C(0^+) - v_L(0^+)}{24} = i_L(0^+)$$

Since $v_C(0^+) = 0$ and $i_L(0^+) = 0$, we get $v_L(0^+) = 0$.
At $t = 0^+$, equation (4.12) becomes

$$\frac{1}{64}\frac{d^2v_C(0^+)}{dt^2} + \frac{1}{2}v_L(0^+) = 0$$

$$\Rightarrow \qquad \frac{1}{64}\frac{d^2v_C(0^+)}{dt^2} + \frac{1}{2} \times 0 = 0$$

$$\Rightarrow \qquad \frac{d^2v_C(0^+)}{dt^2} = 0$$

Differentiating equation (4.12) with respect to t we get

$$\Rightarrow \qquad \frac{1}{64}\frac{d^3v_C}{dt^3} + \frac{1}{2}\frac{dv_L}{dt} = 0 \tag{4.14}$$

Differentiating equation (4.13) with respect to t, we get

$$\frac{\dfrac{dv_C}{dt} - \dfrac{dv_L}{dt}}{24} = \frac{1}{2}v_L$$

At $t = 0^+$, we get

$$\Rightarrow \qquad \frac{\dfrac{dv_C(0^+)}{dt} - \dfrac{dv_L(0^+)}{dt}}{24} = \frac{1}{2}v_L(0^+)$$

$$\Rightarrow \qquad \frac{-128 - \dfrac{dv_L(0^+)}{dt}}{24} = 0$$

$$\Rightarrow \qquad \frac{dv_L(0^+)}{dt} = -128 \text{ volts/ sec}$$

At $t = 0^+$, equation (4.14) becomes

$$\frac{1}{64}\frac{d^3v_C(0^+)}{dt^3} + \frac{1}{2}\frac{dv_L(0^+)}{dt} = 0$$

$$\Rightarrow \qquad \frac{d^3v_C(0^+)}{dt^3} = 4096 \text{ volts/ sec}^3$$

R.P 4.3

In the network of Fig RP 4.3 (a), switch K is closed at $t = 0$. At $t = 0^-$ all the capacitor voltages and all the inductor currents are zero. Three node-to-datum voltages are identified as v_1, v_2 and v_3. Find at $t = 0^+$:

(i) v_1, v_2 and v_3

(ii) $\dfrac{dv_1}{dt}$, $\dfrac{dv_2}{dt}$ and $\dfrac{dv_3}{dt}$

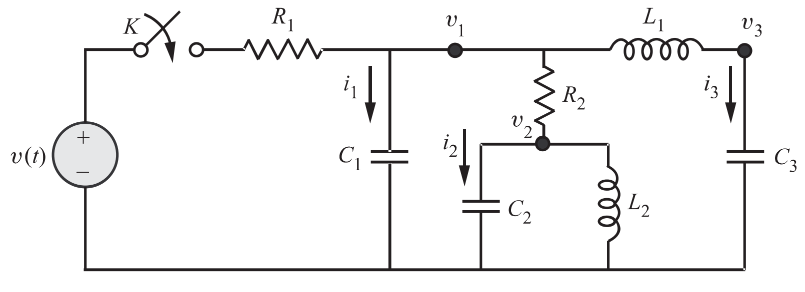

Figure R.P.4.3(a)

SOLUTION

The network at $t = 0^+$ is as shown in Fig RP-4.3 (b).

Since v_C and i_L cannot change instantaneously, we have from the network shown in Fig. RP-4.3 (b),

$$v_1(0^+) = 0$$
$$v_2(0^+) = 0$$
$$v_3(0^+) = 0$$

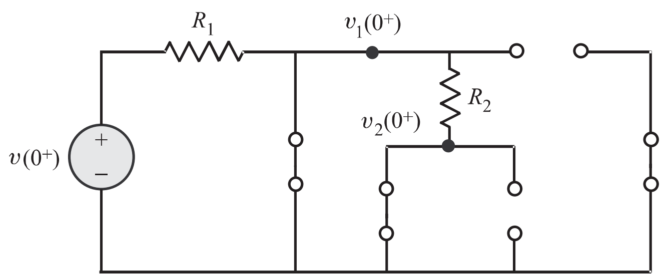

Figure R.P.4.3(b)

For $t \geq 0^+$, the circuit equations are

$$\left.\begin{aligned} v_{C_1} &= \frac{1}{C_1}\int_{0^+}^{t} i_1 dt \\ v_{C_2} &= \frac{1}{C_2}\int_{0^+}^{t} i_2 dt \\ v_{C_3} &= \frac{1}{C_3}\int_{0^+}^{t} i_3 dt \end{aligned}\right\} \qquad (4.15)$$

From Fig. RP-4.3 (b), we can write

$$i_1(0^+) = \frac{v(0^+)}{R_1},$$

$$i_2(0^+) = \frac{v_1(0^+) - v_2(0^+)}{R_2}$$

and

$$i_3(0^+) = 0$$

Differentiating equation (4.15) with respect to t, we get

$$\frac{dv_{C_1}}{dt} = \frac{i_1}{C_1}, \frac{dv_{C_2}}{dt} = \frac{i_2}{C_2} \text{ and } \frac{dv_{C_3}}{dt} = \frac{i_3}{C_3}$$

At $t = 0^+$, the above equations give

$$\frac{dv_1(0^+)}{dt} = \frac{i_1(0^+)}{C_1} = \frac{v(0^+)}{R_1C_1}$$

$$\frac{dv_2(0^+)}{dt} = \frac{i_2(0^+)}{C_2} = \frac{v_1(0^+) - v_2(0^+)}{R_2C_2} = 0$$

and

$$\frac{dv_3(0^+)}{dt} = \frac{i_3(0^+)}{C_3} = 0$$

R.P 4.4

For the network shown in Fig RP 4.4 (a) with switch K open, a steady-state is reached. The circuit paprameters are $R_1 = 10\Omega$, $R_2 = 20\Omega$, $R_3 = 20\Omega$, $L = 1$ H and $C = 1\mu$F. Take $V = 100$ volts. The switch is closed at $t = 0$.

(a) Write the integro-differential equation after the switch is closed.

(b) Find the voltage V_o across C before the switch is closed and give its polarity.

(c) Find i_1 and i_2 at $t = 0^+$.

(d) Find $\dfrac{di_1}{dt}$ and $\dfrac{di_2}{dt}$ at $t = 0^+$.

(e) What is the value of $\dfrac{di_1}{dt}$ at $t = \infty$?

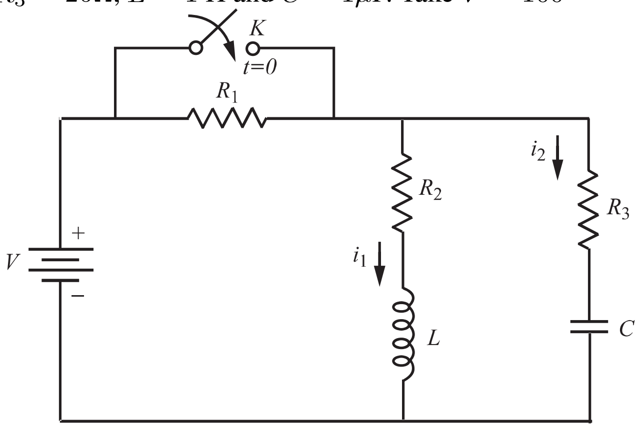

Figure R.P.4.4(a)

SOLUTION

The switch is in open state at $t = 0^-$. The network at $t = 0^-$ is as shown in Fig RP 4.4 (b).

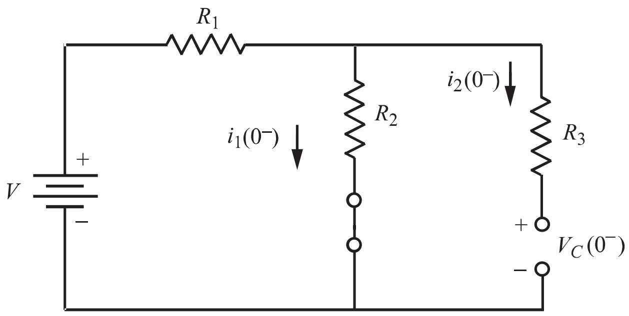

Figure R.P.4.4(b)

$$i_1(0^-) = \frac{V}{R_1 + R_2} = \frac{100}{30} = \frac{10}{3} \text{ A}$$

$$V_C(0^-) = i_1(0^-)R_2 = \frac{10}{3} \times 20 = \frac{200}{3} \text{ volts}$$

Note that L is short and C is open under steady-state condition.

For $t \geq 0^+$ (switch in closed state),

we have
$$20i_1 + \frac{di_1}{dt} = 100 \tag{4.16}$$

and
$$20i_2 + 10^6 \int_{0^+}^{t} i_2 dt = 100 \tag{4.17}$$

Also $$i_1(0^+) = i_1(0^-) = \frac{10}{3} \text{ A}$$

and $$V_C(0^+) = V_C(0^-) = \frac{200}{3} \text{ Volts}$$

From equation (4.16) at $t = 0^+$,

we have
$$\frac{di_1(0^+)}{dt} = 100 - 20 \times \frac{10}{3}$$
$$= \frac{100}{3} \text{ A/sec}$$

From equation (4.17), at $t = 0^+$, we have
$$i_2(0^+) = \frac{1}{20}\left[100 - \frac{200}{3}\right] = \frac{5}{3} \text{ A}$$

Differentiating equation (4.17), we get
$$20\frac{di_2}{dt} + 10^6 i_2 = 0 \tag{4.18}$$

From equation (4.18) at $t = 0^+$, we get
$$\frac{20di_2(0^+)}{dt} + 10^6 i_2(0^+) = 0$$

$$\Rightarrow \quad \frac{di_2(0^+)}{dt} = \frac{-10^6 \times \frac{5}{3}}{20}$$
$$= \frac{-10^6}{12} \text{ A/ sec}$$

At $t = \infty$,
$$i_1(\infty) = \frac{100}{20} = 5 \text{ A}$$
$$\frac{di_1}{dt}(\infty) = 0$$

R.P 4.5

For the network shown in Fig RP 4.5 (a), find $\dfrac{d^2 i_1(0^+)}{dt^2}$.

The switch K is closed at $t = 0$.

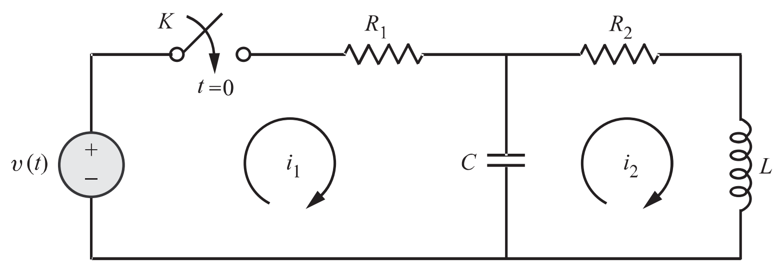

Figure R.P.4.5(a)

SOLUTION

At $t = 0^-$, we have $v_C(0^-) = 0$ and $i_2(0^-) = i_L(0^-) = 0$. Because of the switching property of L and C, we have $v_C(0^+) = 0$ and $i_2(0^+) = 0$. The network at $t = 0^+$ is as shown in Fig RP 4.5 (b).

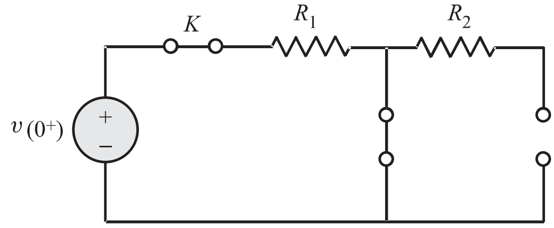

Figure R.P.4.5(b)

Referring Fig RP 4.5 (b), we find that

$$i_1(0^+) = \frac{v(0^+)}{R_1}$$

The circuit equations for $t \geq 0^+$ are

$$R_1 i_1 + \frac{1}{C}\int_{0^+}^{t} (i_1 - i_2)dt = v(t) \tag{4.19}$$

and
$$R_2 i_2 + \underbrace{\frac{1}{C}\int_{0^+}^{t} (i_2 - i_1)dt}_{v_C(t)} + L\frac{di_2}{dt} = 0 \tag{4.20}$$

At $t = 0^+$, equation (4.20) becomes

$$R_2 i_2(0^+) + v_C(0^+) + L\frac{di_2(0^+)}{dt} = 0$$

$$\Rightarrow \qquad \frac{di_2(0^+)}{dt} = 0 \tag{4.21}$$

Differentiating equation (4.19), we get

$$R_1\frac{di_1}{dt} + \frac{1}{C}(i_1 - i_2) = \frac{dv(t)}{dt} \tag{4.22}$$

Letting $t = 0^+$ in equation (4.22), we get

$$R_1\frac{di_1(0^+)}{dt} + \frac{1}{C}\left\{i_1(0^+) - i_2(0^+)\right\} = \frac{dv(0^+)}{dt}$$

$$\Rightarrow \qquad \frac{di_1(0^+)}{dt} = \frac{1}{R_1}\left\{\frac{dv(0^+)}{dt} - \frac{v(0^+)}{R_1 C}\right\} \tag{4.23}$$

Differentiating equation (4.22) gives

$$R_1\frac{d^2i_1}{dt^2}+\frac{1}{C}\left[\frac{di_1}{dt}-\frac{di_2}{dt}\right]=\frac{d^2v(t)}{dt^2}$$

Letting $t=0^+$, we get

$$R_1\frac{d^2i_1(0^+)}{dt^2}+\frac{1}{C}\left[\frac{di_1(0^+)}{dt}-\frac{di_2(0^+)}{dt}\right]=\frac{d^2v(0^+)}{dt^2}$$

$$\Rightarrow \qquad R_1\frac{d^2i_1(0^+)}{dt^2}=-\frac{1}{C}\frac{di_1(0^+)}{dt}+\frac{d^2v(0^+)}{dt^2}$$

$$\Rightarrow \qquad \frac{d^2i_1(0^+)}{dt^2}=-\frac{1}{R_1C}\left\{\frac{1}{R_1}\frac{dv(0^+)}{dt}-\frac{1}{R_1^2C}v(0^+)\right\}+\frac{d^2v(0^+)}{dt^2}$$

R.P 4.6

Determine $v_a(0^-)$ and $v_a(0^+)$ for the network shown in Fig RP 4.6 (a). Assume that the switch is closed at $t=0$.

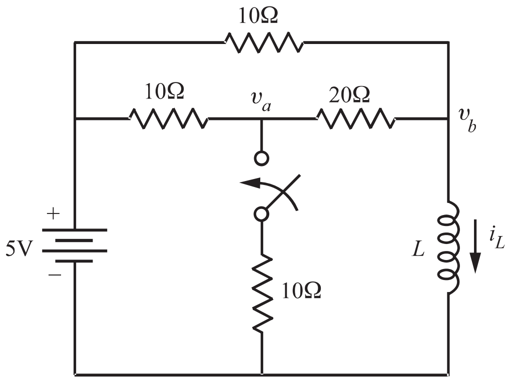

Figure R.P.4.6(a)

SOLUTION

Since L is short for DC at steady state, the network at $t=0^-$ is as shown in Fig. RP 4.6 (b).

Applying KCL at junction a, we get

$$\frac{v_a(0^-)-5}{10}+\frac{v_a(0^-)-v_b(0^-)}{20}=0$$

Since $v_b(0^-)=0$, we get

$$\frac{v_a(0^-)-5}{10}+\frac{v_a(0^-)-0}{20}=0$$

$$\Rightarrow \qquad v_a(0^-)=\frac{0.5}{0.1+0.05}=\frac{10}{3}\text{ volts}$$

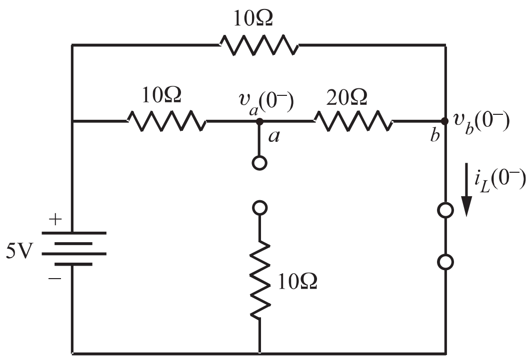

Figure R.P.4.6(b)

Also, $$i_L(0^-) = i_L(0^+) = \frac{v_a(0^-)}{20} + \frac{5}{10}$$
$$= \frac{2}{3}\text{ A}$$

For $t \geq 0^+$, we can write

$$\frac{v_a - 5}{10} + \frac{v_a}{10} + \frac{v_a - v_b}{20} = 0$$

and $$\frac{v_b - v_a}{20} + \frac{v_b - 5}{10} + i_L = 0$$

Simplifying at $t = 0^+$, we get

$$\frac{1}{4}v_a(0^+) - \frac{1}{20}v_b(0^+) = \frac{1}{2}$$

and $$-\frac{1}{20}v_a(0^+) + \frac{3}{20}v_b(0^+) = \frac{-1}{6}$$

Solving we get, $v_a(0^+) = \dfrac{40}{21} = 1.905$ volts

Exercise problems

E.P 4.1

Refer the circuit shown in Fig. E.P. 4.1 Switch K is closed at $t = 0$.

Find $i(0^+)$, $\dfrac{di(0^+)}{dt}$ and $\dfrac{d^2i(0^+)}{dt^2}$.

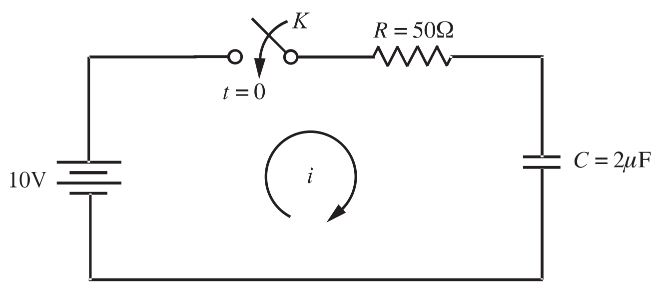

Figure E.P.4.1

Ans: $\boldsymbol{i(0^+) = 0.2\text{A}, \quad \dfrac{di(0^+)}{dt} = -2\times10^3\text{A/sec}, \quad \dfrac{d^2i(0^+)}{dt^2} = 20\times10^6\text{A/sec}^2}$

E.P 4.2

Refer the circuit shown in Fig. E.P. 4.2. Switch K is closed at $t = 0$. Find the values of i, $\frac{di}{dt}$ and $\frac{d^2i}{dt^2}$ at $t = 0^+$.

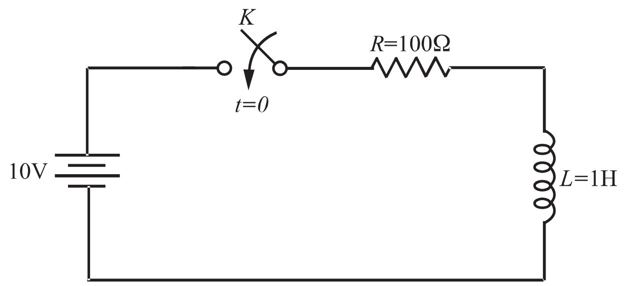

Figure E.P.4.2

Ans: $\boldsymbol{i(0^+) = 0, \quad \frac{di(0^+)}{dt} = 10\ \text{A/sec}, \quad \frac{d^2i(0^+)}{dt^2} = -1000\ \text{A/sec}^2}$

E.P 4.3

Refering to the circuit shown in Fig. E.P. 4.3, switch is changed from position 1 to position 2 at $t = 0$. The circuit has attained steady state before switching. Determine i, $\frac{di}{dt}$ and $\frac{d^2i}{dt^2}$ at $t = 0^+$.

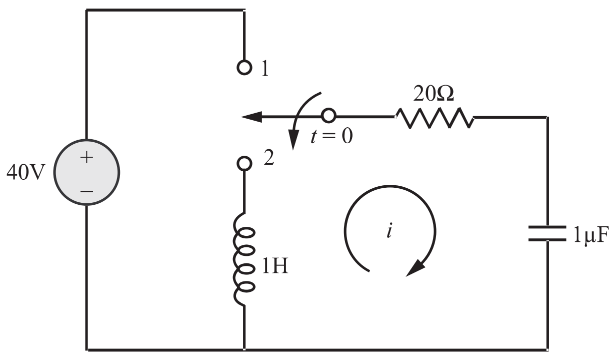

Figure E.P.4.3

Ans: $\boldsymbol{i(0^+) = 0, \quad \frac{di(0^+)}{dt} = -40\ \text{A/sec}, \quad \frac{d^2i(0^+)}{dt^2} = 800\ \text{A/sec}^2}$

E.P 4.4

In the network shown in Fig. E.P.4.4, the initial voltage on C_1 is V_a and on C_2 is V_b such that $v_1(0^-) = V_a$ and $v_2(0^-) = V_b$. Find the values of $\frac{dv_1}{dt}$ and $\frac{dv_2}{dt}$ at $t = 0^+$.

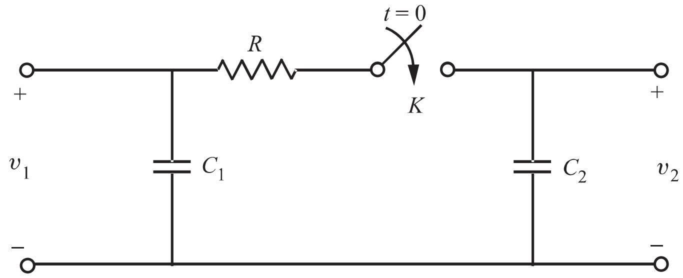

Figure E.P.4.4

Ans: $$\frac{dv_1(0^+)}{dt} = \frac{V_b - V_a}{C_1 R}\,\mathrm{V/sec}, \quad \frac{dv_2(0^+)}{dt} = \frac{V_a - V_b}{C_2 R}\,\mathrm{V/sec}$$

E.P 4.5

In the network shown in Fig E.P. 4.5, switch K is closed at $t = 0$ with zero capacitor voltage and zero inductor current. Find $\frac{d^2v_2}{dt^2}$ at $t = 0^+$.

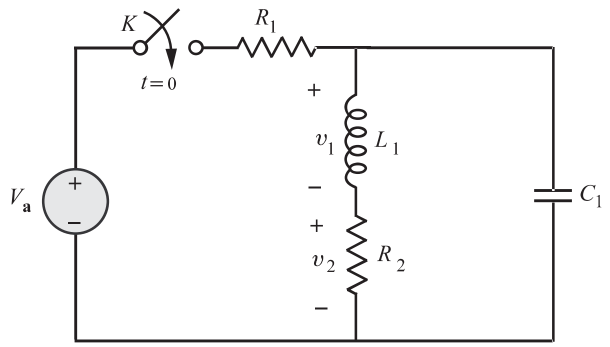

Figure E.P.4.5

Ans: $$\frac{d^2v_2(0^+)}{dt^2} = \frac{R_2 V_a}{R_1 L_1 C_1}\,\mathrm{V/sec^2}$$

E.P 4.6

In the network shown in Fig. E.P. 4.6, switch K is closed at $t = 0$. Find $\dfrac{d^2v_1}{dt^2}$ at $t = 0^+$.

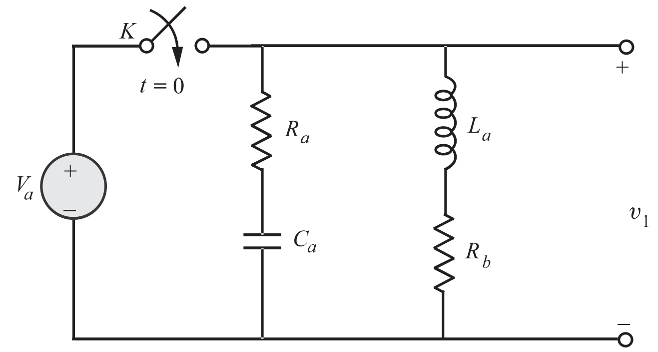

Figure E.P.4.6

Ans: $\boldsymbol{\dfrac{d^2v_1(0^+)}{dt^2} = 0\ \mathrm{V/sec^2}}$

E.P 4.7

The switch in Fig. E.P. 4.7 has been closed for a long time. It is open at $t = 0$. Find $\dfrac{di(0^+)}{dt}$, $\dfrac{dv(0^+)}{dt}$, $i(\infty)$ and $v(\infty)$.

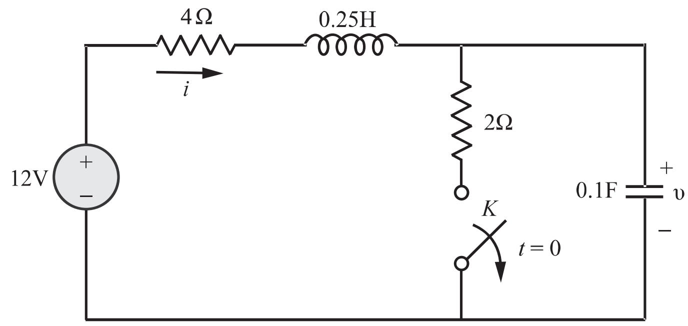

Figure E.P.4.7

Ans: $\boldsymbol{\dfrac{di(0^+)}{dt} = 0\mathrm{A/sec}}$, $\boldsymbol{\dfrac{dv(0^+)}{dt} = 20\mathrm{A/sec}}$, $\boldsymbol{i(\infty) = 0\mathrm{A}}$, $\boldsymbol{v(\infty) = 12\mathrm{V}}$

E.P 4.8

In the circuit of Fig E.P. 4.8, calculate $i_L(0^+)$, $\dfrac{di_L(0^+)}{dt}$, $\dfrac{dv_C(0^+)}{dt}$, $v_R(\infty)$, $v_C(\infty)$ and $i_L(\infty)$.

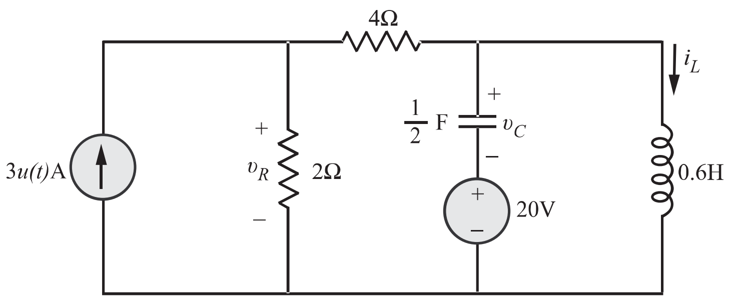

Figure E.P.4.8

Ans: $\boldsymbol{i_L(0^+) = 0\ \mathrm{A},\ \dfrac{di_L(0^+)}{dt} = 0\ \mathrm{A/sec}}$

$$\boldsymbol{\frac{dv_C(0^+)}{dt} = 2\ \mathrm{V/sec},\quad v_R(\infty) = 4\mathrm{V},\quad v_C(\infty) = -20\mathrm{V},\quad i_L(\infty) = 1\mathrm{A}}$$

E.P 4.9

Refer the circuit shown in Fig. E.P. 4.9. Assume that the switch was closed for a long time for $t < 0$. Find $\dfrac{di_L(0^+)}{dt}$ and $i_L(0^+)$. Take $v(0^+) = 8$ V.

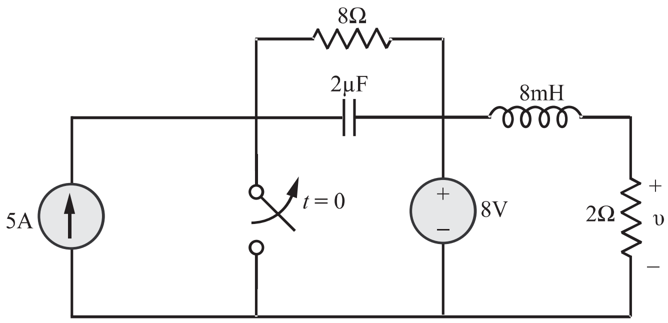

Figure E.P.4.9

Ans: $\boldsymbol{i_L(0^+) = 4\ \mathrm{A},\ \dfrac{di_L(0^+)}{dt} = 0\ \mathrm{A/sec}}$

E.P 4.10

Refer the network shown in Fig. E.P. 4.10. A steady state is reached with the switch K closed and with $i = 10$A. At $t = 0$, switch K is opened. Find $v_2(0^+)$ and $\dfrac{dv_2(0^+)}{dt}$.

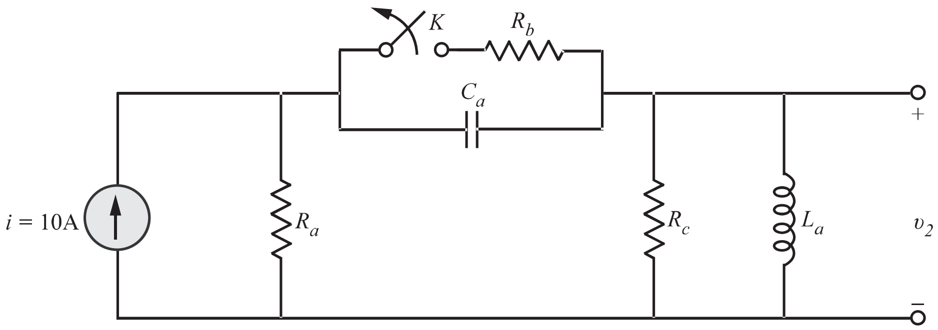

Figure E.P.4.10

Ans: $\boldsymbol{v_2(0^+) = 0,} \qquad \boldsymbol{\dfrac{dv_2(0^+)}{dt} = \dfrac{10R_aR_c}{C_a(R_a+R_b)(R_a+R_c)}}$ V/ sec.

E.P 4.11

Refer the network shown in Fig. E.P. 4.11. The network is in steady state with switch K closed. The switch is opened at $t = 0$. Find $v_k(0^+)$ and $\dfrac{dv_k(0^+)}{dt}$.

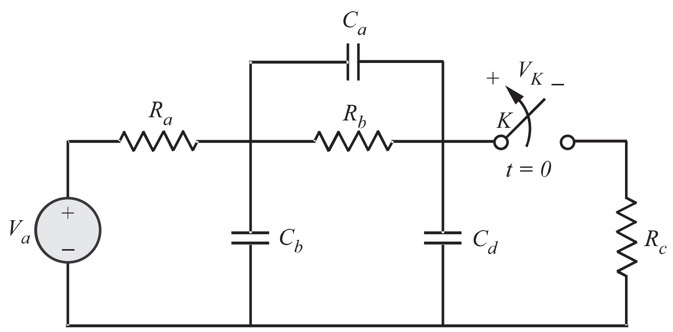

Figure E.P.4.11

Ans: $\boldsymbol{v_k(0^+) = \dfrac{V_aR_c}{R_a+R_b+R_c}}$ **Volts,**

$$\boldsymbol{\frac{dv_k(0^+)}{dt} = \frac{V_a(C_a+C_b)}{(R_a+R_b+R_c)(C_aC_d+C_bC_a+C_bC_d)}\text{V/ sec}}$$

E.P 4.12

Refer the network shown in Fig. E.P. 4.12. Find $\dfrac{d^2i_1(0^+)}{dt^2}$.

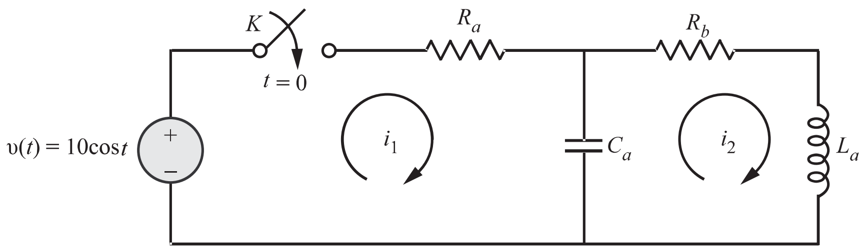

Figure E.P.4.12

Ans: $$\frac{d^2 i_1(0^+)}{dt^2} = \frac{1}{R_a}\left[-10 + \frac{10}{R_a^2 C_a^2}\right] \text{A/sec}^2$$

E.P 4.13

Refer the circuit shown in Fig. E.P. 4.13. Find $\frac{di_1(0^+)}{dt}$. Assume that the circuit has attained steady state at $t = 0^-$.

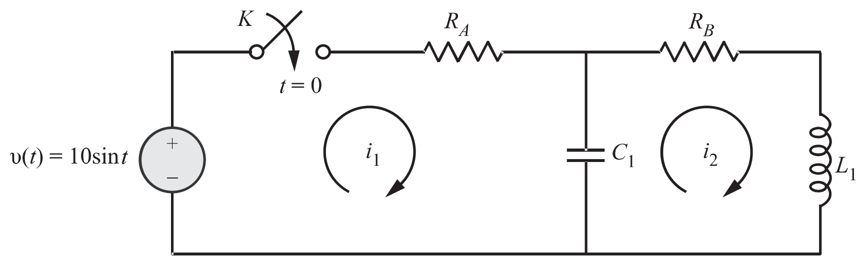

Figure E.P.4.13

Ans: $$\frac{di_1(0^+)}{dt} = \frac{10}{R_A} \text{A/sec}$$

E.P 4.14

Refer the network shown in Fig. E.P.4.14. The circuit reaches steady state with switch K closed. At a new reference time, $t = 0$, the switch K is opened. Find $\frac{dv_1(0^+)}{dt}$ and $\frac{d^2 v_2(0^+)}{dt^2}$.

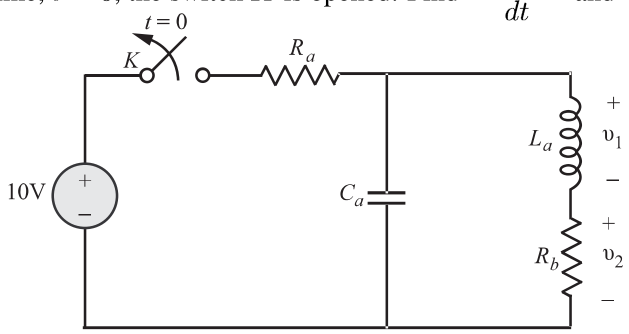

Figure E.P.4.14

Ans: $\dfrac{dv_1(0^+)}{dt} = \dfrac{-10}{C_a(R_a + R_b)}$ **V/sec,** $\dfrac{d^2v_2(0^+)}{dt^2} = \dfrac{-10R_b}{L_aC_a(R_a + R_b)}$ **V/sec²**

E.P 4.15

The switch shown in Fig. E.P. 4.15 has been open for a long time before closing at $t = 0$. Find: $i_0(0^-), i_L(0^-)$ $i_0(0^+), i_L(0^+), i_0(\infty), i_L(\infty)$ and $v_L(\infty)$.

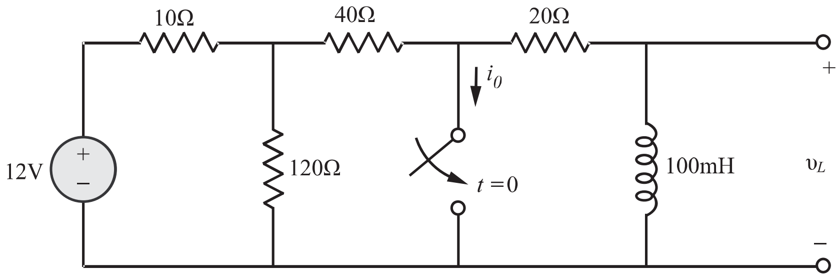

Figure E.P.4.15

Ans: $\boldsymbol{i(0^-) = 0,\ \ i_L(0^-) = 160\text{mA},\ \ i_0(0^+) = 65\text{mA},\ \ i_L(0^+) = 160\text{mA},}$
$\boldsymbol{i_0(\infty) = 225\text{mA}, i_L(\infty) = 0, v_L(\infty) = 0}$

E.P 4.16

The switch shown in Fig. E.P. 4.16 has been closed for a long time before opeing at $t = 0$.
Find: $i_1(0^-), i_2(0^-), i_1(0^+), i_2(0^+)$. Explain why $i_2(0^-) \neq i_2(0^+)$.

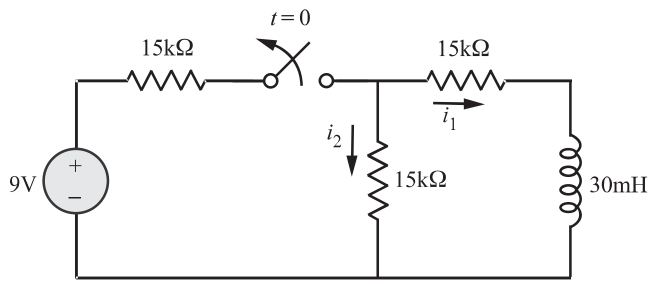

Figure E.P.4.16

Ans: $\boldsymbol{i_1(0^-) = i_2(0^-) = 0.2\text{mA}, i_2(0^+) = -i_1(0^+) = -0.2\text{mA}}$

E.P 4.17

The switch in the circuit of Fig E.P.4.17 is closed at $t = 0$ after being open for a long time. Find:

(a) $i_1(0^-)$ and $i_2(0^-)$

(b) $i_1(0^+)$ and $i_2(0^+)$

(c) Explain why $i_1(0^-) = i_1(0^+)$

(d) Explain why $i_2(0^-) \neq i_2(0^+)$

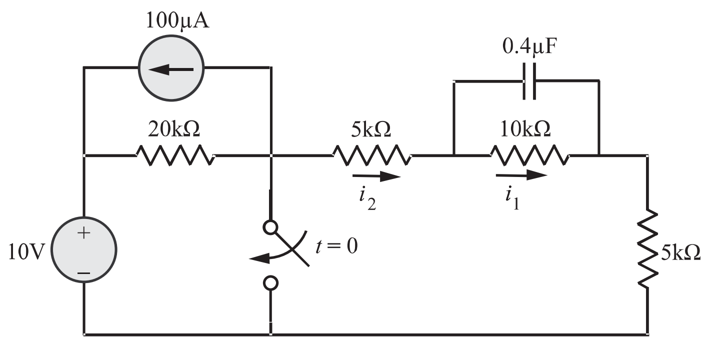

Figure E.P.4.17

Ans: $\mathbf{i_1(0^-) = i_2(0^-) = 0.2\ mA,\ i_1(0^+) = 0.2\ mA,\ i_2(0^+) = –0.2mA}$

Outcomes:

1. Analyze the transient behavior of RC and RL circuits.
2. Determine the transient response of second-order systems and understand the concepts of under damped, over damped, and critically-damped circuits.
3. Explain transient phenomena and analyze the transient behavior of simple circuits.
4. Explain concepts such as steady-state response, transient response and total response as they apply to electronic circuits;
5. Describe the transient behavior of simple RC and RL circuits;
6. Predict the transient response of generalized first-order systems from a knowledge of its initial and final values;
7. Sketch increasing or decreasing exponential waveforms and identify key characteristics;
8. Describe the output of simple RC and RL circuits in response to a square-wave input;
9. Outline the transient behavior of various forms of second-order system.

Resources:

1. http://www.ee.ic.ac.uk/hp/staff/dmb/courses/ccts1/ccts1.htm
2. http://nptel.ac.in/courses/108105053/pdf/L-10(GDR)(ET)%20((EE)NPTEL).pdf
3. http://www.calvin.edu/~svleest/circuitExamples/TransientAnalysis/theory.htm